Carleson measures and elliptic boundary value problems

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Abstract. In this article, we highlight the role of Carleson measures in elliptic boundary value problems, and discuss some recent results in this theory. The focus here is on the Dirichlet problem, with measurable data, for second order elliptic operators in divergence form. We illustrate, through selected examples, the various ways Carleson measures arise in characterizing those classes of operators for which Dirichlet problems are solvable with classical non-tangential maximal function estimates.

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1. Introduction

Measures of Carleson type were introduced by L. Carleson in [9] and [10] to solve a problem in analytic interpolation, via a formulation that exploited the duality between Carleson measures and non-tangential maximal functions (defined below). Carleson measures have since become one of the most important tools in harmonic analysis, playing a fundamental role in the study of singular integral operators in particular, through their connection with BMO, the John-Nirenberg space of functions of bounded mean oscillation. We aim to describe, through some specific examples, the ubiquitous role of measures of this type in the theory of boundary value problems, especially with regard to sharp regularity of "elliptic" measure, the probability measure arising in the Dirichlet problem for second order divergence form elliptic operators. Perhaps the first connection between Carleson measures and boundary value problems was observed by C. Feffeman in [18], namely that every BMO function on $\mathbb{R}^n$ has a harmonic extension to the upper half space $\mathbb{R}^{n+1}_+$ which satisfies a certain Carleson measure condition. This established an important link between solutions to boundary value problems for the Laplacian and the function space BMO. It may be surprising to see the extent to which this link exists for operators other than the Laplacian, and in the context of more general domains.

In order to define Carleson measures, we introduce the geometric notion of a Carleson region above a cube. If $Q \subset \mathbb{R}^n$ is a cube with side length $l(Q)$ set $T_Q = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in I, 0 < t < l(Q)\}$, a cube sitting above its boundary face Q. (The notation T_Q comes from an equivalent formulation involving "tents" over cubes.)

Definition 1.1. The measure $d\mu$ is a Carleson measure in the upper half space $\mathbb{R}^{n+1}_+$ if there exists a constant C such for all cubes $Q \subset \mathbb{R}^n$, $\mu(T(Q)) < C|Q|$, where $|Q|$ denotes the

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Lebesgue measure of the cube Q.

The classical theory of harmonic functions in the upper half space, or the ball, considers solutions to the Dirichlet problem with measurable, specifically L^p, data. Given a function $f \in L^p(\mathbb{R}^n)$, the convolution of f and the Poisson kernel is an absolutely convergent integral when $1 < p < \infty$, giving meaning to the harmonic extension $u(x,t)$ of an L^p function. And the sense in which this extension u converges to its boundary values is "non-tangential". That is, for every $x_0 \in \mathbb{R}^n$, one can define a non-tangential approach region to x_0, $\Gamma_a(x_0) = \{(x,t) : |x - x_0| < at\}$. Then if $u(x,t)$ is the Poisson extension of $f \in L^p(\mathbb{R}^n)$, for almost every x_0, $u(x,t) \to f(x_0)$ as $(x,t) \in \Gamma_a(x_0)$ approaches x_0. Moreover, one has a non-tangential maximal function estimate, specified below, which yields solvability and uniqueness of this L^p Dirichlet problem.

The result of C. Fefferman about harmonic functions, which proved to be a powerful tool in harmonic function theory, is this: if $u(x,t)$ is the Poisson extension of $f \in BMO$, then $d\mu = t|\nabla u|^2 dxdt$ is a Carleson measure in the upper half space $\mathbb{R}^{n+1}_+$. The converse also holds for functions that are not too large at ∞.

In the last several decades, there have been many significant developments in the theory of boundary value problems with data in L^p spaces, for harmonic (or poly-harmonic) functions defined in very general domains, and for solutions to second order divergence form (and higher order) elliptic operators with non-smooth coefficients. We will highlight a selection of these developments in which the role of Carleson measures has been decisive.

For simplicity of notation, we will formulate the results in the upper half plane, $\mathbb{R}^{n+1}_+$, but in fact these results are more naturally formulated on Lipschitz domains - see the cited references for this generality. In some cases, the perturbation results hold in more general (chord-arc) domains: [41–43].

