

BMO solvability and the A_∞ condition for elliptic operators

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Abstract

We establish a connection between the absolute continuity of elliptic measure associated to a second order divergence form operator with bounded measurable coefficients with the solvability of an endpoint BMO Dirichlet problem.

1 Introduction

We begin an investigation into the precise connection between solvability of certain endpoint (BMO) boundary value problems for second order elliptic operators and the absolute continuity of the elliptic measure associated to these operators.

To establish the context for the results here, let us begin by recalling the methods for establishing absolute continuity which appeared in [13]. There, the following was shown. Let $L = \operatorname{div}A\nabla$ be an elliptic operator with bounded measurable coefficients whose matrix is real but not necessarily symmetric. If all bounded solutions possess a certain approximation property in a domain D , then the elliptic measure is in fact absolutely continuous in D . The approximation property is one which is well known for harmonic functions in the disc or the upper half space.

Definition 1.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n and let $L = \operatorname{div}A\nabla$, an elliptic divergence form operator whose matrix has coefficients which are bounded and measurable. A weak solution u to $Lu = 0$ in Ω , with $\|u\|_\infty \leq 1$, is said to be ϵ -approximable if there exists a $\varphi \in C^\infty(\Omega)$ such that $\|u - \varphi\|_\infty < \epsilon$ in Ω and such that for all surface balls $\Delta(r, Q) = \partial\Omega \cap B(r, Q)$,*

$$\int_{T(\Delta(r, Q))} |\nabla\varphi| dX \leq C_\epsilon \sigma(\Delta),$$

where $T(\Delta(r, Q)) = B(r, Q) \cap \Omega$ is the Carleson region associated to $\Delta(r, Q)$, and C_ϵ depends also on the Lipschitz character of Ω .

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Carleson measures are defined in Section 2, below. See [10] for Garnett's proof of the ϵ -approximability property of harmonic functions in \mathbb{R}_+^2 , and [4] for the same result for harmonic functions in Lipschitz domains.

In [13] it was shown that if an operator L has the property that all its bounded solutions in a domain Ω are ϵ -approximable, then the elliptic measure associated to L is not only absolutely continuous, but in fact belongs to the Muckenhoupt class A^∞ with respect to surface measure on the boundary of Ω . The A^∞ property of elliptic measure implies solvability of the Dirichlet problem with data in some, possibly large, L^p class with respect to surface measure on the boundary of the domain. Recall that the solvability of the Dirichlet problem with data in L^p is expressed in terms of an a priori inequality relating nontangential maximal functions of solutions to the data on the boundary.

In order to use this criterion to prove absolute continuity of elliptic measure, we need to find a means of checking that approximability holds for solutions. It turns out that Dahlberg's proof in [4] can be adapted to verify approximability under certain assumptions, and this led to the second main result of [13]: if the nontangential maximal function and the square function associated to the operator have a equivalent norms in L^2 (or any L^p) with respect to surface measure on the boundary of every Lipschitz subdomain of Ω , then ϵ -approximability holds. An additional subtlety, or technicality, has been introduced here. Notice that one must verify that solutions to L on the larger domain have this equivalence property on all (Lipschitz) subdomains. On the one hand, it seems that verifying the fine properties of the elliptic measure on the boundary of Ω shouldn't require knowledge that depends on boundaries of Lipschitz subdomains. On the other hand, to say that the Dirichlet problem is solvable in some L^p space does require that the solutions converge to their data nontangentially: and the associated maximal functions and square functions give rise to nontangential approach regions and sawtooth regions inside the domain, which are intrinsically connected to solvability.

We conjecture however that the equivalence between the norms of these two operators on solutions can be replaced by the assumption of solvability of an endpoint boundary value problem on the given domain. It is very interesting to tie the A^∞ property of elliptic measure to solvability of a particular endpoint Dirichlet boundary value problem. Since A^∞ is itself an endpoint space (of A^p) it is natural to look to endpoint spaces like BMO . We essentially solve this problem here, except for the subtlety of dealing with conditions which must hold on Lipschitz subdomains. It remains to find a means of eliminating this assumption, if possible. Ideally, one would want to prove A^∞ for a second order divergence form operator by testing for solvability of an endpoint Dirichlet problem on a single function or a simple class of functions. Thus the real goal of this continuing investigation is a $T(1)$ type characterization for absolute continuity of elliptic measure.

The reader who is unfamiliar with the basic properties of solutions to divergence form operators with bounded measurable coefficients, such as the contents of [18] or [1] is referred to the introduction of [13].

2 Definitions and Statements of Main Theorems

Let us begin by defining introducing Carleson measures and square functions on domains which are locally given by the graph of a function. We shall therefore assume that our domains are Lipschitz.

Definition 2.1. $\mathbb{Z} \subset \mathbb{R}^n$ is an M -cylinder of diameter d if there exists a coordinate system (x, t) such that

$$\mathbb{Z} = \{(x, t) : |x| \leq d, -2Md \leq t \leq 2Md\}$$

and for $s > 0$,

$$s\mathbb{Z} = \{(x, t) : |x| < sd, -2Md \leq t \leq 2Md\}.$$

Definition 2.2. $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with Lipschitz ‘character’ (M, N, C_0) if there exists a positive scale r_0 and at most N cylinders $\{Z_j\}_{j=1}^N$ of diameter d , with $\frac{r_0}{c_0} \leq d \leq c_0 r_0$ such that

(i) $8Z_j \cap \partial\Omega$ is the graph of a Lipschitz function ϕ_j ,

$$\|\phi_j\|_\infty \leq M, \phi_j(0) = 0,$$

(ii)

$$\partial\Omega = \bigcup_j (Z_j \cap \partial\Omega)$$

(iii)

$$Z_j \cap \Omega \supset \left\{ (x, t) : |x| < d, \text{dist}((x, t), \partial\Omega) \leq \frac{d}{2} \right\}.$$

If $Q \in \partial\Omega$ and

$$B_r(Q) = \{x : |x - Q| \leq r\}$$

then $\Delta_r(Q)$ denotes the surface ball $B_r(Q) \cap \partial\Omega$ and $T(\Delta_r) = \Omega \cap B_r(Q)$ is the called the Carleson region above $\Delta_r(Q)$.

Definition 2.3. Let $T(\Delta_r)$ be a Carleson region associated to a surface ball Δ_r in $\partial\Omega$. A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$,

$$\mu(T(\Delta_r)) \leq C\sigma(\Delta_r).$$

For such measure μ we denote by $\|\mu\|_{Car}$ the number

$$\|\mu\|_{Car} = \sup_{\Delta \subset \partial\Omega} (\sigma(\Delta)^{-1} \mu(T(\Delta)))^{1/2}.$$

Definition 2.4. A cone of aperture a is a non-tangential approach region for $Q \in \partial\Omega$ of the form

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \leq a \text{ dist}(X, \partial\Omega)\}.$$

Sometimes it is necessary to truncate the height of Γ by h . Then $\Gamma_h(Q) = \Gamma(Q) \cap B_h(Q)$.

In what follows, L will be a shorthand for $L = \operatorname{div}A\nabla$ where the matrix A has bounded measurable coefficients $a_{i,j}$ and is strongly elliptic: there exists λ such that uch that for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\lambda|\xi|^2 \leq \sum a_{i,j}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2.$$

Definition 2.5. *If $\Omega \subset \mathbb{R}^n$, and u is a solution to L , the square function in $Q \in \partial\Omega$ relative to a family of cones Γ is*

$$Su(Q) = \left(\int_{\Gamma(Q)} |\nabla u(X)|^2 \delta(X)^{2-n} dX \right)^{1/2}.$$

and the non-tangential maximal function at Q relative to Γ is

$$Nu(Q) = \sup\{|u(X)| : X \in \Gamma(Q)\}.$$

Here $\delta(X) = \operatorname{dist}(X, \partial\Omega)$. We also consider truncated versions of these operators which we denote by $S_h u(Q)$ and $N_h(Q)$, respectively; the only difference in the definition is that the nontangential cone $\Gamma(Q)$ is replaced by the truncated cone $\Gamma_h(Q)$.

