Canonical Heights, Nef Divisors, and Arithmetic Degrees

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Canonical Heights on Abelian Varieties

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Let $A/\bar{\mathbb{Q}}$ be an abelian variety. The (quadratic part of the) **Néron–Tate canonical height** on A relative to a divisor D is given by the limit

$$\hat{h}_{A,D}(x) = \lim_{n\to\infty} \frac{1}{n^2} h_{A,D}(nx).$$

It has well-known properties, such as

$$\hat{h}_{A,D}(nx) = n^2 \hat{h}_{A,D}(x)$$

and

$$\hat{h}_{A,D}(x) = h_{A,D}(x) + O(1).$$

Further, if ***D* is ample**, then

$$\hat{h}_{A,D}(x) = 0 \quad \Longleftrightarrow \quad x \in A_{\text{tors}}.$$

Goal. Generalize to divisors D that are **nef**.

Canonical Heights for Divisors in $\mathrm{Div}(A) \otimes \mathbb{R}$

The map $\hat{h}_{A,D} : A(\bar{\mathbb{Q}}) \to \mathbb{R}$ is a quadratic form. Let

$$\langle x, y \rangle_{A,D} = \hat{h}_{A,D}(x+y) - \hat{h}_{A,D}(x) - \hat{h}_{A,D}(y)$$

be the associated bilinear form.

The map $\hat{h}_{A,D}$ may be extended $\mathbb{R}$-linearly to a quadratic form

$$\hat{h}_{A,D} : A(\bar{\mathbb{Q}}) \otimes \mathbb{R} \longrightarrow \mathbb{R}$$

for divisors $D \in \mathrm{Div}(A) \otimes \mathbb{R}$. Further, if D is ample, then $\hat{h}_{A,D}$ is positive definite on $A(\bar{\mathbb{Q}}) \otimes \mathbb{R}$.

Explicitly, if

$$x = \sum_i a_i x_i \in A(\bar{\mathbb{Q}}) \otimes \mathbb{R} \quad \text{and} \quad D = \sum_k c_k D_k \in \mathrm{Div}(A) \otimes \mathbb{R},$$

then

$$\hat{h}_{A,D}(x) = \langle x, x \rangle_{A,D} = \sum_{i,j,k} a_i a_j c_k \langle x_i, x_j \rangle_{A,D_k}.$$

Nef Divisors on (Abelian) Varieties

A divisor $D \in \mathrm{Div}(X) \otimes \mathbb{R}$ is said to be **nef (Numerically Effective)** if

$$D \cdot C \geq 0 \quad \text{for all curves } C \subset X.$$

The **Nef Cone** is the cone in $\mathrm{NS}(X) \otimes \mathbb{R}$ generated by all nef divisors.

An alternative definition is that the Nef Cone is the real closure of the ample cone. So

$$D \text{ is nef} \quad \Longleftrightarrow \quad nD + H \text{ is ample}$$

for every ample divisor H and every $n \geq 1$.

If $D \in \mathrm{Div}(A) \otimes \mathbb{R}$ is nef, then one easily sees that

$$\hat{h}_{A,D}(x) \geq 0 \quad \text{for all } x \in A(\bar{\mathbb{Q}}).$$

But it is no longer clear which points have $\hat{h}_{A,D}(x) = 0$.

Canonical Heights for Nef Divisors

Nef Canonical Height Theorem. (SK–JS) Let $A/\bar{\mathbb{Q}}$ be an abelian variety, and let $D \in \operatorname{Div}(A) \otimes \mathbb{R}$ be a non-zero nef divisor. Then there is a unique connected abelian subvariety $B_D \subsetneq A$ such that

$$\{x \in A(\bar{\mathbb{Q}}) : \hat{h}_{A,D}(x) = 0\} = B_D(\bar{\mathbb{Q}}) + A(\bar{\mathbb{Q}})_{\text{tors}}.$$

This theorem generalizes the classical result for ample divisors. Further, if A is *simple*, then the classical equivalence

$$\hat{h}_{A,D}(x) = 0 \iff x \in A_{\text{tors}} \quad \text{holds for nef divisors,}$$

since the simplicity of A forces $B_D = 0$.

Why should one care about nef heights? One answer involves arithmetic degrees, which leads us to the second part of the talk.