2. Definitions and background

A divergence form elliptic operator

$$L := -\operatorname{div} A(x)\nabla,$$

defined in $\mathbb{R}^{n+1}$, where A is a (possibly non-symmetric) $(n+1)\times(n+1)$ matrix of bounded real coefficients, satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \le \langle A(x)\xi,\xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \le \lambda^{-1}, \tag{2.1}$$

for some $\lambda > 0$, and for all $\xi \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$. As usual, the divergence form equation is interpreted in the weak sense, i.e., we say that $Lu = 0$ in a domain Ω if $u \in W^{1,2}_{loc}(\Omega)$ and

$$\int A\nabla u \cdot \nabla\Psi = 0\,,$$

for all $\Psi \in C_0^\infty(\Omega)$.

For notational simplicity, Ω will henceforth be the half-space $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^n \times (0,\infty)\}$ even though the results are more naturally formulated on Lipschitz domains. See the cited references for this generality.

The solvability of the Dirichlet problem for L with data in $L^p(dx)$ is a function of a precise relationship between the elliptic measure ω associated to L and Lebesgue measure.

The elliptic measure associated to L is analogous to the harmonic measure: it is the representing measure for solutions to L with continuous data on the boundary.

Definition 2.2. A non-negative Borel measure ω defined on $\mathbb{R}^n$ is said to belong to the class A_∞ if there are positive constants C and θ such that for every cube Q, and every Borel set $F \subset Q$, we have

$$\omega(F) \leq C \left(\frac{|F|}{|Q|} \right)^\theta \omega(Q). \tag{2.3}$$

A real variable argument shows that a measure, ω, belongs to $A_\infty(dx)$ if and only if it is absolutely continuous with respect to Lebesgue measure and there is an exponent $q > 1$ such that the Radon-Nikodym derivative $k := d\omega/dx$ satisfies

$$\left(\fint_Q k(x)^q dx \right)^{1/q} \leq C \fint_Q k(x)\, dx \,, \tag{2.4}$$

uniformly for every cube Q. This property is called a reverse-Hölder estimate of order q.

If ω is the elliptic measure associated to an operator L, then the existence of such a $q > 1$ is, in turn, equivalent to the solvability of the Dirichlet problem for L with boundary data $f \in L^p$ (for p dual to q), in the sense of non-tangential convergence and non-tangential estimates on the boundary. These non-tangential estimates are expressed in terms of L^p bounds on two classical operators associated to solutions: the square function

$$S^\alpha(u)(x) := \left(\iint_{|x-y|<\alpha t} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2} , \tag{2.5}$$

and the non-tangential maximal function

$$N_*^\alpha(u)(x) := \sup_{(y,t):|x-y|<\alpha t} |u(y,t)| \tag{2.6}$$

Precisely, the elliptic measure satisfies a reverse Hölder estimate of order q if and only if the following L^p Dirichlet problem is solvable, for p dual to the exponent q:

$$\begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \lim_{t\to 0} u(\cdot,t) = f \text{ in } L^p(\mathbb{R}^n) \text{ and n.t.} \\ \|N_*(u)\|_{L^p(\mathbb{R}^n)} < C\|f\|_p. \end{cases} \tag{D_p}$$

Here, the notation "$u \to f$ n.t." means that $\lim_{(y,t)\to(x,0)} u(y,t) = f(x)$, for $a.e.\, x \in \mathbb{R}^n$, where the limit runs over $(y,t) \in \Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$. The constant C depends only on ellipticity and dimension.

We will usually suppress the dependence on the aperture α, since the choice of aperture does not affect the range of available L^p estimates.

Solutions to L are said to satisfy De Giorgi-Nash-Moser bounds when the following local Hölder continuity estimates hold. Assume that $Lu = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense and

$B_{2R}(X) \subset \mathbb{R}^{n+1}_+$, $X \in \mathbb{R}^{n+1}_+$, $R > 0$. Then

$$|u(Y) - u(Z)| \leq C \left(\frac{|Y-Z|}{R}\right)^{\mu} \left(\int_{B_{2R}(X)} |u|^2 \frac{\mathrm{dx}}{|B_2R(X)}\right)^{\frac{1}{2}}, \quad \text{for all} \quad Y, Z \in B_R(X), \tag{2.7}$$

for some constants $\mu > 0$ and $C > 0$. In particular, one can show that for any $p > 0$

$$|u(Y)| \leq C \left(\int_{B_{2R}(X)} |u|^p \frac{\mathrm{dx}}{|B_2R(X)|}\right)^{\frac{1}{p}}, \quad \text{for all} \quad Y, Z \in B_R(X). \tag{2.8}$$

The De Giorgi-Nash-Moser bounds always hold when the coefficients of the underlying equation are real [14, 40, 44], and the constants depend quantitatively only upon ellipticity and dimension. We will assume that for the complex equations considered later on (t-independent coefficients), that solutions satisfy the De Giorgi-Nash-Moser bounds, which may not in general obtain ([25, 39]).