Definition 2.6. *The Dirichlet problem with the $L^p(\partial\Omega, d\sigma)$ data is solvable for L if the solution u for continuous boundary data f satisfies the estimate*

$$\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)},$$

where the implied constant depends only the ellipticity of the operator, the p , and the Lipschitz constant of the domain as measured by the triple of Definition 2.2.

Definition 2.7. *We say that a function $f : \partial\Omega \rightarrow \mathbb{R}$ belongs to BMO with respect to the surface measure $d\sigma$ if*

$$\sup_{I \subset \partial\Omega} \sigma(I)^{-1} \int_I |f - f_I|^2 d\sigma < \infty.$$

Here $f_I = \sigma(I)^{-1} \int_I f d\sigma$. We denote by $\|f\|_{BMO(p)}$ the number

$$\|f\|_{BMO(p)} = \sup_{I \subset \partial\Omega} \left(\sigma(I)^{-1} \int_I |f - f_I|^p d\sigma \right)^{1/p}.$$

It can be shown for any $1 \leq p < \infty$ that $\|f\|_{BMO(2)} < \infty$ if and only if $\|f\|_{BMO(p)} < \infty$. Moreover, $\|\cdot\|_{BMO(p)}$ and $\|\cdot\|_{BMO(2)}$ are equivalent in the sense that there is a constant $C > 0$ such that the inequality

$$C^{-1}\|f\|_{BMO(p)} \leq \|f\|_{BMO(2)} \leq C\|f\|_{BMO(p)} \tag{2.1}$$

holds for any BMO function f .

This definition can be modified further. Instead of using the difference $f - f_I$ in the definition of the BMO norm one can take

$$\|f\|_{BMO^*(p)} = \sup_{I \subset \partial\Omega} \inf_{c_I \in \mathbb{R}} \left(\sigma(I)^{-1} \int_I |f - c_I|^p d\sigma \right)^{1/p}. \quad (2.2)$$

Again, it can be shown that this gives an equivalent norm, i.e., there is $C > 0$ such that

$$C^{-1} \|f\|_{BMO^*(p)} \leq \|f\|_{BMO(2)} \leq C \|f\|_{BMO^*(p)}.$$

Definition 2.8. *The Dirichlet problem with the BMO data is solvable for L if the solution u for continuous boundary data f satisfies*

$$\| |\nabla u|^2 \delta(X) dX \|_{Car} \approx \|f\|_{BMO(2)}^2.$$

Equivalently,

$$\sup_{\Delta \subset \partial\Omega} \sigma(\Delta)^{-1} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX \approx \sup_{I \subset \partial\Omega} \sigma(I)^{-1} \int_I |f - f_I|^2 d\sigma. \quad (2.3)$$

The implied constants depend only the ellipticity of the operator, the p , and the Lipschitz constant of the domain as measured by the triple of Definition 2.1.

We now state our main theorems.

Theorem 2.1. *Let Ω be a Lipschitz domain and L be a divergence form elliptic operator with bounded coefficients satisfying the strong ellipticity hypothesis. Assume that the elliptic measure $d\omega_L$ associated with L is in $A_\infty(\partial\Omega, d\sigma)$. Then the Dirichlet problem with the any BMO data is solvable for L and the estimate (2.3) holds. The constants in this estimate only depend on the Lipschitz character of the domain Ω and the ellipticity constant of L .*

To prove the reverse direction we require a condition slightly stronger than (2.3). We will assume (2.3) not only on $\partial\Omega$ but also for all restrictions of solutions u on Ω to Lipschitz subdomains $\Omega' \subset \Omega$. We suspect, but have not been able to show, that this additional assumption is unnecessary.

Theorem 2.2. *Let Ω be a Lipschitz domain and L be a divergence form elliptic operator with bounded coefficients satisfying the strong ellipticity hypothesis. Assume that there exists $C > 0$ such that for all Lipschitz domains $\Omega' \subset \Omega$ of same Lipschitz character and any continuous solution $u \in C(\bar{\Omega})$ of the equation $Lu = 0$ in Ω we have*

$$C^{-1} \|\tilde{f}\|_{BMO(d\sigma)} \leq \sup_{\Delta \subset \partial\Omega'} \sigma(\Delta)^{-1} \iint_{T(\Delta)} |\nabla u|^2 \delta_{\Omega'}(X) dX \leq C \|\tilde{f}\|_{BMO(d\sigma)}, \quad (2.4)$$

where $\tilde{f} = u|_{\partial\Omega'}$, and $\delta_{\Omega'}(X) = \text{dist}(X, \partial\Omega')$. Then the elliptic measure $d\omega_L$ associated with L belongs to $A_\infty(\partial\Omega, d\sigma)$.

3 Proof of the statements

The proof of Theorem 2.1 uses ideas in Fabes-Neri [8], where the authors showed that the *BMO* Dirichlet problem was solvable for the Laplacian in Lipschitz domains.

Proof of Theorem 2.1. By our assumption, since $d\omega_L \in A_\infty(\partial\Omega, d\sigma)$, by [?] there is $p_0 > 1$ such that the Dirichlet problem (D_p) for L is solvable for all $p_0 < p \leq \infty$.

Consider $f \in BMO(\partial\Omega)$. We will establish that

$$\iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX \leq C\sigma(\Delta) \|f\|_{BMO}^2. \quad (3.5)$$

Consider any $\Delta \subset \partial\Omega$ a surface ball of radius r . Let us denote by $\tilde{\Delta}$ an enlargement of Δ such that $3\Delta \subset \tilde{\Delta} \subset 5\Delta$. We will write the solution u of the Dirichlet problem for boundary data f as $u_1 + u_2 + u_3$, where u_1, u_2 solve

$$\begin{aligned} Lu_1 &= 0, & u_1|_{\partial\Omega} &= (f - f_{\tilde{\Delta}})\chi_{\tilde{\Delta}}, \\ Lu_2 &= 0, & u_2|_{\partial\Omega} &= (f - f_{\tilde{\Delta}})\chi_{\partial\Omega \setminus \tilde{\Delta}}, \\ u_3 &= f_{\tilde{\Delta}} \text{ in } \Omega. \end{aligned}$$

Here $f_{\tilde{\Delta}}$ denotes, as before, the average of f over the set $\tilde{\Delta}$ and $\chi_{\tilde{\Delta}}$ is the characteristic function of the set $\tilde{\Delta}$.