Arithmetic Degree

Rational Maps and Arithmetic Degrees

We now consider a (smooth projective) variety and a rational map

$$f : X \dashrightarrow X$$

defined over $\bar{\mathbb{Q}}$, and we fix a height

$$h_{X,H} : X(\bar{\mathbb{Q}}) \longrightarrow [1,\infty) \quad \text{with } H \in \operatorname{Div}(X) \text{ ample.}$$

The **arithmetic degree** of a point $x \in X(\bar{\mathbb{Q}})$ is

$$\alpha_f(x) = \lim_{n\to\infty} h_{X,H}\bigl(f^n(x)\bigr)^{1/n} \quad \text{(if the limit exists).}$$

The arithmetic degree measures the arithmetic complexity of the orbit $\mathcal{O}_f(x) = \{x, f(x), f^2(x), \ldots\}$. Thus

$$\lim_{n\to\infty} \frac{\#\{y \in \mathcal{O}_f(x) : h_{X,H}(y) \le B\}}{\log B} = \frac{1}{\log \alpha_f(x)}.$$

Arithmetic Degree Conjectures

Conjecture. Assume that the orbit $\mathcal{O}_f(x)$ exists.
(a) The limit defining $\alpha_f(x)$ exists.
(b) $\alpha_f(x)$ is an algebraic integer.
(c) The set $\{\alpha_f(x) : x \in X(\bar{\mathbb{Q}})\}$ is a finite set.

In general, when f is a rational map, these conjectures seem hard. But they are true for morphisms.

Theorem. (SK–JS) If $f : X \to X$ is a morphism, then $\alpha_f(x)$ exists, and $\{\alpha_f(x) : x \in X(\bar{\mathbb{Q}})\}$ is a finite set of algebraic integers.

The proof uses canonical heights for Jordan blocks for the action of f^* on $\mathrm{Pic}(X) \otimes \mathbb{C}$. These heights generalize the classical canonical heights associated to eigendivisors of f^*.

Jordan Block Canonical Heights

Canonical Heights for Jordan Blocks

Canonical heights exist if $f^*D \sim \lambda D$, but in general the action of f^* need not be diagonalizable. So we look at Jordan blocks.

Let $|\lambda| > 1$, and suppose that the divisors

$$D_0, D_1, D_2, \ldots \in \operatorname{Div}(X) \otimes \mathbb{C}$$

satisfy linear equivalences

$$\begin{aligned} f^*D_0 &\sim \lambda D_0 \\ f^*D_1 &\sim D_0 + \lambda D_1 \\ f^*D_2 &\sim \qquad\quad D_1 + \lambda D_2 \\ &\vdots \qquad\qquad\quad \ddots \qquad \ddots \end{aligned}$$

Then in general, the limit

$$\lim_{n\to\infty} \lambda^{-n} h_{D_k}\bigl(f^n(x)\bigr)$$

does ***not*** converge. So we add correction terms.

Canonical Heights for Jordan Blocks

Theorem. Consider the following limits, defined recursively for $k = 0, 1, 2, \ldots,$,

$$\hat{h}_{D_k} = \lim_{n\to\infty} \left(\lambda^{-n} h_{D_k} \circ f^n - \sum_{i=1}^{k} \binom{n}{i} \lambda^{-i} \hat{h}_{D_{k-i}} \right).$$

Then $\hat{h}_{D_k}(x)$ converges. The resulting functions satisfy:

(a) $\hat{h}_{D_k} = h_{D_k} + O(1)$.

(b) $\hat{h}_{D_k} \circ f = \lambda \hat{h}_{D_k} + \hat{h}_{D_{k-1}}$.

If $|\lambda| > \overline{\alpha}_f(x)^{1/2}$, a similar statement is true with the linear equivalences replaced by algebraic equivalences, although the normalization condition (a) becomes weaker.

The proof is an adaptation of Tate's telescoping sum argument.

Dynamical Degrees and Arithmetic Degrees

Dynamical Degrees

A fourth, and deeper, part of the conjecture relates arithemtic degrees to geometrically defined dynamical degrees.

For a dominant rational map $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ of projective space, the **dynamical degree of f** is

$$\delta_f = \lim_{n\to\infty} \Bigl(\deg(f^n)\Bigr)^{1/n}.$$

N.B. In general, $\deg(f^n) \ne (\deg f)^n$.

Example The map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad f(x : y : z) = (xy : xz : z^2),$$

has dynamical degree

$$\delta_f = \frac{1+\sqrt{5}}{2}.$$

Dynamical Degrees (continued)

Let $N = \dim(X)$ and choose an ample divisor H on X. Then the **dynamical degree of f** is

$$\delta_f = \lim_{n\to\infty} \left((f^n)^* H \cdot H^{N-1} \right)^{1/n}.$$

N.B. In general, $(f^n)^* \neq (f^*)^n$.