3. Perturbations of elliptic operators

In this section, we briefly discuss some background which will motivate certain topics treated later, and for which Carleson measure estimates have played a decisive role.

In the upper half space, the Dirichlet problem is uniquely solvable for the Laplacian when the boundary data belongs to $L^p(dx)$, $1 < p < \infty$, in the sense that the Poisson extension $u(x,t)$ of f satisfies the estimate $\|N(u)\|_p \leq C\|f\|_p$. The same holds for solutions to $L := -\operatorname{div} A(x)\nabla$, when coefficients of A are smooth, or even just C^1 ([20]). However, without some regularity assumptions, the elliptic measure associated to L may be singular with respect to Lebesgue measure ([7])), and no estimate of this type will hold.

Many interesting examples of elliptic operators in divergence form arise as pullbacks of the Laplacian from a change of variable. From the viewpoint of complex function theory, it is natural to consider boundary behavior of harmonic functions in domains other than the ball or the upper half space. One approach to solving boundary value problems for harmonic functions in, say, a domain above a graph, is to invoke a change variables, mapping the harmonic function v to a solution u of a new divergence form elliptic operator, L. Thus, if the domain were bounded by a smooth curve, an appropriate change of variables results in a real symmetric divergence form operator with smooth coefficients. But if the boundary of the domain is not regular, the resulting operator has non-smooth coefficients, and the problem has not become easier. For a variety of reasons, including scale invariance and naturally arising geometric constructions, attention focused on the class of Lipschitz domains. In [12], Dahlberg showed that harmonic measure on any Lipschitz domain belonged to A_∞ with respect to the surface measure on the boundary. In fact, he showed that the L^2 Dirichlet problem, D_2, was solvable, but that D_p was not uniformly solvable on all Lipschitz domains when $p < 2$. More recently, the theory has developed to include a body of results for non-graph domains described by geometric conditions (non-tangentially accessible, chord-arc, Reifenberg flat.).

Consider the following example of a particularly straightforward change of variables. The domain is the region above a graph $t = \phi(x)$, where $\phi(x)$ is Lipschitz. The change of variables, $(x,t) \to (x, t-\phi(x))$, "flattens" it to the upper half space. Under this change of variable, from the Lipschitz domain to $\mathbb{R}^{n+1}_+$, harmonic functions are mapped to solutions of a symmetric elliptic divergence form operator L whose coefficients involve the Jacobian of this transformation and are therefore merely bounded and measurable. However, the coefficients have one redeeming feature: they are independent of the transverse variable t. Jerison and Kenig (JK) discovered how to put Dahlberg's result in a larger context when they showed that D_2 was solvable in $\mathbb{R}^{n+1}_+$ for all elliptic symmetric t-independent operators. Their well known result was based on an L^2 identity (a "Rellich" identity) which decisively used these three properties of the (real) operator L: symmetry, ellipticity, t-independence of the coefficients. Specifically, if $Lu = 0$, and $\vec{e}$ is the unit normal at the boundary of $\mathbb{R}^{n+1}_+$, then

$$\mathrm{div}(A\nabla u.\nabla u\, \vec{e}) = 2\,\mathrm{div}(D_{n+1}(u)A\nabla u). \tag{3.1}$$

Integrating this identity and applying the divergence theorem results in a boundary identity that can be used to show that the normal and tangential derivatives of a solution are comparable in L^2 norm. This boundary identity scales to show that the elliptic measure is not only absolutely continuous but satisfies a reverse Hölder condition of order two. Therefore, the Dirichlet problem with data in L^2 is solvable.

Many subsequent advances in the theory of boundary value problems for real symmetric elliptic equations and systems were based on variants of this Rellich identity.