We first estimate u_1 . We claim that

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX \leq C \int_{\tilde{\Delta}} S_r^2(u_1) d\sigma. \quad (3.6)$$

Let us denote by Δ_X the set $\{Q \in \partial\Omega; X \in \Gamma(Q)\}$. It follows that $\sigma(\Delta_X \cap \tilde{\Delta}) \approx \delta(X)^{n-1}$. Hence

$$\begin{aligned} \iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX &\leq C \iint_{T(\Delta)} \delta(X)^{2-n} |\nabla u_1|^2 \sigma(\Delta_X \cap \tilde{\Delta}) dX \\ &\leq C \int_{Q \in \tilde{\Delta}} \int_{\Gamma_r(Q)} \delta(X)^{2-n} |\nabla u_1|^2 dX d\sigma \\ &\leq C \int_{\tilde{\Delta}} S_r^2(u_1) d\sigma. \end{aligned} \quad (3.7)$$

By Hölder inequality for sufficiently large p (such that the L^p Dirichlet problem is solvable on Ω)

$$\begin{aligned} \int_{\tilde{\Delta}} S_r^2(u_1) d\sigma &\leq \sigma(\tilde{\Delta})^{\frac{p-2}{p}} \left(\int_{\tilde{\Delta}} S^p(u_1) d\sigma \right)^{2/p} \\ &\leq C\sigma(\Delta)^{\frac{p-2}{p}} \left(\int_{\tilde{\Delta}} |u_1|^p d\sigma \right)^{2/p}. \end{aligned} \quad (3.8)$$

The last inequality uses solvability of the Dirichlet problem in L^p , which implies that the L^p norm the square function is comparable to the L^p norm of the boundary data. We put (3.6) and (3.8) together to obtain an estimate

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX \leq C\sigma(\Delta)^{\frac{p-2}{p}} \left(\int_{\tilde{\Delta}} |f - f_{\tilde{\Delta}}|^p d\sigma \right)^{2/p} \leq C\sigma(\Delta) \|f\|_{BMO(p)}^2. \quad (3.9)$$

This is the desired estimate for u_1 . Now we handle u_2 . This function is a solution of the equation $Lu_2 = 0$ with Dirichlet boundary data $f_2 := f - (f - f_{\tilde{\Delta}})\chi_{\tilde{\Delta}}$. Let us call by f_2^+ and f_2^- the positive and negative part of the function f_2 , that is $f_2 = f_2^+ - f_2^-$ and $f_2^+, f_2^- \geq 0$. We denote by u_2^\pm the solution of the Dirichlet problem

$$Lu_2^\pm = 0 \quad \text{in } \Omega, \quad u_2^\pm|_{\partial\Omega} = f_2^\pm.$$

Hence $u_2^\pm \geq 0$ and $u_2 = u_2^+ - u_2^-$. We claim the following

Lemma 3.1. *There exist $C > 0$ depending only on the ellipticity of the operator L such that for any $X \in \Omega$*

$$\left(\delta(X)^{-n} \int_{B(X, \delta(X)/2)} |\nabla u_2^\pm(Y)|^2 dY \right)^{1/2} \leq \frac{C}{\delta(X)} \int_{\partial\Omega} f_2^\pm(Q) d\omega^X(Q). \quad (3.10)$$

Here ω^X is the elliptic measure for the operator L at the point X .

This statement is a consequence of the Poincaré inequality that allows to estimate the integral of a gradient by an average of $(u_2^\pm - u_2^\pm(X))^2$ over slightly larger ball and by Harnack inequality that implies $u_2^\pm(Y) \approx u_2^\pm(X)$ for $Y \in B(X, \delta(X)/2)$. Notice that the integral $\int_{\partial\Omega} f_2^\pm(Q) d\omega^X(Q)$ equals to the value of u_2^\pm at the point X .

Let us set

$$v_2(X) = \int_{\partial\Omega} |f_2(Q)| d\omega^X(Q) = \int_{\partial\Omega} (f_2^+(Q) + f_2^-(Q)) d\omega^X(Q). \quad (3.11)$$

It follows that $v^2(X) = u_2^+(X) + u_2^-(X)$.

Lemma 3.2. *There exist $C, \varepsilon > 0$ depending only on the ellipticity constant of the operator L such that for all $x \in T(\Delta)$:*

- $v_2(X) \leq C\|f\|_{BMO}$
- $v_2(X) \leq C\|f\|_{BMO} \left(\frac{\delta(X)}{r} \right)^\varepsilon$. Here r is the radius of the surface ball Δ .

We postpone the proof of this lemma until we show how it gives us the desired estimate.

To to that we consider a standard ‘dyadic’ decomposition of the Carleson region $T(\Delta)$. What this means is that $T(\Delta)$ can be written as a union of disjoint regions I_n , $n = 1, 2, 3, \dots$ such that for each region I_n the diameter of the region $d = \text{diam}(I_n)$ is

comparable to the distance $\text{dist}(I_n, \partial\Omega)$ and the volume of the region is comparable to d^n . For each region I_n we denote by x_n a point inside I_n . It follows that

$$\begin{aligned} \iint_{T(\Delta)} |\nabla u_2^\pm|^2 \delta(X) dX &\leq \sum_n \iint_{I_n} |\nabla u_2^\pm|^2 \delta(X) dX \leq C \sum_n \delta(x_n) \frac{|u_2^\pm(x_n)|}{\delta(x_n)^2} \delta(x_n)^n \\ &\leq C \int_{T(\Delta)} \frac{|u_2^\pm(X)|^2}{\delta(X)} dX \leq C \left(r^{-2\varepsilon} \int_{T(\Delta)} \delta(X)^{2\varepsilon-1} dX \right) \|f\|_{BMO}^2. \end{aligned} \quad (3.12)$$

Here we used Lemma 3.1 for the last estimate in the first line of (3.12) and Lemma 3.2 for the last estimate in the second line (clearly $u_2^\pm(X) \leq v_2(X)$).

Since $r^{-2\varepsilon} \int_{T(\Delta)} \delta(X)^{2\varepsilon-1} dX \leq Cr^{n-1} \approx \sigma(\Delta)$ we see that (3.9) and (3.12) together implies the estimate (3.5) we sought (function u_3 is constant, hence the required estimate hold trivially).

Proof of Lemma 3.2. The first estimate of the lemma, namely that $v_2(X) \leq C\|f\|_{BMO}$, essentially follows from Lemma on p.35 in [8]. As stated there

$$v_2(X) = \int_{\partial\Omega \setminus \tilde{\Delta}} |f - f_{\tilde{\Delta}}| K(X, Q) d\sigma(Q),$$

for some kernel $K(X, Q)$ (a Radon-Nykodim derivative of the elliptic measure ω^X). Fabes and Neri then use then fact that $K \in B^2(d\sigma)^1$ to establish the estimate. By looking at their proof we see that it is enough to have $K \in B^q$ for some $q > 1$. This holds, as we assume that $\omega^X \in A_\infty(d\sigma) = \bigcup_{q>1} B_q(d\sigma)$.

The further improvement in the estimate $v_2(X) \leq C\|f\|_{BMO} \left(\frac{\delta(X)}{r}\right)^\varepsilon$ is a consequence of Di Giorgi-Nash-Moser theory. Nonnegative solutions u of L in the region $T(\tilde{\Delta})$ which vanish on 2Δ satisfy

$$u(X) \leq C \left(\frac{|X - Q|}{r} \right)^\varepsilon \sup_{T(2\Delta)} u, \quad \text{for any } X \in T(\Delta).$$

Here ε only depends on the ellipticity constant of the operator L and Q is the center of the ball Δ . (See for example (1.9) in [13] for reference). From this the estimate follows as we can move point Q around (within Δ) as our function vanishes on $\tilde{\Delta} \supset 3\Delta$.

Now we prove the reverse estimate to (3.5). We want to show that

$$\|f\|_{BMO^*(d\sigma)}^2 \leq C \sup_{\Delta \subset \partial\Omega} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) \frac{dX}{\sigma(\Delta)}. \quad (3.13)$$

In this case it is more convenient to use (2.2) to define BMO norm. We first prove the following

$$\sup_{\Delta \subset \partial\Omega} \inf_{c_\Delta} \sigma(\Delta)^{-1} \int_\Delta |f - c_\Delta| d\sigma \leq C \sup_{\Delta \subset \partial\Omega} \left(\inf_{c_\Delta} \omega(\Delta)^{-1} \int_\Delta |f - c_\Delta|^p d\omega \right)^{1/p}. \quad (3.14)$$

¹We denote by B^q the class of Gehring weights. The weights in this class satisfy the reverse Hölder inequality with exponent q .