Alternative Definition.

$$\delta_f = \lim_{n\to\infty} \left(\begin{array}{l} \text{Spectral raduis of } (f^n)^* \\ \text{acting on } \mathrm{NS}(X) \otimes \mathbb{R} \end{array} \right)^{1/n}.$$

(If f is a morphism, then don't need to take a limit.)

Conjecture. (Bellon, Viallet 1999)
δ_f is an algebraic integer.

A Fundamental Inequality

Theorem. (SK–JS)

$$\overline{\alpha}_f(x) := \limsup_{n\to\infty} h_{X,H}\bigl(f^n(x)\bigr)^{1/n} \le \delta_f$$

for all $x \in X(\bar{\mathbb{Q}})$.

If $\mathrm{Pic}(X) = \mathbb{Z}$, e.g., $X = \mathbb{P}^N$, then the proof is easy. And still not so hard if $\operatorname{rank} \mathrm{NS}(X) = 1$. But if

$$\operatorname{rank} \mathrm{NS}(X) \ge 2,$$

the proof is more intricate, even if f is a morphism.

Shu will discuss this theorem and its proof in the next talk.

Arithmetic Degree Conjectures (continued)

Conjecture.

(d) $\mathcal{O}_f(x)$ Zariski dense in $X \Longrightarrow \alpha_f(x) = \delta_f$.

The intuition is:

$$\begin{bmatrix}\text{If the } f\text{-orbit of } x \\ \text{is maximal in a ge-} \\ \text{ometric sense}\end{bmatrix} \Longrightarrow \begin{bmatrix}\text{Then the } f\text{-orbit} \\ \text{of } x \text{ is maximal in} \\ \text{an arithmetic sense}\end{bmatrix}.$$

Theorem. (SK–JS) Let $f : A \to A$ be an endomorphism of an abelian variety defined over $\bar{\mathbb{Q}}$. Then Conjecture (d) is true, i.e.,

$$\mathcal{O}_f(x) \text{ Zariski dense in } A \quad \Longrightarrow \quad \alpha_f(x) = \delta_f.$$

The proof uses the Nef Canonical Height Theorem.

An Eigenvalue Lemma and Application

The first step in applying nef canonical heights is:

Lemma. (Birkhoff, AMM, 1967) Let V be a finite dimensional $\mathbb{R}$-vector space, let $T : V \to V$ be a linear transformation with spectral radius λ, and let $\Gamma \subsetneq V$ be a closed cone having the property that $T(\Gamma) \subset \Gamma$. Then there is a non-zero vector $\boldsymbol{v} \in \Gamma$ satisfying $T\boldsymbol{v} = \lambda\boldsymbol{v}$.

Corollary. Let $f : X \to X$ be a morphism. Then there is a non-zero nef $D_f \in \mathrm{NS}(X) \otimes \mathbb{R}$ satisfying

$$f^* D_f \equiv \delta_f D_f.$$

Proof: We apply the lemma to

$$V = \mathrm{NS}(X) \otimes \mathbb{R}, \quad T = f^*, \quad \Gamma = \text{the Nef Cone.}$$

The map f^* preserves the Nef Cone, since

$$f^* D \cdot C = D \cdot f_* C \geq 0.$$

Nef Canonical Heights and Orbits on Abelian Varieties

Using $\hat{h}_{A,D_f}\bigl(f^n(x)\bigr) = \delta_f^n \hat{h}_{A,D_f}(x)$, it is is not hard to show that

$$\hat{h}_{A,D_f}(x) > 0 \quad \Longrightarrow \quad \alpha_f(x) = \delta_f.$$

On the other hand, the Nef Height Theorem gives

$$\hat{h}_{A,D_f}(x) = 0 \quad \Longrightarrow \quad \mathcal{O}_f(x) \subset B_D + A_{\text{tors}}$$

with $B_D \subsetneq A$.

From this is is easy to check that

$$\hat{h}_{A,D_f}(x) = 0 \quad \Longrightarrow \quad \overline{\mathcal{O}_f(x)} \ne A.$$

Hence

$$\overline{\mathcal{O}_f(x)} = A \implies \hat{h}_{A,D_f}(x) > 0 \implies \alpha_f(x) = \delta_f,$$

which proves Conjecture (d) for abelian varieties. (Neither of the converses is true.)

Proof Sketch
of the
Nef Canonical Height Theorem

Proof Sketch of the Nef Canonical Height Theorem

The proof relies on the classical theory of divisors and endomorphisms of abelian varieties.

A divisor class $[D] \in \mathrm{NS}(A)$ induces a homomorphism

$$\phi_D : A \longrightarrow \hat{A} = \mathrm{Pic}^0(A), \qquad \phi_D(x) = [T_x^* D - D],$$

where $T_x(y) = y + x$ is translation-by-x.

