Solving $Ax = b$

We previously saw how to solve $Ax = 0$.

This was equivalent to solving $\text{rref}(A)\hat{x} = 0$, where

$A\hat{x} = \hat{b}$ is not equivalent to solving $b = \text{rref}(A)$, because row operations taking $A$ to $b$
also affect $\hat{b}$ (but they don't affect $0$).

$p = \text{rref}(A) \implies \hat{b} = \tilde{b}$ exactly when $E\hat{x} = E\hat{b}$, if $E$ is invertible, (when $\hat{b} = (0, \tilde{b} = 0 + \lambda)$

Example:

$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} 0 \\ 6 \\ 7 \end{bmatrix}$. Want to solve

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 7 \end{bmatrix}$$

We use the augmented matrix $[A \hat{b}]$ and reduce to a matrix $[R \hat{d}]$.

$$[A \hat{b}] = \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}$$

So

$p$ pivot columns of $A$ are col 1, col 3, so pivt

Vars are $x_1, x_3$, free vars are $x_2, x_4$. 

$\text{rref}(A)$
We get one particular solution by setting free variables $x_2$ and $x_4 = 0$ and solving for $x_1$ and $x_3$:

\[
\begin{align*}
 Ax &= \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 4 & 4 & 3 & 3 \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
 x_1 + 3x_2 + 2x_3 &= 1, \\
 x_3 + 4x_4 &= 0.
\end{align*}
\]

So a particular solution is $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

If now $y \in N(A) = \text{null}(A)$, then

\[
\tilde{R}(x + y) = Ax + Ay = \tilde{d} + \tilde{c} = \tilde{d},
\]

It turns out that all solutions to $Ax = \tilde{d}$ are of the form $X_0 + y$, where $y \in N(A)$.

#### Compute $N(A)$

From above, we solve for pivot variables in terms of free variables.

\[
\begin{align*}
 X_1 &= -x_4, \\
 X_2 &= -3x_2 - 2x_4, \\
 X_3 &= -4x_4.
\end{align*}
\]

So

\[
\begin{align*}
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ -1 \end{bmatrix}.
\end{align*}
\]

So the complete solution to our original $Ax = b$ is

\[
\begin{align*}
 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix},
\end{align*}
\]

where $x_2$ and $x_4$ vary over all real numbers.
Note that this set is not a subspace of \( \mathbb{R}^4 \) because it does not contain \( \mathbf{0} \) (since \( \mathbf{Ax} = \mathbf{0} \neq \mathbf{b} \)).

Let's consider the system \( \mathbf{Ax} = \mathbf{b} \) for
\[
\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}, \quad \text{general } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix},
\]

\[
\begin{bmatrix} \mathbf{A} \mathbf{x} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \\ 0 & 0 & 1 & 4 & b_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 1 & 4 & b_3 - b_1 \\ 0 & 0 & 1 & 4 & b_4 \end{bmatrix}
\]

Want to solve \( x_1 + 3x_2 + 2x_3 = b_1 \),
\( x_3 + 4x_4 = b_2 \),
\( 0x_1 + 0x_2 + 0x_3 + 0x_4 = b_3 - b_1 \).

So we need \( b_3 = b_1 + b_2 \).

To find a particular solution, set \( x_2, x_4 = 0 \),

solve to get \( x_3 = b_2, x_1 = b_1 \).

So the complete solution to \( \mathbf{A} \mathbf{x} = \mathbf{b} \) is
\[
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ -4 \end{bmatrix},
\]

\( \mathbf{N}(\mathbf{A}) \).

Note: If \( \mathbf{A} \) is invertible, the only solution to \( \mathbf{Ax} = \mathbf{b} \) is \( \mathbf{A}^{-1} \mathbf{b} \) b/c \( \mathbf{AA}^{-1} = \mathbf{I} \). This is consistent with the complete solution being \( \mathbf{x} = \mathbf{x}_p + \mathbf{x}_n \) where \( \mathbf{x}_p \) is a particular solution and \( \mathbf{x}_n \in \mathbf{N}(\mathbf{A}). \) Since \( \mathbf{A} \) is invertible, \( \mathbf{x}_n = \mathbf{0} \), and our only particular solution is \( \mathbf{x}_p = \mathbf{A}^{-1} \mathbf{b} \).

In general, if all columns of \( \mathbf{A} \) are pivot columns, then \( \mathbf{N}(\mathbf{A}) = \mathbf{0} \) since there are no free variables so no special solutions.
Linear Independence and Bases

Idea: Given a vector space $V$, I want to find vectors $\vec{v}_1, ..., \vec{v}_n \in V$ such that every element of $V$ is a linear combination of $\vec{v}_1, ..., \vec{v}_n$.

Ex: $V = \mathbb{R}^2$ is all 2-dim. vectors

Let $\vec{v}_1 = (1, 0)$, $\vec{v}_2 = (1, 1)$, $\vec{v}_3 = (0, 1)$

Linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are

$$a(1) + b(1) + c(0) = (a+b, 0).$$

This gives me any vector in $\mathbb{R}^2$. I can pick $a, b$ so that $a+b=d$ and then pick $c$ so that $at2b+c=e$.

We say that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span $V$. A set of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ fill $V$. However, $\vec{v}_1$ and $\vec{v}_2$ also span $V$.

$$a(1) + b(1) = (a+b, a+2b)$$

also gives me any vector in $\mathbb{R}^2$ because we can solve $a+b=d \rightarrow b=e-d$, $a+2b=e \rightarrow a = 2d-e$.

Clearly neither $\vec{v}_1$ nor $\vec{v}_2$ spans $V$ by itself.

So $\vec{v}_1, \vec{v}_2$ are a minimal spanning set. Such a set is called a basis. $\vec{v}_1, \vec{v}_2$ are a basis for $V$.

The dimension of a vector space $V$ is the number of elements in any basis for $V$. All bases of a vector space have the same number of elements.
Another example of a basis for $\mathbb{R}^2$ is $(e)$ and $(f)$, because any vector $(g)$ can be written as $(h) + e(1)$, and neither $(i)$ nor $(j)$ span $\mathbb{R}^2$ by itself.

**Definition**

A set of vectors $v_1, v_2, \ldots, v_n$ are called **linearly independent** if $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ only happens when $c_1 = 0, c_2 = 0, \ldots, c_n = 0$.

Equivalently, $v_1, v_2, \ldots, v_n$ are linearly independent if the nullspace of the matrix $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ is just $\mathbf{0}$, i.e., if $A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \text{N}(A)$, then $A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1v_1 + \cdots + c_nv_n = 0$.

If $v_1, \ldots, v_n$ are not linearly independent, they are called **linearly dependent**. In this case, there exists a linear combination $c_1v_1 + \cdots + c_nv_n = 0$ with the $c_i$'s not all 0.

Equivalently, $v_1, \ldots, v_n$ are dependent if one of them is a linear combination of the others.

The zero vector can never be part of a linearly independent set, i.e., $c_0 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ for any choice of $c_i$.

**Examples**

Two vectors in a vector space are independent exactly when one is a multiple of the other.

If $c_1v_1 = v_2$, then $c_1v_1 - v_2 = 0$.

And if $c_1v_1 + c_2v_2 = 0$ with, say, $c_1 > 0$, then $v_1 = \frac{-c_2}{c_1}v_2$. 