Here $\omega = \omega^{X_0}$ is the elliptic measure for the operator L at some (fixed) interior point X_0 . This inequality implies that a BMO function with respect to the surface measure σ is also a BMO function with respect to the elliptic measure ω . Indeed, Let $d\sigma = k d\omega$. The fact $\omega \in A_\infty(d\sigma)$ implies that $\sigma \in A_\infty(d\omega) = \bigcup_{q>1} B_q(d\omega)$. Hence there exists $q > 1$ such that k satisfies the reverse Hölder inequality

$$\left(\omega(\Delta)^{-1} \int_{\Delta} k^q d\omega \right)^{1/q} \leq C \omega(\Delta)^{-1} \int_{\Delta} k d\omega \quad \text{for all } \Delta \subset \partial\Omega. \quad (3.15)$$

It follows

$$\begin{aligned} & \sigma(\Delta)^{-1} \int_{\Delta} |f - c_{\Delta}| d\sigma = \sigma(\Delta)^{-1} \int_{\Delta} |f - c_{\Delta}| k d\omega \\ & \leq \sigma(\Delta)^{-1} \left(\int_{\Delta} k^q d\omega \right)^{1/q} \left(\int_{\Delta} |f - c_{\Delta}|^p d\omega \right)^{1/p} \end{aligned} \quad (3.16)$$

$$\begin{aligned} & \leq C \sigma(\Delta)^{-1} \omega(\Delta)^{1/q-1} \left(\int_{\Delta} k d\omega \right) \left(\int_{\Delta} |f - c_{\Delta}|^p d\omega \right)^{1/p} \\ & = C \left(\omega(\Delta)^{-1} \int_{\Delta} |f - c_{\Delta}|^p d\omega \right)^{1/p}. \end{aligned} \quad (3.17)$$

This gives (3.14). It also follows that it suffices to prove (3.13) with $d\omega$ measure on the left-hand side instead of $d\sigma$.

In what follows we use the following lemma from [?].

Lemma 3.3. *Let X_0 be a fixed point inside a Lipschitz domain Ω , ω^{X_0} the elliptic measure for an operator L at X_0 and $G(\cdot, \cdot)$ the Green's function for L . Then for any open surface ball $\Delta_r \subset \partial\Omega$ or radius r such that $\delta(X_0) \geq 2r$ and*

$$G(X_0, Y) r^{n-2} \approx \omega(\Delta_r), \quad (3.18)$$

where $Y \in \Omega$ such that $\text{dist}(Y, \Delta_r) \approx \delta(Y) = r$. The precise constants in the estimate (3.18) only depends on the ellipticity of L and Lipschitz character of domain Ω .

The following lemma is crucial for the proof.

Lemma 3.4. *There exists $C > 0$ such that for all $f \in BMO(d\omega)$*

$$\|f\|_{BMO^*(d\omega)} \leq C \sup_{\Delta \subset \partial\Omega} \left(\iint_{T(\Delta)} |\nabla u|^2 G(X_0, X) \frac{dX}{\omega(\Delta)} \right)^{1/2}. \quad (3.19)$$

Assume for the moment the Lemma is true. By using Lemma 3.3 we get that

$$\iint_{T(\Delta)} |\nabla u|^2 G(X_0, X) dX \leq C \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \omega(\Delta_X) dX, \quad (3.20)$$

where Δ_X is as before the set $\{Q \in \partial\Omega; X \in \Gamma(Q)\}$. By changing the order of integration we get that

$$\iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \omega(\Delta_X) dX \leq \int_{\bar{\Delta}} S_r^2 u(Q) d\omega(Q). \quad (3.21)$$

Combining (3.19)-(3.21) we get that

$$\|f\|_{BMO^*(d\omega)} \leq \sup_{\Delta \subset \partial\Omega} \left(\int_{\Delta} S_r^2 u(Q) \frac{d\omega(Q)}{\omega(\Delta)} \right)^{1/2}. \quad (3.22)$$

Now we use the same trick as above to change measure back from ω to σ . Again using reverse Hölder inequality (now for k^{-1}) we get that

$$\sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^2 u(Q) \frac{d\omega(Q)}{\omega(\Delta_r)} \right)^{1/2} \leq C \sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^q u(Q) \frac{d\sigma(Q)}{\sigma(\Delta_r)} \right)^{1/q} \quad \text{for some } q > 2.$$

Finally, there exists $C > 0$

$$\begin{aligned} \sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^q u(Q) \frac{d\sigma(Q)}{\sigma(\Delta_r)} \right)^{1/q} &\leq C \sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^2 u(Q) \frac{d\sigma(Q)}{\sigma(\Delta_r)} \right)^{1/2} \\ &= C \sup_{\Delta \subset \partial\Omega} \left(\iint_{T(\Delta)} |\nabla u|^2 \delta(X) \frac{dX}{\sigma(\Delta)} \right)^{1/2}. \end{aligned} \quad (3.23)$$

The first estimate in (3.23) follows from the BMO John-Nirenberg argument (same way as (2.1) is established). This concludes the proof of Theorem 2.1 (modulo Lemma 3.4).

Proof of Lemma 3.4. We fix a surface ball $\Delta \subset \partial\Omega$ of radius r and center Q . As before we consider a point X_0 inside Ω such that $\delta(X_0) \geq 5r$. Finally, let us denote by \mathcal{D} the domain $\Omega \cap B(Q, 4r)$. We pick a point $X \in \mathcal{D}$ such that $\text{dist}(X, \partial\mathcal{D}) \approx 2r$. We denote by ν the elliptic measure for operator L on the domain \mathcal{D} with pole at X .

We study relations between measures ω and ν . The following Lemma holds

Lemma 3.5. *For any measurable set $E \subset \Delta$*

$$\frac{\omega(E)}{\omega(\Delta)} \leq C\nu(E), \quad (3.24)$$

where the constant $C > 0$ only depends on the ellipticity constant and Lipschitz character of the domain Ω .

It suffices to establish (3.24) for all balls $\Delta' \subset \Delta$, as the general statement for all measurable sets E follows by a covering lemma. For both balls Δ' and Δ we find points Y' and Y , respectively such that $\text{dist}(Y', \partial\Delta') \approx \delta(Y') = r'$ and $\text{dist}(Y, \partial\Delta) \approx \delta(Y) = r$, where r' and r are radii of these balls. According to Lemma 3.3

$$\omega(\Delta') \approx G_{\Omega}(X_0, Y')(r')^{n-2}, \quad \text{and} \quad \nu(\Delta') \approx G_{\mathcal{D}}(X, Y')(r')^{n-2}.$$

Hence

$$\frac{\omega(\Delta')}{\nu(\Delta')} \approx \frac{G_\Omega(X_0, Y')}{G_{\mathcal{D}}(X, Y')} \approx \frac{G_\Omega(X_0, Y)}{G_{\mathcal{D}}(X, Y)}.$$

The last relation comes from the comparison principle for two positive solutions $v(\cdot) = G_\Omega(X_0, \cdot)$ and $w(\cdot) = G_{\mathcal{D}}(X, \cdot)$ that vanish at the boundary. Finally,

$$\frac{\omega(\Delta')}{\nu(\Delta')} \approx \frac{G_\Omega(X_0, Y)}{G_{\mathcal{D}}(X, Y)} \approx \frac{\omega(\Delta)r^{n-2}}{\nu(\Delta)r^{n-2}},$$

again by using Lemma 3.3. However, $\nu(\Delta) = O(1)$, since the measure ν is doubling, and $\nu(\partial\mathcal{D}) = 1$. Hence Lemma 3.5 follows.