The theory of perturbations of elliptic operators arose from several separate points of view. One source was T. Kato's interest in the analyticity of square roots of complex second order divergence form elliptic operators, which led to a question about analyticity of small L^∞ perturbations of self-adjoint elliptic operators. There is extensive literature on this subject which we are not going to delve into in this article. (See [4] for the solution to Kato's conjecture.) Another, and related, source of interest, stemmed from the the discovery that independence in the t variable in $\mathbb{R}^{n+1}_+$ (or similarly, of the radial variable in the unit ball) endows the elliptic measure ω with good properties. One may then try to relax this condition and understand more precisely the relationship between the smoothness that is required in the t direction and good estimates for elliptic measure. This was the approach taken in [13, 19, 30, 34], and see also [2, 3, 26] for later developments in perturbation theory. Dahlberg, [13], imposed a "vanishing" condition on the Carleson discrepancy between the coefficients and proved strong results about preservation of reverse Hölder estimates for the elliptic measure. An entirely new approach to the vanishing Carleson condition was taken in [2] that provided major extensions of the perturbation theory to complex coefficient operators.

Consider an operator $L_1 := -\,\mathrm{div}\, A(x,t)\nabla$, in $\mathbb{R}^{n+1}_+$, regarded as a perturbation of $L_0 := -\,\mathrm{div}\, A(x,0)\nabla$, and suppose one asks for some quantitative conditions on $|A(x,t) - A(x,0)|$ that yield good estimates for the elliptic measure ω_{L_1}. More generally, one can formulate the question as follows: what are the optimal conditions on the difference of the coefficients such that the perturbation L_1 of a "good" operator L_0, not necessarily t-independent, also satisfies good estimates for solvability of a boundary value problem. In [19], optimal conditions were found.

Theorem 3.2. *Let* $L_0 = \mathrm{div}\, A_0\nabla$ *and* $L_1 = \mathrm{div}\, A_1\nabla$ *and define the disagreement function* $a(x,t)$ *by*

$$a(x,t) = sup\{|A_0(y,s) - A_1(y,s)| : |y-x| < t, t/2 < s < 2t\}. \tag{3.3}$$

If $a^2(x,t)t^{-1}dxdt$ *is a Carleson measure, then* $\omega_{L_0} \in A_\infty$ *implies* $\omega_{L_1} \in A_\infty$.

4. Linking A_∞ to Carleson measure estimates

Prior to the approach taken in [32], the regularity of elliptic measure for an operator L was essentially derived either from a Rellich identity, or as a consequence of the perturbation theory. There were two obvious classes of operators of interest where these L^2-identities were not valid: operators with complex coefficients and operators with non-symmetric coefficients. In the case of operators with complex coefficients, one of the most compelling outstanding questions was the Kato conjecture. This decades-old problem was finally resolved in the series of papers [4, 5, 24]. The solution of the Kato conjecture is a long story, summarized well in C. Kenig's review [31]. We will only mention that the solution also relied on a critical use of Carleson measures. The situation regarding (non-symmetric) t-independent operators is discussed in the next section.

In [32], it was shown that the elliptic measure associated to adivergence form operator $L := -\operatorname{div} A(x)\nabla$, belongs to the class A_∞ if and only if every bounded solution could (locally) be approximated arbitrarily well by a continuous function whose gradient satisfied a Carleson measure condition. This criteria was dubbed "ϵ-approximability", and was immediately applied to t-independent operators in dimension two.

Definition 4.1. Let $u \in L^\infty(\mathbb{R}^{n+1}_+)$, with $\|u\|_\infty \leq 1$. Given $\epsilon > 0$, we say that u is ϵ-**approximable** if for every cube $Q_0 \subset \mathbb{R}^n$, there is a $\varphi = \varphi_{Q_0} \in W^{1,1}(T_{Q_0})$ such that

$$\|u - \varphi\|_{L^\infty(T_{Q_0})} < \epsilon\,, \tag{4.2}$$

and

$$\sup_{Q \subset Q_0} \frac{1}{|Q|} \iint_{T_Q} |\nabla \varphi(x,t)|\, dxdt \leq C_\epsilon\,, \tag{4.3}$$

where C_ϵ depends also upon dimension and ellipticity, but not on Q_0.

To motivate this definition, we recall that harmonic functions in the upper half space possess the property of ϵ-approximability ([21, 45]). Although bounded harmonic functions in $\mathbb{R}^{n+1}_+$ satisfy an L^2-Carleson measure condition, the (technically more desirable) L^1-Carleson condition fails to hold. It turns out that the approximation property is a good substitute for certain applications. In [11], Dahlberg showed that ϵ-approximability holds for bounded harmonic functions on Lipschitz domains as well. His proof used the previously established equivalence in L^p-norm between the square function and the non-tangential maximal function on LIpschitz domains.

