Math 520 Practice Problems for the Final

1. True or False: If $A$ is a (square) diagonalizable matrix, then there is a matrix $B$ so that $B^2 = A$. Explain. Let $A = \begin{bmatrix} 1 & -4 & 2 \\ 3 & -4 & 0 \\ 3 & -1 & -3 \end{bmatrix}$. Find $B$ so that $B^2 = A$.

2. More T/F. If true, why? If false, provide a counterexample, and think about what changes would make the statement true.
   
   (a) The vectors $(2i, 2 + 3i)$ and $(2i - 2, -1 + 5i)$ are linearly independent.
   
   (b) If $A$ is Markov and $A^\infty$ exists, then $A$ has exactly one eigenvalue $\lambda$ (counted with algebraic multiplicity) with $|\lambda| = 1$.
   
   (c) If $A^T = -A$, then all eigenvalues of $A$ are purely imaginary ($= bi, b \in \mathbb{R}$).
   
   (d) For any matrix $A$, $N(A) = N(A^T A)$.
   
   (e) For any matrix $A$, rank($A$) = rank($A^T A$).

3. Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 8 \end{bmatrix}$. Compute $A^+$ and $(AA^+)^2$.

4. Let $A = \begin{bmatrix} 0 & s \\ 1 & 1 - s \end{bmatrix}$. For which $s$ with $0 \leq s \leq 1$ does $A^\infty = \lim_{k \to \infty} A^k$ exist?

5. Find a basis for the orthogonal complement of the one-dimensional subspace of $\mathbb{C}^3$ spanned by $(1 + i, 1, 2i)$.

6. Find a symmetric matrix $A$ so that

$$\mathbf{x}^T A \mathbf{x} = 4 \left( \frac{x_1}{\sqrt{14}} + \frac{2x_2}{\sqrt{14}} + \frac{3x_3}{\sqrt{14}} \right)^2 + 2 \left( \frac{3x_1}{\sqrt{10}} - \frac{x_3}{\sqrt{10}} \right)^2.$$ 

7. Let $r(t)$ be the number of robins at time $t$, and let $w(t)$ be the number of worms at time $t$. Assume that the robins and worms are governed by the relationship

$$r' = r + 2w \quad \quad w' = -3r + 6w.$$ 

Initially, there are 160 robins and 210 worms. Compute the limit

$$\lim_{t \to \infty} \frac{r(t)}{w(t)}.$$ 

8. The Jibonacci numbers $z_k$ are defined by the formula

$$z_{k+2} = 3z_{k+1} - 2z_k,$$

and $z_0 = 0, z_1 = 1$. Find $z_{100}$. 

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9. Find the intersection of the two spaces in $\mathbb{R}^4$:

$$X = \{(0, 1, 1, 0) + a(-3, 1, 0, 0) + b(1, 0, 1, 0) + c(0, 0, 0, 1)\}$$

$$Y = \{(1, 2, -1, 0) + d(1, 0, -2, 0) + e(0, 1, 1, 0) + f(0, 0, 2, 1)\}$$

(Here, $a$, $b$, $c$, $d$, $e$, and $f$ run over all real numbers.)

10. For the following matrices $A$, $B$ and $C$ which matrix decompositions exist? Do not compute them.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Which of the following exist?

- $LU$
- $QR$
- $SAS^{-1}$
- $Q\Lambda Q^T$
- $QH$
- $R^T R$
- $SVD$