By Lemma 3.5 we see that for any $c_\Delta \in \mathbb{R}$

$$\int_\Delta |f - c_\Delta|^2 \frac{d\omega}{\omega(\Delta)} \leq C \int_\Delta |f - c_\Delta|^2 d\nu \leq C \int_{\partial\mathcal{D}} |u - c_\Delta|^2 d\nu. \quad (3.25)$$

Since ν is the natural (elliptic) measure for the domain \mathcal{D} it follows that the $L^2(d\nu)$ Dirichlet problem is always solvable in this domain. This implies the the $L^2(d\nu)$ norm of the square function is comparable with the $L^2(d\nu)$ of the (normalized) boundary data, i.e.,

$$\inf_{c_\Delta \in \mathbb{R}} \int_{\partial\mathcal{D}} |u - c_\Delta|^2 d\nu \approx \int_{\partial\mathcal{D}} S^2 u d\nu \approx \iint_{\Omega \setminus B_{r/8}(X)} |\nabla u(Y)|^2 G_{\mathcal{D}}(X, Y) dY. \quad (3.26)$$

Finally, we claim that

$$G_{\mathcal{D}}(X, Y) \leq G_\Omega(X, Y) \approx \frac{G_\Omega(X_0, Y)}{\omega(\Delta)}, \quad \text{for all } Y \in \Omega \setminus B_{r/8}(X). \quad (3.27)$$

Combining the estimates (3.25)-(3.27) we obtain Lemma 3.4. The first estimate of (3.27) is simply a maximum principle, as $G_{\mathcal{D}}(X, Y)$ vanishes on the whole $\partial\mathcal{D}$, and $G_\Omega(X, Y)$ is positive at the portion of this boundary. Both functions have same pole at X . The relation $G_\Omega(X, Y) \approx \frac{G_\Omega(X_0, Y)}{\omega(\Delta)}$ can be established as follows. For $Y \in \Omega \setminus B_{r/8}(X)$ such that $\delta(Y) \geq r$ Lemma 3.3 implies that $G_\Omega(X_0, Y) \approx r^{n-2}\omega(\Delta)$. On the other hand $G_\Omega(X, Y) \approx r^{n-2}$ as Y is of distance r from the pole and also r away from the boundary. For Y near the boundary we use the comparison principle (since both function vanish at $\partial\Omega$). This gives

$$\frac{G_\Omega(X, Y)}{G_\Omega(X_0, Y)} \approx \frac{G_\Omega(X, Y')}{G_\Omega(X_0, Y')}$$

for all $Y, Y' \in \Omega \setminus B_{r/8}(X)$. This establishes (3.27) and concludes the proof of Theorem 2.1. \square

Proof of Theorem 2.2. As follows from Theorem 2.10 of [13] it suffices to prove that there exists $C > 0$ such that for all Lipschitz domains $\Omega' \subset \Omega$ of same Lipschitz

character and any continuous solution $u \in C(\bar{\Omega})$ of the equation $Lu = 0$ in Ω vanishing at some fixed point the estimate

$$C^{-1} \int_{\partial\Omega'} N^2(u) d\sigma \leq \iint_{\Omega'} |\nabla u(X)|^2 \delta_{\Omega'}(X) dX \leq C \int_{\partial\Omega'} N^2(u) d\sigma \quad (3.28)$$

holds. This estimate means that the square function and nontangential maximal functions are equivalent on domains Ω' for solutions u in the whole domain Ω . It has been shown in [13] that if this holds, then $\omega \in A_\infty(d\sigma)$.

We start with an easier estimate ' $S \leq N$ ' (this is the second estimate of (3.28)). By the same argument as in [5] it suffices to establish the following lemma (good λ -inequality)

Lemma 3.6. *Suppose that $Su(P) \leq \lambda$ for some P in a surface ball $\Delta = \Delta(P_0, r) \subset \partial\Omega'$. Then there exists $c > 0$ depending only on the Lipschitz character of the domain $\partial\Omega'$, ellipticity constant of the operator L and C in the estimate (2.4) such that for any $\gamma > 0$*

$$\sigma(\{Q \in \Delta; Su(Q) > 2\lambda, Nu(Q) \leq \gamma\lambda\}) \leq c\gamma^2\sigma(\Delta). \quad (3.29)$$

Proof. Without loss of generality we may assume that the aperture of the nontangential cone used to define N is twice as large as the aperture of the cone for the square function S . Let $E = \{Q \in \Delta; Su(Q) \geq 2\lambda, Nu(Q) \leq \gamma\lambda\}$ and consider a point $Q \in E$. Since $S^2u(P) \leq \lambda^2$ it follows that

$$\iint_{\Gamma(Q) \setminus \Gamma(P)} |\nabla u(X)|^2 \delta_{\Omega'}^{2-n}(X) dX > 3\lambda^2, \quad (3.30)$$

where $\Gamma(Q), \Gamma(P)$ are the nontangential cones for S corresponding to these two points. We would like to estimate the integral (3.30) over the region $(\Gamma(Q) \setminus \Gamma(P)) \cap \mathcal{C}_{2^j r}$, $j = 0, 1, 2, \dots$, where

$$\mathcal{C}_t = \{X \in \Omega'; \text{dist}(X, \partial\Omega') \in [t, 2t)\}.$$

We note that the volume of the region $R_j = (\Gamma(Q) \setminus \Gamma(P)) \cap \mathcal{C}_{2^j r}$ is comparable to $(2^j)^{n-1} r^n$. We also notice that there exists an integer k depending only on the Lipschitz character of the domain $\partial\Omega'$ such that R_j can be covered by a finite union of k balls of radius $2^j r$ whose distance to Q and $\partial\Omega'$ is comparable to $2^j r$. Thus

$$R_j \subset \bigcup_{i=1}^k B_j^i, \quad (3.31)$$

where B_j^i , $i = 1, 2, \dots, k$ are the balls covering R_j . Let $(B_j^i)^*$ be slightly larger ball than B_j^i such that the union of $(B_j^i)^*$ belongs to the nontangential cone $\Gamma(Q)$ for the operator N . This is possible, as we have chosen the aperture for this cone to be larger, than the cone for the square function S . Hence by Poincaré inequality

$$\iint_{B_j^i} |\nabla u(X)|^2 dX \leq (2^j r)^{-2} \iint_{(B_j^i)^*} u(X)^2 dX \leq (2^j r)^{n-2} \gamma^2 \lambda^2, \quad (3.32)$$

since $Nu(Q) \leq \gamma\lambda$. We'll also use Meyers's result [15] which implies that for some $p > 1$ depending only on the ellipticity constant of the operator

$$\left(\frac{1}{|B_j^i|} \iint_{B_j^i} |\nabla u(X)|^{2p} dX \right)^{1/2p} \leq \left(\frac{C}{|(B_j^i)^*|} \iint_{(B_j^i)^*} |\nabla u(X)|^2 dX \right)^{1/2}. \quad (3.33)$$