Theorem 4.4 ([32]). *Let $L := -\operatorname{div} A(x)\nabla$, be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in $\mathbb{R}^{n+1}_+$. Then there exists an ϵ, depending on the ellipticity constant of L such that if every solution to $Lu = 0$ in $\mathbb{R}^{n+1}_+$ with $|u| \leq 1$ is ϵ-approximable then ω belongs to A_∞.*

We will now sketch the main steps in the proof in [32] of this result, and then describe a recent modification of these ideas that yields a much stronger statement. The references give details, including certain technicalities, that we shall not describe in detail here.

The A_∞ class has many equivalent characterizations, and it will be convenient to work with this one:

Given any $\eta > 0$, there exists a $\delta > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and any $E \subset Q$, we have that $|E|/|Q| < \eta$ whenever $\omega(E)/\omega(Q) < \delta$.

The main idea in the proof of Theorem 4.4 is as follows. Fix a cube Q of side length r, and suppose that E is a set whose elliptic measure, $\omega(E)$, is small. Let ϕ denote the ϵ-approximation of u. If E has sufficiently small measure, it will be shown that a truncated L^1-version of the square function of ϕ is large. That is, the r-truncated $A_r(\phi)(x) := \left(\iint_{|x-y|<t<r} |\nabla\phi(y,t)| \frac{dydt}{t^n}\right)^{1/2}$ will be be larger than some prescribed value $k = k(\epsilon)$. The desired conclusion will follow from the Carleson measure estimate by integrating:

$$|E|k^2 < \int_E A_r^2(\phi)(x)dx < \int_Q A_r^2(\phi)(x)dx < \iint_{T_Q} |\nabla\varphi(x,t)|\, dxdt$$

By the Carleson measure property, this latter expression is bounded by a constant C_ϵ times $|Q|$. and thus $|E|/|Q| < \eta$ where $\eta \approx 1/k^2$.

In order to show that $A(\phi)$ is large on sets of small elliptic measure, a solution u to $Lu = 0$ was constructed with the property that u that oscillates by at least some fixed value a large number of times in cones over points $x \in E$. Because u can be approximated arbitrarily well by ϕ, this entailed that ϕ also oscillates a large number of times. This lower bound on oscillation translated, via interior estimates, into an estimate from below for $\nabla\phi$ in disjoint layers of a truncated cone over x.

There are several constructions that drive this proof, the first of which is Christ's construction of dyadic grids on spaces of homogeneous type. Thus $Q \subset \mathbb{R}^n$ possesses a *dyadic grid* adapted to ω, which is a collection of subsets $\{I_{j,l}\}$ of $Q \subset \mathbb{R}^n$ such that for each fixed $j \geq 0$,

(1) $\mathbb{R}^n = \bigcup_l I_{j,l}$, and $I_{j,l_1} \cap I_{j,l_2} = \emptyset$ if $l_1 \neq l_2$.

(2) Each $I_{j,l}$ contains $B(2^{-j}, x_l)$, and is contained in an M-fold dilate $B(M2^{-j}, x_l)$, where $B(2^{-j}, x_l)$ denotes the ball of radius 2^{-j} about the point $x_l \in \mathbb{R}^n$.

(3) If $I_{j,l} \cap I_{j',l} \neq \emptyset$ then either $I_{j,l} \subset I_{j',l}$ or $I_{j',l} \subset I_{j,l}$. Moreover, there exists a $C_M < 1$ such that $\omega(I_{j,l}) < C_M \omega(I_{j',l})$whenever $I_{j,l} \subset I_{j',l}$.

(4) Any open set $\mathcal{O}$ can be decomposed as $\mathcal{O} = \bigcup I_{j,l}$ where the $I_{j,l}$ are non-overlapping. For each $I_{j,l}$ in this decomposition, there exists a point $p_{j,l}$ such that the distance from $p_{j,l}$ to $I_{j,l}$ is comparable to diam($I_{j,l}$).

Definition 4.5. Let ϵ be small and given. If $E \subset Q$, a *good ϵ-cover of E of length k* is a collection of nested open sets $\{\mathcal{O}_i\}_{i=1}^k$ with $E \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} ... \subset \mathcal{O}_0 \subset Q$ where each $\mathcal{O}_i = \bigcup S_l^i$ such that

(1) each S_l^i belongs to the dyadic grid, and

(2) for all $0 < i < k$, $\omega(\mathcal{O}_i \cap S_l^{i-1}) < \epsilon\omega(S_l^{i-1})$.

