A real unitary matrix is orthogonal.

A unitary matrix has an orthonormal basis of eigenvectors.

The Fast Fourier Transform

If \( f(x) \) is a function such that \( f(x+2\pi) = f(x) \) (it is periodic with period \( 2\pi \)), \( f \) has a Fourier series
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}
\]

Note that \( e^{inx} \) is periodic with period \( 2\pi \) for any \( n \)

\[
e^{i(n+2\pi)x} = e^{inx} e^{2\pi i x} = e^{inx} \]

with

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.
\]

What we need: a Fourier series is a series

\[
\sum_{n=-\infty}^{\infty} c_n e^{inx}
\]

We will deal with Fourier series of the form

\[
Ce^{i0x} + c_1 e^{ix} + c_2 e^{i2x} + \ldots + c_{n-1} e^{i(n-1)x},
\]

and we want the values \( f\left(\frac{2\pi k}{n}\right) \) for \( k = 0, 1, \ldots, n-1 \)

Example: \( n = 3 \):

\[
f(x) = c_0 + c_1 e^{ix} + c_2 e^{i2x}
\]

\[
f(0) = c_0 + c_1 + c_2
\]

\[
f\left(\frac{2\pi}{3}\right) = c_0 + c_1 e^{\frac{2\pi i}{3}} + c_2 e^{\frac{4\pi i}{3}}
\]

\[
f\left(\frac{4\pi}{3}\right) = c_0 + c_1 e^{\frac{4\pi i}{3}} + c_2 e^{\frac{8\pi i}{3}}
\]
Let \( w = e^{\frac{2\pi i}{3}} \)

Then
\[
y_0 = f(0) = c_0 + c_1 + c_2
\]
\[
y_1 = f\left(\frac{2\pi i}{3}\right) = c_0 + c_1 w + c_2 w^2
\]
\[
y_2 = f\left(\frac{4\pi i}{3}\right) = c_0 + c_1 w^2 + c_2 w^4
\]

So
\[
\begin{bmatrix}
  y_0 \\
  y_1 \\
  y_2
\end{bmatrix} =
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & w & w^2 \\
  1 & w^2 & w^4
\end{bmatrix}
\begin{bmatrix}
  c_0 \\
  c_1 \\
  c_2
\end{bmatrix}
\]

\( F \) is the Fourier matrix for \( n=3 \).

\( F \) is symmetric but not Hermitian.

\( F^* F = 3 I \), so \( F^{-1} = \frac{1}{3} F^* \), and
\[
= \frac{1}{3} F \text{ (bsc } F \text{ symm)}
\]

So
\[
\frac{F}{\sqrt{3}} \text{ is unitary } \left( \frac{F}{\sqrt{3}} \left( \frac{F}{\sqrt{3}} \right)^* = \frac{F F^*}{3} = I \right).
\]

For a general \( n \), the Fourier matrix \( F_n \) is

\( w = e^{\frac{2\pi i}{n}} \)

\[
\begin{bmatrix}
  1 & 1 & \cdots & 1 \\
  1 & w & w^2 & \cdots & w^{n-1} \\
  1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2}
\end{bmatrix}
\]

\( F_n F_n^* = n I \), \( \frac{F_n}{\sqrt{n}} \) is unitary.

When using Fourier matrices, keep in mind that \( w^n = 1 \) and
\( w^{-1} + w^{-2} + \cdots + 1 = 0 \) (bsc
\[
0 = w^n - 1 = (w-1)(w^{n-1}+\cdots+1)
\]
Application of Pseudo Inverse

Recall, we can only solve $Ax = b$ when $b \in \text{Col}(A)$.  
When we can't solve $Ax = b$, we look for $\hat{x}$ so that 
$Ax = b$ is as close as possible to $b$. This happens when 
$Ax - \hat{b} \perp \text{Col}(A)$, which is  
when $AT(Ax - \hat{b}) = 0$, i.e. 
\[
ATAx = AT\hat{b} \quad \text{(see notes on projections for details)}
\]
which can always be solved for $x^\dagger$.

We saw when $A$ has indep. columns, $ATA$ is invertible, and $x = (ATA)^{-1}AT\hat{b}$. In general, there may be many solutions for $x$. One such solution will always be $x^\dagger = A^+ \hat{b}$, i.e.,
\[
\hat{b} = \text{proj}_{\text{Col}(A)} \hat{b} + \text{proj}_{\text{N}(AT)} \hat{b} \quad \text{w/c} \quad \text{Col}(A^\perp) = \text{N}(AT),
\]

Therefore, 
\[
ATAx = AT\hat{b}
\]
\[
= AT(ATAx - \hat{b})
\]
\[
= AT(\text{proj}_{\text{Col}(A)} \hat{b} - \hat{b})
\]
\[
= AT(-\text{proj}_{\text{N}(AT)} \hat{b}) = -AT(\text{proj}_{\text{N}(AT)} \hat{b})
\]
\[
= 0
\]
So 
\[
ATAx = AT\hat{b}
\]