Math 60, Midterm 1.

Show all your work for full credit. 48 points total.

1. Evaluate the following expressions. (4 points each)

(a) \[ \int 2x^5 + 3x + 2 \, dx = \frac{1}{3}x^6 + \frac{3}{2}x^2 + 2x + C \]

(b) \[ \int_0^1 2x^3(x^4 + 2)^3 \, dx \]

\[ = \int_2^3 u^3 \cdot \frac{1}{2} \, du \]

\[ = \frac{1}{2} \left( \frac{1}{4} u^4 \right)_2^3 \]

\[ = \frac{1}{2} \cdot \frac{1}{4} \left( 3^4 - 2^4 \right) \]

\[ = \frac{1}{8} (81 - 16) = \frac{65}{8} \]
(c) \int_{-1}^{1} (x^2 + 3)^2 \, dx
\begin{align*}
&= \int_{-1}^{1} X^4 + 6X^2 + 9 \, dx \\
&= 2 \left[ \frac{1}{5} X^5 + 2X^3 + 9X \right]_0^1 \\
&= 2 \left( \frac{1}{5} + 2 + 9 \right) = \boxed{22.8} \\
\end{align*}

(d) \int \frac{\ln x}{x} \, dx
\begin{align*}
&= \frac{1}{2} \ln^2 x + C \\
&= \boxed{\frac{1}{2} (\ln x)^2 + C}
\end{align*}
\[
(e) \frac{d}{dx} \left( \int_2^{x^2} \frac{1}{t^3} \, dt \right) = \frac{1}{(x^2)^3} \cdot \frac{d}{dx} \left( x^6 \right) \\
= \frac{2x^5}{x^6} = \left( \frac{2}{x} \right)
\]
2. Let $f(x) = 2x$ throughout this problem.

(a) List three antiderivatives of $f$ and sketch the graph of each of them on the same coordinate axes. (4 points)

(b) Suppose we know that $F$ is an antiderivative of $f$ and that $F(1) = 3$. What is $F''$? (4 points)

\[ F(x) = x^2 + C \]

\[ 3 = F(1) = 1^2 + C \quad \Rightarrow \quad C = 2 \]

\[ F(x) = x^2 + 2 \]
3. (a) Sketch the graph of \( y = |t| - 1 \) and label the points where the graph crosses the \( t \) and \( y \)-axes. (2 points)

(b) For \( x \geq -1 \), which value of \( x \) does the function \( f(x) = \int_{-1}^{x} |t| - 1 \, dt \) achieve its minimum? Explain. (4 points)

The graph of \( y = |t| - 1 \) is the area between the graph of \( y = |t| - 1 \) and the \( t \)-axis from \( t = -1 \) to \( t = x \), but we count the area as negative if it is below the \( t \)-axis. We get all our negative area from \( t = -1 \) and \( t = 1 \), after which point we have positive area (b/c the graph is then above the \( t \)-axis (see graph in part-a)).

So \( x = 1 \)
4. (a) Using the Trapezoid Rule with two trapezoids, approximate the area under the curve \( y = \ln x \) from \( x = 2 \) to \( x = 4 \). (4 points)

\[
\Delta = \frac{4 - 2}{2} = 1
\]

Area of two trapezoids is

\[
\frac{1}{2} \left( \frac{\ln(2) + \ln(3)}{2} \right) + \frac{1}{2} \left( \frac{\ln(3) + \ln(4)}{2} \right)
\]

\[
= \frac{1}{2} \left( \ln(2) + \ln(3) + \ln(4) \right)
\]

(b) Verify that \( \int \ln x \, dx = x \ln x - x + C \) by differentiating \( x \ln x - x \).

(3 points)

\[
= \frac{d}{dx} \left( x \ln x - x \right) = x \frac{d}{dx} (\ln x) + \frac{d}{dx} x \ln x - \frac{d}{dx} x
\]

\[
= x \cdot \frac{1}{x} + \ln x - 1 = \ln x
\]
(c) Using 4b, compute $\int_2^4 \ln x \, dx$. (4 points)

$$\left. x \ln x - x \right|_2^4 = (4 \ln(4) - 4) - (2 \ln(2) - 2) = \ln(2^3) - 2 \ln(2) = 3 \ln(2) - 2 \ln(2) = \ln(2)$$

(d) Is your answer from 4a greater than or less than your answer from 4c? Explain (drawing a picture that shows the quantities that your answers from 4a and 4c represent will help). (3 points)

Answer in part a) is less because trapezoids lie entirely under the curve and part c) measures all of area under curve $y = \ln x$. Better picture.