Note that a good ϵ-cover has the property that each S_j^i is properly contained in some S_l^{i-1}, as well as the further nesting property that for $k > i > m > 0$, $\omega(S_j^m \cap \mathcal{O}_i) < \epsilon^{i-m}\omega(S_j^m)$.

Lemma 4.6 ([32]). *Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\omega(E) < \delta$, then E has a good ϵ-cover of length k where $k \to \infty$ as $\omega(E) \to 0$.*

The good ϵ-cover of length k is used to construct the boundary data f which will give rise to a bounded, oscillating solution u to L. Set:

$$f = \sum_{i=0}^{k} (-1)^i \mathcal{X}_{\mathcal{O}_i}. \tag{4.7}$$

and let u be the solution to $Lu = 0$, with $u(x, 0) = f$.

Note that $f \leq 1$, and so $0 \leq u \leq 1$. For each point $x \in E$, we find a sequence of points, $X_m = (x_m, t_m)$ in the cone $\Gamma(x)$ with the property that, for $0 < m < k$ even, $u(X_m) > c_1$, and for $0 < m < k$ odd, $u(X_m) < c_2$ and $c_1 - c_2 > c(\epsilon)$. To define these X_m, collect the dyadic grid cubes $S_l^m \subset \mathcal{O}_m$ that contain the given point x. Let $l(S)$ denote the side length of S. The point X_m, when m is even, is essentially any point in the top half of the Carleson region over S_l^m. When m is odd, the point $X_m = (x_m, t_m)$ will also be in this Carleson region, but t_m will be closer to the boundary, that is, $t \approx \eta l(S_l^m)$. (In order to make sure that these points X_m descend in the cone, i.e., have the property that $t_m < \rho t_{m-1}$ for some $\rho < 1$, we may have to skip a finite number of levels m. Details are in [32].)

We give a rough sketch of these estimates. Recall the integral representation of solutions: $u(x, t) = \int K(x, t; y, 0) f(y) d\omega(y)$.

Fix an even m. We can then write $u(X_m) = u_1(X_m) + u_2(X_m)$ where $u_1(x, 0) = f_1(x) := \sum_{i=0}^{m} \mathcal{X}_{\mathcal{O}_i}.$. Moreover, since $u > 0$, we have that, for some c_1 depending only ellipticity,

$$u(X_m) > \int K(x_m, t_m; y, 0) f(y) d\omega(y) \geq c \frac{1}{\omega(S_l^m)} \int_{S_l^m} f(y) d\omega(y).$$

Because m is even, the function $f_1 = 1$ on S_l^m, and so $u_1 > c_1'$. By the nesting property of the cover,

$$\frac{1}{\omega(S_l^m)} \int_{S_l^m} f_2(y) d\omega(y) < \frac{1}{\omega(S_l^m)} \sum_{i=m+1}^{k} \omega(\mathcal{O}_i \cap S_l^m) < 2\epsilon,$$

and thus $u(X_m) > c_1' - 2\epsilon > c_1$. When m is odd, the boundary function f is split similarly, and a more technical analysis is needed to show that the main term is indeed given by f_1, which vanishes on the dyadic cube S_l^m. Since (x_m, t_m) was chosen so that $t_m \approx l(S_l^m)$, the Hölder decay of the solution near the boundary where it vanishes will be used to show that $u(x_m, t_m) < c_2 < c_1 - \epsilon$, if ϵ and η are chosen appropriately.

In conclusion, one can extract from this construction a sequence of points $\{x_m, t_m\}_{m=0}^k \in \Gamma(x)$ such that $|u(x_m, t_m) - u(x_{m-1}, t_{m-1})| > \epsilon$, and such that $t_m < \rho t_{m-1}$. One can then derive a lower bound for the L^1-square function $A(u)$, and likewise for $A(\phi)$ where ϕ is the approximate to u.

This approximation theorem, and its proof, yielded several applications to specific classes of operators ([15, 22, 36, 37]): [22] is explained in more detail in the next section. Since one cannot expect the actual solution to L to satisfy an L^1-Carleson condition (as the approximate does), this program left open the question of the role of classical Carleson measure estimates for solutions.