It follows that

$$\begin{aligned} & \iint_{R_j} |\nabla u(X)|^2 \delta_{\Omega'}^{2-n}(X) dX \leq C(2^j r)^{2-n} \sum_{i=1}^k \iint_{R_j \cap B_j^i} |\nabla u(X)|^2 dX \\ & \leq C(2^j r)^{2-n} \sum_{i=1}^k |R_j \cap B_j^i|^{1-1/p} \left(\iint_{R_j \cap B_j^i} |\nabla u(X)|^{2p} dX \right)^{1/p} \\ & \leq C(2^j r)^{2-n} \sum_{i=1}^k |R_j \cap B_j^i|^{1-1/p} |B_j^i|^{1/p} \left(\frac{1}{|B_j^i|} \iint_{B_j^i} |\nabla u(X)|^{2p} dX \right)^{1/p} \\ & \leq C(2^j r)^{2-n} \sum_{i=1}^k |R_j \cap B_j^i|^{1-1/p} |B_j^i|^{1/p} \frac{1}{|B_j^i|} \iint_{B_j^i} |\nabla u(X)|^2 dX \\ & \leq C(2^j r)^{2-n} \sum_{i=1}^k \left(\frac{(2^j)^{n-1} r^n}{(2^j)^n r^n} \right)^{1-1/p} \iint_{B_j^i} |\nabla u(X)|^2 dX \leq C 2^{-j(1-1/p)} \gamma^2 \lambda^2. \end{aligned} \quad (3.34)$$

It follows that

$$\begin{aligned} & \iint_{\Gamma(Q) \setminus \Gamma(P), \delta_{\Omega'}(X) \geq r} |\nabla u(X)|^2 \delta_{\Omega'}^{2-n}(X) dX = \sum_{j \geq 0} \iint_{R_j} |\nabla u(X)|^2 \delta_{\Omega'}^{2-n}(X) dX \\ & \leq C \sum_{j \geq 0} 2^{-j(1-1/p)} \gamma^2 \lambda^2 \leq C \gamma^2 \lambda^2. \end{aligned} \quad (3.35)$$

This and (3.30) implies that for $\gamma \leq \gamma_0$

$$S_r^2 u(Q) = \iint_{\Gamma_r(Q)} |\nabla u(X)|^2 \delta_{\Omega'}^{2-n}(X) dX > \lambda^2, \quad (3.36)$$

where S_r is the truncated square function (at the height r) and $Q \in E$. From this

$$\begin{aligned} \sigma(E) & \leq \sigma(\{Q \in \Delta; S_r u(Q) \geq \lambda, Nu(Q) \leq \gamma\lambda\}) \\ & \leq \frac{1}{\lambda^2} \int_E S_r^2 u(Q) d\sigma(Q). \end{aligned} \quad (3.37)$$

Consider now a Lipschitz domain \mathcal{D} such that

$$\bigcup_{Q \in E} \Gamma_r(Q) \subset \mathcal{D} \subset \bigcup_{Q \in E} \Gamma_{2r}(Q),$$

where $\Gamma_r(Q), \Gamma_{2r}(Q)$ are the truncated nontangential cones on $\partial\Omega'$ used to define the nontangential maximal function N . It follows that for all $X \in \mathcal{D}$ we have $|u(X)| \leq \gamma\lambda$. Also $E \subset \partial\mathcal{D}$ and $\sigma(\partial\mathcal{D}) \leq C\sigma(\Delta)$. It follows, we can convert the boundary integral over E in (3.37) into a solid integral over \mathcal{D} :

$$\begin{aligned} \sigma(E) &\leq \frac{1}{\lambda^2} \int_E S_r^2 u(Q) d\sigma(Q) \leq \frac{C}{\lambda^2} \iint_{\mathcal{D}} |\nabla u(X)|^2 \delta_{\mathcal{D}}(X) dX \\ &\leq \frac{C}{\lambda^2} \sigma(\partial\mathcal{D}) \|u|_{\partial\mathcal{D}}\|_{BMO(d\sigma)}. \end{aligned} \quad (3.38)$$

The last inequality follows from (2.4), as \mathcal{D} is a domain that satisfies assumptions of Theorem 2.2. Since the BMO norm can be controlled by the L^∞ norm this finally gives

$$\sigma(E) \leq \frac{C}{\lambda^2} \sigma(\partial\mathcal{D}) \|u|_{\partial\mathcal{D}}\|_{BMO(d\sigma)} \leq \frac{c}{\lambda^2} \sigma(\Delta) (\gamma\lambda)^2 = c\gamma^2 \sigma(\Delta). \quad (3.39)$$

The reverse estimate ‘ $N \leq S$ ’ (this is the first estimate of (3.28)) is harder and requires normalization of the solution, since S is invariant under addition of a constant, i.e., u and $u + C$ have the same square function values, but N is not. Again, we as in [5] it suffices to establish appropriate good λ -inequality.

Lemma 3.7. *Suppose that $Nu(P) \leq \lambda$ for some P in a surface ball $\Delta = \Delta(P_0, r) \subset \partial\Omega'$. Let*

$$q(\gamma, Q) = \sup_{\Delta' \ni Q} \left\{ \frac{\sigma(\{Su > \gamma\lambda\} \cap \Delta')}{\sigma(\Delta')}; \text{ for all surface balls } \Delta' \subset 2\Delta \subset \partial\Omega' \right\}, \quad (3.40)$$

Then there exists $c > 0$ depending only on the Lipschitz character of the domain $\partial\Omega'$, ellipticity constant of the operator L and C in the estimate (2.4) such that for any $\gamma > 0$

$$\sigma(\{Q \in \Delta; Nu(Q) > 5\lambda, Su(Q) \leq \gamma\lambda, q(\gamma, Q) \leq 1/2\}) \leq c\gamma^2 \sigma(\Delta). \quad (3.41)$$

Proof. Let $E = \{Q \in \Delta; Nu(Q) > 5\lambda, Su(Q) \leq \gamma\lambda, q(\gamma, Q) \leq 1/2\}$. It is enough to assume the ball Δ is small so that we can write the set $\partial\Omega' \cap B(P_0, 5r)$ in local coordinates as $\{(x, t); \psi(x) > t\}$ for a Lipschitz function ψ .

As before we may assume that we use nontangential cones $\Gamma(Q)$ of aperture a to define the nontangential maximal function N . In local coordinates for $Q = (x, \psi(x))$

$$\Gamma(Q) = \Gamma(x, \psi(x)) = \{(x', t'); a(t' - \psi(x)) > |x - x'|, t' > \psi(x)\}.$$

We use cones of larger aperture b ($b \geq 2a$) to define the square function S . Following [13] (page 266) we define function h by

$$h(x) = \sup\{t > \psi(x); \sup_{\Gamma(x, t)} |\tilde{u}| \geq 3\lambda\}, \quad (3.42)$$

where $\Gamma(x, t)$ is a cone of aperture a with vertex at (x, t) , i.e., $\Gamma(x, t) = \{(x', t'); a(t' - t) > |x - x'|, t' > t\}$ and $\tilde{u}(X) = u(X) - u(X_0)$. We choose X_0 to be a point

above P_0 distance $\frac{k}{2}r$ away, i.e., in local coordinates is $P_0 = (p_0, \psi(p_0))$, then $X_0 = (p_0, \psi(p_0) + \frac{k}{2}r)$. The precise choice of k will be done a bit later, now we will only note that X_0 is sufficiently high up so that $X_0 \in \Gamma(P)$, hence $|u(X_0)| \leq \lambda$.

By Lemma 3.13 of [13] function h is Lipschitz with Lipschitz constant $\frac{1}{a}$, hence

$$\Omega_h = \{(x, t); t > h(x)\}$$

defines a Lipschitz domain. Also, since we know that $Nu(P) \leq \lambda$ it is clear that at this point $N\tilde{u}(P) \leq 2\lambda$, hence $h = 0$. From this, using the fact that h is Lipschitz, we see that $h \leq \frac{2}{a}r$ for points in Δ .

