Areas + Indefinite Integrals

We can approximate the area under the curve $y = f(x)$ from $a$ to $b$ with rectangles.

Here we are using left hand endpoints to find the heights of our rectangles.

$[a, b]$ has been divided into $n$ subintervals of (equal) length $\frac{b-a}{n}$. Let $\Delta = \frac{b-a}{n}$.

The sum of the areas of our $n$ rectangles is:

$$\sum_{i=0}^{n} f(x_i) \Delta$$

where $\Delta$ is the width of any rectangle and $f(x_i)$ is the height of the $i^{th}$ rectangle.

If we used right hand endpoints instead, the sum would be $\sum_{i=1}^{n} f(x_i) \Delta$.

**Definition**

We define the integral $\int_{a}^{b} f(x) \, dx$ of $f(x)$ from $a$ to $b$ to be

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta,$$

where $\Delta = \frac{b-a}{n}$, $x_0 = a$, $x_n = b$, $x_i = a + i \Delta$.

If this limit exists,

If the limit exists, $f$ is called integrable.
- Just like \( f'(x) \) may not exist, \( \int_a^b f(x) \, dx \) may not exist either.
- When the graph of \( y = f(x) \) is above the \( x \)-axis:

  \[
  \int_a^b f(x) \, dx = \text{area under the curve } y = f(x)
  \]
  from \( x = a \) to \( x = b \).

**Example**

\( y = x \), from \( x = 1 \) to \( x = 3 \):

\[
\Delta = \frac{3-1}{n}
\]

\[
x_i = 1 + \frac{2i}{n}
\]

\[
\text{Area} = \lim_{n \to \infty} \frac{n}{n} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \frac{2}{n}
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) \frac{2}{n}
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \left[ \frac{n^2}{2} + \frac{n}{2} \frac{2}{3} \right]
\]

Can also compute this area using geometry.
Example
- Express the area under the curve from $x=0$ to $x=1$ as an integral.

\[ y = 1 - x^2 \]
\[ \int_0^1 (1 - x^2) \, dx \]

**Ex** What is \[ \int_{-1}^1 \sqrt{1-x^2} \, dx \]?

**Answer**

Area = \[ \frac{1}{2} \pi (1)^2 \]
\[ = \pi/2 \]

Properties of the Definite Integral

\[ \int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx \]

\[ \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

If \( f(x) \geq 0 \) for all \( x \) in \([a, b]\), then \( \int_a^b f(x) \, dx \) is the area under the graph of \( f \) from \( a \) to \( b \).

\[ \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \] if \( f(x) \leq g(x) \) for all \( x \) in \([a, b]\).

\[ \int_a^a f(x) \, dx = 0 \]

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]