Standard Proposition. If D is ample, then the map ϕ_D is an isogeny (has finite kernel).

We fix an ample divisor $H \in \mathrm{Div}(A)$. Then we get an inclusion

$$\Phi : \mathrm{NS}(A) \otimes \mathbb{Q} \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}, \quad \Phi_D = \phi_H^{-1} \circ \phi_D.$$

Having fixed H, we also get a **Rosati involution**

$$\mathrm{End}(A) \longrightarrow \mathrm{End}(A), \quad \alpha \longmapsto \alpha' = \phi_H^{-1} \circ \hat{\alpha} \circ \phi_H.$$

A Characterization of Nef Divisors

Standard facts (from Mumford's *Abelian Varieties*).

Theorem. $\mathrm{End}(A)\otimes\mathbb{R}$ is a product of matrix algebras,

$$\mathrm{End}(A)\otimes\mathbb{R}\cong\prod M_{d_i}(\mathbb{K}),$$

with $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Further, the Rosati involution corresponds to the conjugate transpose involution on the matrix algebras.

Theorem.

(a) $\mathrm{Image}(\Phi)=\{\alpha\in\mathrm{End}(A)\otimes\mathbb{Q}:\alpha'=\alpha\}$.

(b) D is nef if and only if Φ_D is positive semi-definite.

Corollary. If D is nef, then

$$\Phi_D=\alpha'\circ\alpha\quad\text{for some }\alpha\in\mathrm{End}(A)\otimes\mathbb{R}.$$

Canonical Heights and Divisor-Induced Maps

The proof of the Nef Canonical Height Theorem also relies on two canonical height formulas that are of independent interest.

Proposition. Let

$$x, y \in A(\bar{\mathbb{Q}}) \otimes \mathbb{R}, \ \ \alpha \in \mathrm{End}(A) \otimes \mathbb{R}, \ D \in \mathrm{Div}(A) \otimes \mathbb{R}.$$

(a) $\langle x, y \rangle_{A,D} = \bigl\langle x, \Phi_D(y) \bigr\rangle_{A,H}.$

(b) $\bigl\langle \alpha(x), y \bigr\rangle_{A,H} = \bigl\langle x, \alpha'(y) \bigr\rangle_{A,H}.$

Part (a) relates canonical heights for arbitrary divisors to canonical heights relative to H.

Part (b) says that the Rosati involution is the adjoint for the ample canonical height pairing.

Proof of the Nef Canonical Height Theorem

Since the divisor D is nef, we can write $\Phi_D = \alpha' \circ \alpha$ for some $\alpha \in \operatorname{End}(A) \otimes \mathbb{R}$. Then

$$\begin{aligned} 2\hat{h}_{A,D}(x) &= \langle x, x\rangle_{A,D} \\ &= \bigl\langle x, \Phi_D(x)\bigr\rangle_{A,H} \\ &= \bigl\langle x, \alpha' \circ \alpha(x)\bigr\rangle_{A,H} \\ &= \bigl\langle \alpha(x), \alpha(x)\bigr\rangle_{A,H} \\ &= 2\hat{h}_{A,H}\bigl(\alpha(x)\bigr). \end{aligned}$$

Hence

$$\begin{aligned} \hat{h}_{A,D}(x) = 0 \quad &\Longleftrightarrow \quad \hat{h}_{A,H}\bigl(\alpha(x)\bigr) = 0 \\ &\Longleftrightarrow \quad \alpha(x) = 0 \text{ in } A(\bar{\mathbb{Q}}) \otimes \mathbb{R}. \end{aligned}$$

To conclude, we write α as an $\mathbb{R}$-linear combination of elements of $\operatorname{End}(A)$ and apply a linear algebra lemma.

Additional Comments, Directions, Questions

- Conjecture (d) is true for endomorphisms of $\mathbb{G}_m^N$, i.e.,

$$\mathcal{O}_f(x) \text{ Zariski dense in } \mathbb{G}_m^N \quad \Longrightarrow \quad \alpha_f(x) = \delta_f.$$

  The proof uses local heights and linear forms in logs, so is rather different from the abelian variety proof. But both proofs conclude with a linear algebra lemma.

- It would be interesting to prove Conjecture (d) for semi-abelian varieties.

- Other recent work on the Conjecture includes a result by M. Jonsson and E. Wulcan for polynomial maps $\mathbb{A}^2 \to \mathbb{A}^2$ of small topological degree.

I want to thank you for your attention, and Julie and Liang-Chung for inviting me to speak.

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