In [16], it was shown that the A_∞ propoerty of elliptic measure is equivalent to the existence of Carleson measure estimates for solutions with boundary data in BMO. The result was proven in Lipschitz domains (and will likely hold for chord-arc domains as well).

Theorem 4.8 ([16]). *Let $L := -\operatorname{div} A(x)\nabla$, be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in $\mathbb{R}_+^{n+1}$. Then $\omega \in A_\infty$ if and only if, for every solution u to $Lu = 0$ with boundary data $f \in BMO$, we have the Carleson measure estimate:*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} t |\nabla u(x, t)|^2 \, dxdt \leq C ||f||_{BMO}^2, \tag{4.9}$$

The proof of Theorem 4.8 used a dual formulation of the A_∞ condition:

Given any $\eta > 0$, there exists a $\delta > 0$ such that for any cube $Q \subset \mathbb{R}^n$ and any $E \subset Q$, we have that $\omega(E)/\omega(Q) < \eta$ whenever $|E|/|Q| < \delta$.

To verify this condition, a construction of [28] was invoked to produce, for any such E, a BMO function $f \geq \mathcal{X}_E$ with small BMO norm. An upper estimate on $\omega(E)/\omega(Q)$ in terms of the (small) Carleson measure bound on f required a lemma in [34]. See [16] for details.

In turn, this left open the question of whether the A_∞ property of elliptic measure could be characterized by the existence of Carleson measure conditions for solutions to bounded data, as opposed to data in the larger class, BMO.

The solution u, with boundary data f as in (4.7), has only $A(u)$ large on the set $E \subset Q$ when $\omega(E)$ is small, but not necessarily $S(u)$ large as well. To see why, suppose Q has side length 1, and cut the cone into dyadic layers: $\Gamma_j(x) = \{(y,t) \in \Gamma(x) : 2^{-j} < t < 2^{-j+1}\}$.

We write

$$S(u)(x) = \sum_j \int_{\Gamma_j(x)} t^{1-n} |\nabla u|^2 dydt$$

Each piece $\int_{\Gamma_j(x)} t^{1-n}|\nabla u|^2 dydt$ is a scaled average of the gradient of u which, by a Poincaré estimate, can be bounded from below by the oscillation of u over this dyadic layer of the cone. However, this construction doesn't yield any information about the oscillation of u on such dyadic regions because there is no control on the distance between the the points $\{x_m, t_m\}_{m=0}^k \in \Gamma(x)$ that belong to different levels $\mathcal{O}_m$.

The linking of A_∞ to Carleson measure estimates for L^∞ functions, is the subject of [33]. Essentially, one can use the same cover, and define a new function f as follows. Each $\mathcal{O}_m$ is a union of dyadic intervals S_l^m, and each S_l^m has a (bounded) number of immediate dyadic subintervals. For each S_l^m choose one of its dyadic children and call it $\tilde{S}_l^m$. If m is even, define f_m to take the value 1 on $\bigcup_l (S_l^m \setminus \tilde{S}_l^m)$ and 0 elsewhere. If m is odd, we define f_m to "zero out" the values of f_{m-1}: $f_m = -1$ where $f_m = 1$ and is) elsewhere. Now set $f = \sum_{m=0}^k f_m$ and let u be the solution to $Lu = 0$ with boundary data f. On each even level m, f takes on both the values 0 and 1 on dyadic children. Thus, arguments modeled on those of [32] will yield the following: for some $C, c > 0$, and every $x \in E$, there are sequences $\{x_m, t_m\}_{m=0}^k$ with $ct_{m-1} < t_m < Ct_{m-1}$ for which $|u(x_m, t_m) - u(x_{m-1}, t_{m-1})| > \epsilon$.

From this construction, it can be concluded that if solutions to L with bounded data satisfy classical Carleson measure estimates, then the elliptic measure associated to L is A_∞, and thus the Dirichlet problem with data in L^p is uniquely solvable for some $p > 1$. As a corolloary, we see that solutions with BMO data posses C. Fefferman-type Carleson estimates if and only if solutions with L^∞ data posses these Carleson estimates.