We note now that although we used nontruncated nontangential cones to define function h , we may as well use truncated version of this function

$$h_T(x) = \sup\{t \in (\psi(x), T]; \sup_{\Gamma(x,t)} |\tilde{u}| \geq 3\lambda\}, \quad (3.43)$$

since $h = h_T$ for all points in 2Δ , provided we take the truncation T in the form $T = 10r$.

To see this consider any $Q \in 2\Delta$ and assume that there is a point $X = (x, t) \in \Gamma(Q)$ such that $\tilde{u}(X) > 2\lambda$ and $t > 10r$. Since $Nu(P) \leq \lambda$ we can find a point $Y \in \Gamma(P)$ such that $|u(Y)| \leq \lambda$ and the distance $|X - Y| \leq 3r$. Clearly $\delta_{\Omega'}(X) \approx \delta_{\Omega'}(Y) \geq 7r$. Let B be a ball of radius $\frac{1}{2}\delta_{\Omega'}(X)$ with center at X and \tilde{B} a ball of radius $\frac{3}{4}\delta_{\Omega'}(X)$ with center at X . Since the operator L is elliptic there is $C > 0$ depending only on the ellipticity constant (c.f. [?]) such that

$$\text{osc}_B u \leq C\delta_{\Omega'}(X) \frac{1}{|\tilde{B}|} \iint_{\tilde{B}} |\nabla u(Z)| dZ. \quad (3.44)$$

Clearly, for sufficiently large aperture b that can be chosen independent of the point X , point Q or the ball Δ we will have $\tilde{B} \subset \Gamma^b(Q)$ for a cone with vertex at Q and aperture b . Hence the fact that $Su(Q) \leq \gamma\lambda$ allows us to control the righthand side of (3.44). We get that

$$\begin{aligned} \text{osc}_B u &\leq C\delta_{\Omega'}^{1-n}(X) \iint_{\tilde{B}} |\nabla u(Z)| dZ \leq C\delta_{\Omega'}^{(2-n)/2}(X) \left(\iint_{\tilde{B}} |\nabla u(Z)|^2 dZ \right)^{1/2} \\ &\leq C \left(\iint_{\Gamma^a(Q)} |\nabla u(Z)|^2 \delta_{\Omega}^{2-n}(Z) dZ \right)^{1/2} \leq C\gamma\lambda. \end{aligned} \quad (3.45)$$

It follows that for all $\gamma > 0$ small we see that $\text{osc}_B u \leq \lambda/2$. From this

$$|u(X)| \leq |u(Y)| + |u(X) - u(Y)| \leq \lambda + \text{osc}_B u < 2\lambda.$$

Hence $\tilde{u}(X) < 3\lambda$. It follows that $h = h_{10r}$ on 2Δ .

We construct a new Lipschitz domain $\mathcal{D} \subset \Omega'$ such that

$$\bigcup_{Q \in E} \Gamma_{kr}^b(Q) \subset \mathcal{D} \subset \bigcup_{Q \in E} \Gamma_{(k+1)r}^b(Q),$$

where we use nontangential cones of aperture b truncated at height kr in this definition for $k = \max\{10, \frac{4}{a}\}$.

Finally we denote by \mathcal{D}_h the Lipschitz domain

$$\mathcal{D}_h = \Omega_h \cap D.$$

Notice that we constructed the set \mathcal{D} from truncated cones of sufficient height so that the intersection with the set Ω_h is nonempty and the interior of \mathcal{D}_h will contain a ball with center at the point X_0 of radius comparable with r , i.e. $|\mathcal{D}_h| \approx r^n$. The fact that \mathcal{D}_h is a Lipschitz domain follows from the fact that it is an intersection of Lipschitz sets. We claim that

$$\sigma(E) \leq \frac{C}{\lambda^2} \int_{\partial\mathcal{D}_h} M_{\partial\mathcal{D}_h}(\tilde{u}^2) d\sigma, \quad (3.46)$$

where $C > 0$ only depends on the ellipticity constant of the operator, Lipschitz constant of Ω' and γ . The function $M_{\partial\mathcal{D}_h}$ is the classical Hardy-Littlewood maximal function on $\partial\mathcal{D}_h$:

$$M_{\partial\mathcal{D}_h}(w)(Q) = \sup_{Q \in \Delta' \subset \partial\mathcal{D}_h} \frac{1}{\sigma(\Delta')} \int_B |w| d\sigma, \quad \text{where } \Delta' \subset \partial\mathcal{D}_h \text{ is a surface ball.} \quad (3.47)$$

To see that (3.46) holds we establish the following lemma.

Lemma 3.8. *For every $Q \in E$ there is a surface ball $\Delta' \subset 2\Delta \subset \partial\Omega'$ such that $Q \in 4\Delta'$ and for every $R = (y, \psi(y)) \in \Delta'$*

$$R' = (y, h(y)) \in \partial\mathcal{D}_h \quad \text{and} \quad \tilde{u}(R') > \lambda. \quad (3.48)$$

Clearly, (3.48) guarantees that for every $Q \in \Delta$, we have $M_{\partial\mathcal{D}_h}(\tilde{u}^2)(Q) > c'\lambda^2$, hence (3.46) holds.

To see this lemma holds consider any $Q \in E$ and assume that in local coordinates we can write Q as $Q = (x, \psi(x))$. Since $Nu(Q) > 5\lambda$, we can claim that $N\tilde{u}(Q) > 4\lambda$. This implies that here $h(x) > 0$ and $Q' = (x, h(x))$ is a point above Q laying on $\partial\mathcal{D}_h$ such that $N\tilde{u}(Q') = 3\lambda$. At all points above Q' function N is less than 3λ , hence the boundary of the nontangential cone $\Gamma(Q') = \{(x', t'); a(t' - h(x)) > |x' - x|\}$ must contain a point $P'_1 = (x_1, t_1)$ such that $|\tilde{u}(P'_1)| = 3\lambda$. It is also clear that $t_1 = h(x_1)$. We may therefore conclude that $P'_1 \in \partial\mathcal{D}_h$, as P'_1 is an interior point of \mathcal{D} since is constructed from nontangential cones of aperture $b \geq 2a$ and $P'_1 \in \Gamma^a(Q)$ for a cone of aperture a . Let us denote by P_1 the point directly below P'_1 lying on $\partial\Omega'$, i.e., $P_1 = (x_1, \psi(x_1))$. We claim the the distance between Q and P_1 cannot be too big. Indeed, we have seen above that $\tilde{u}(Y) < 2\lambda$ for points in $\Gamma(Q)$ of distance bigger than $10r$ from Q . This implies that for $a \leq 1/20$ this distance will be less than $10ra \leq r/2$, i.e., $P_1 \in \frac{3}{2}\Delta$.

Let us define $\Delta' = \{(z, \psi(z)); |z - x_1| < |x - x_1|/4\}$. It follows that $\Delta' \subset 2\Delta$ and $Q \in 4\Delta'$. Consider any point $R = (z, \psi(z)) \in \Delta'$ and let $R' = (z, h(z))$ be the point above R on $\partial\Omega_h$. We claim that R' is an interior point of \mathcal{D} . Indeed, this is trivial, as h

is a Lipschitz function of Lipschitz constant at most $\frac{1}{a}$, hence the fact that $P'_1 \in \Gamma^a(Q)$ implies the same for R' . But $\Gamma^a(Q) \subset \mathcal{D}$. This proves that $R' = (z, h(z)) \in \partial\mathcal{D}_h$.