Theorem 4.10 ([33]). *Let $L := -\operatorname{div} A(x)\nabla$, be an elliptic divergence form operator, not necessarily symmetric, with bounded measurable coefficients, defined in $\mathbb{R}^{n+1}_+$. Then $\omega \in A_\infty$ if and only if, for every solution u to $Lu = 0$ with boundary data $f \leq 1$, we have the Carleson measure estimate:*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} t|\nabla u(x,t)|^2 \, dxdt \leq C. \tag{4.11}$$

5. Application to time-independent operators

The ϵ-approximability theorem of [32] was established by showing the equivalence in L^p norm between the the non-tangential maximal function and the square function, and invoking a stopping time construction due to Dahlberg ([11]). Examples were given to demonstrate that, for $p \to \infty$, there exists elliptic operators in this class for which D_p is not solvable.. In other words, no stronger conclusion than A_∞ of the elliptic measure can be concluded from ϵ-approximability. A more precise study of these counterexamples was undertaken in [1], where it was shown that the boundary equation method and the Lax-Milgram method may construct different solutions, thus underscoring the differences between the symmetric and the non-symmetric situation.

As an application of the consequences of norm equivalence between non- tangential maximal function and the square function, [32] contained a proof that two-dimensional t-independent divergence form non-symmetric elliptic operators had elliptic measure belonging to A_∞. This was a first step in establishing regularity of elliptic measure without recourse to L^2 identities of Rellich type. Although the proof only worked in $\mathbb{R}^2$, it worked under a surprisingly flexible condition on the matrix.

Theorem 5.1 ([32]). *Let $L := -\operatorname{div} A(x)\nabla$ be an elliptic operator in $\mathbb{R}^2$ with bounded measurable coefficients. Suppose that there exists a fixed unit vector $\vec{e}$ such that $A(x,t) = A((x;,t) \cdot \vec{e})$. Then the elliptic measure ω_L belongs to A_∞ in a domain in any Lipschitz domain in $\mathbb{R}^2$.*

At this point, we note that the development of the theory of non-symmetric operators has had several motivations. First of all, the boundary value problem for general non-symmetric elliptic operators cannot be solved in L^2, and L^p solvability requires a different approach than that of Rellich identities. Second, the well-posedness results for equations with real non-symmetric coefficients and associated estimates on solutions are the first step towards understanding operators with complex coefficients in the non-Hermitian case, a case of interest for Kato's analyticity program. Finally, many problems arising in homogenization theory have non-symmetric coefficients [6]. Solving the Dirichlet problem with data in L^p is the first step in the study of the uniform bounds, independent of the scaling parameter in homogenization theory, in the absence of symmetry ([6]).

It is therefore desirable to develop approaches to solving L^p boundary value problems that are neither perturbative nor rely on symmetry of the matrices. However, the proof of Theorem 5.1 did not generalize to higher dimensions, as it relied on a special change of variable to put the matrix of coefficients in upper triangular form. It took almost fifteen years, and the development of the tools used to solve Kato's conjecture (the square root estimates), to be able to prove this result in all dimensions.

Theorem 5.2 ([22]). *Let L be a divergence form elliptic operator as above, with t-independent coefficients. Then there is a $p < \infty$ such that the Dirichlet problem D_p is well-posed; equivalently, for each cube $Q \subset \mathbb{R}^n$, the L-harmonic measure $\omega_L \in A_\infty(Q)$, with constants that are uniform in Q.*

The proof in [22] proceeded, as in two dimensions, by establishing A_∞ of the elliptic measure as consequence of ϵ-approximability of bounded solutions. The boundedness in norm of the non-tangential maximal function by the square function had previously been established (globally) in [2] so the main contribution of [22] was the converse, which had the immediate corollary:

Corollary 5.3 ([22]). *Under the same hypotheses as in Theorem 5.2, for a bounded solution u, we have the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} |\nabla u(x,t)|^2 t dt dx \leq C \|u\|_{L^\infty(\Omega)}, \tag{5.4}$$

where C depends only upon dimension and ellipticity.

Theorem 4.10 implies that this Carleson measure estimate alone is now sufficient to conclude A_∞, somehwat simplifying the proof of A_∞ for this class of elliptic measures.

In [32], it was shown that the equivalence between non-tangential maximal functions and square functions implied A_∞, for that equivalence was necessary to prove ϵ-approximation of bounded solutions. We see now that only half of this information is required, namely the bounds on the square function in terms of the non-tangential maximal function.

Remark 5.5. Most of the discussion in this article has centered on the Dirichlet problem. Over the years, there has been a parallel development for boundary value problems such as the Neumann and the regularity problems for second order operators, and for higher order operators and elliptic systems. There is a vast literature on the solvabilty of these (even more) challenging problems, which is beyond the scope of the present article.

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