It remains to establish that $\tilde{u}(R') > \lambda$. Let us remind ourselves that $P'_1, R' \in \Gamma^a(Q)$, $\Gamma^b(Q) \subset \Omega'$ and $\text{dist}(P'_1, Q) \approx \text{dist}(R', Q) \approx d_{\Omega'}(P'_1)$. Hence the geometry of this situation implies that we can find a ball B of radius approximately $d_{\Omega'}(P'_1)$ such that $P'_1, R' \in B$ and $\tilde{B} = tB \subset \Gamma^b(Q) \subset \Omega'$ for some $t = t(a, b) > 1$. Here tB denotes an enlarged ball by factor of t . This means we can again use (3.44). This gives

$$\text{osc}_B \tilde{u} \leq C \delta_{\Omega'}(P'_1) \frac{1}{|\tilde{B}|} \iint_{\tilde{B}} |\nabla \tilde{u}(Z)| dZ. \quad (3.49)$$

As in (3.45) we use Hölder inequality and the fact that $Su(Q) \leq \gamma\lambda$ to get that

$$\text{osc}_B \tilde{u} \leq C\gamma\lambda. \quad (3.50)$$

Hence for all $\gamma > 0$ small we have that $\text{osc}_B \tilde{u} \leq \lambda$ which gives

$$|\tilde{u}(R')| \geq |\tilde{u}(P'_1)| - \text{osc}_B \tilde{u} \geq 3\lambda - \lambda > \lambda.$$

This finishes the proof of Lemma 3.8 and establishes (3.46).

By the properties of Hardy-Littlewood maximal function (3.46) immediately implies that for some larger constant $C > 0$:

$$\sigma(E) \leq \frac{C}{\lambda^2} \int_{\partial\mathcal{D}_h} \tilde{u}^2 d\sigma = \frac{C}{\lambda^2} \int_{\partial\mathcal{D}_h} |u - u(X_0)|^2 d\sigma. \quad (3.51)$$

We may now use (2.4) as the assumptions are satisfied. This gives

$$\sigma(E) \leq \frac{C\sigma(\partial\mathcal{D}_h)}{\lambda^2} \sup_{\Delta \subset \partial\mathcal{D}_h} \frac{1}{\sigma(\Delta')} \iint_{T(\Delta')} |\nabla u(X)|^2 \delta_{\mathcal{D}_h}(X) dX. \quad (3.52)$$

From this the claim (3.39) follows, provides we show that

$$\frac{1}{\sigma(\Delta')} \iint_{T(\Delta')} |\nabla u(X)|^2 \delta_{\mathcal{D}_h}(X) dX \leq C\gamma^2\lambda^2, \quad (3.53)$$

for a constant $C > 0$ independent of Δ' . Indeed, if (3.53) holds then using (3.52) and the fact that $\sigma(\partial\mathcal{D}_h) \approx \sigma(\Delta)$ gives us $\sigma(E) \leq C \frac{\sigma(\Delta)}{\lambda^2} \lambda^2 \gamma^2 = C\gamma^2\sigma(\Delta)$.

To establish (3.53) we need to analyze two cases. Consider any surface ball $\Delta' \subset \partial\mathcal{D}_h$, let us denote by ρ the radius of this ball.

Case 1. $\rho \leq 10 \text{dist}(T(\Delta'), \partial\Omega')$. In this case we have

$$\begin{aligned} & \frac{1}{\sigma(\Delta')} \iint_{T(\Delta')} |\nabla u(X)|^2 \delta_{\mathcal{D}_h}(X) dX \leq \frac{\rho}{\sigma(\Delta')} \iint_{T(\Delta')} |\nabla u(X)|^2 dX \\ & \leq C\rho^{1-n} \iint_B |\nabla u(X)|^2 dX \leq C\rho^{-n} \iint_{2B} |u(X) - u(Z)|^2 dX. \end{aligned} \quad (3.54)$$

Here B is a ball of radius 2ρ containing $T(\Delta')$ with center at Z and $2B$ the double of this ball. The last inequality is the Poincaré inequality. Now as follows from the definition of the domain \mathcal{D}_h there is a point $Q \in E$ such that $3B \subset \Gamma^b(Q)$. Therefore again we may use (3.44) and obtain as in (3.45) that $\text{osc}_B u \leq C\gamma\lambda$. From this

$$C\rho^{-n} \iint_{2B} |u(X) - u(Z)|^2 dX \leq C\rho^{-n}\rho^n(\gamma\lambda)^2 \leq C\gamma^2\lambda^2. \quad (3.55)$$

This establishes the claim for the case 1.

Case 2. $\rho > 10 \text{dist}(T(\Delta'), \partial\Omega')$. Let us denote by $\tilde{\Delta}$ the subset of $\partial\Omega'$ such that

$$\tilde{\Delta} = \{Q \in \partial\Omega'; \Gamma^b(Q) \cap T(\Delta') \neq \emptyset\}. \quad (3.56)$$

The geometry of this case implies that for some $C = C(b) > 0$ (b is the aperture of the nontangential cone $\Gamma^b(Q)$) we have $\sigma(\tilde{\Delta}) \subset C\sigma(\Delta')$.

For $X \in T(\Delta')$ let us denote by Δ_X the set

$$\Delta_X = \{Q \in \partial\Omega'; X \in \Gamma^b(Q)\}. \quad (3.57)$$

Clearly, $\sigma(\Delta_X) \approx d_{\Omega'}^{n-1}(X)$ and $\tilde{\Delta} = \bigcup_{X \in T(\Delta')} \Delta_X$. The set Δ_X might not be completely contained in $2\Delta = \Delta(P_0, 2r)$ but $\Delta_X \cap 2\Delta$ contain a substantial part of the original set, i.e., $\sigma(\Delta_X \cap 2\Delta) \approx \sigma(\Delta_X)$. Hence, from the definition of the function q in (3.40) it follows that there exists $c > 0$ independent of $X \in T(\Delta')$ such that

$$\frac{\sigma(\{Q; Su(Q) \leq \gamma\lambda\} \cap \Delta_X)}{\sigma(\Delta_X)} \geq c > 0. \quad (3.58)$$

It follows that we can write

$$\begin{aligned} & \iint_{T(\Delta')} |\nabla u(X)|^2 \delta_{\mathcal{D}_h}(X) dX \leq \iint_{T(\Delta')} |\nabla u(X)|^2 \delta_{\Omega'}(X) dX \\ & \leq C \iint_{T(\Delta')} |\nabla u(X)|^2 \delta_{\Omega'}^{2-n}(X) \sigma(\{Q; Su(Q) \leq \gamma\lambda\} \cap \Delta_X) dX, \end{aligned} \quad (3.59)$$

by (3.58) and the fact that $\sigma(\Delta_X) \approx d_{\Omega'}^{n-1}(X)$. This means we can now change the order of integration and integrate the square function Su over the set $\tilde{\Delta} \cap \{Su \leq \gamma\lambda\}$. Hence the last term of (3.59) is less or equals to

$$\leq \int_{\tilde{\Delta} \cap \{Su \leq \gamma\lambda\}} S^2 u(Q) d\sigma(Q) \leq \int_{\tilde{\Delta} \cap \{Su \leq \gamma\lambda\}} (\gamma\lambda)^2 d\sigma(Q) \leq C\sigma(\tilde{\Delta})\gamma^2\lambda^2. \quad (3.60)$$

This together with the fact that $\sigma(\tilde{\Delta}) \subset C\sigma(\Delta')$ proves (3.53). Hence Lemma 3.7 is proved and ' $N \leq S$ ' holds on $\partial\Omega'$.

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