BMO solvability and the A_∞ condition for elliptic operators

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Abstract

We establish a connection between the absolute continuity of elliptic measure associated to a second order divergence form operator with bounded measurable coefficients with the solvability of an endpoint BMO Dirichlet problem. We show that these two notions are equivalent. As a consequence we obtain an end-point perturbation result, i.e., the solvability of the BMO Dirichlet problem implies L^p solvability for all $p > p_0$.

1 Introduction

We shall prove an equivalence between solvability of certain endpoint (BMO) Dirichlet boundary value problems for second order elliptic operators and a quantifiable absolute continuity of the elliptic measure associated to these operators. More precisely, we consider here the Dirichlet problem for divergence form (not necessarily symmetric) elliptic operators $L = \mathrm{div} A\nabla$, where $A = (a_{ij}(X))$ is a matrix of bounded measurable functions for which there exists a $\lambda > 0$ such that $\lambda^{-1}|\xi|^2 < \sum a_{ij}\xi_i\xi_j < \lambda|\xi|^2$. The L^p Dirichlet problem for L asks for solvability in a domain Ω, in the sense of non-tangential convergence and a priori L^p estimates, of the problem: $Lu = 0$ in Ω with $u = f$ on $\partial\Omega$.

Let us recall ([13]) a fundamental property of the harmonic extension to $\mathbb{R}^+_n$ of functions of bounded mean oscillation on $\mathbb{R}^n$: If $f \in BMO$, then the Poisson extension $u(x,t) = P_t * f(x)$ has the property that $t|\nabla u|^2 dxdt$ is a Carleson measure. (Carleson measures are defined in Section 2, below.) In fact the Carleson measure norm of this extension and the BMO norm of f are equivalent.

In [12], this fundamental property was shown to hold for the harmonic functions in the class of Lipschitz domains. The key fact here is that harmonic measure on Lipschitz domains is always mutually absolutely continuous with respect to surface measure, by a well known result of [4].

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In [20], further connections between Carleson measure properties of solutions to very general second order divergence form elliptic equations and absolute continuity were established. There it was shown that if all bounded solutions to $L = \mathrm{div}A\nabla$ are arbitrarily well approximated by continuous functions satisfying an L^1 version of the Carleson measure property, then in fact the elliptic measure belongs to A_∞ with respect to surface measure. This approximation property was shown (in [20]) to follow from a certain norm equivalence between two different classical quantities associated to the solution of an elliptic equation: the nontangential maximal function, measuring size, and the square function, measuring the size of oscillations.

These results, from the Carleson measure properties of harmonic functions in the upper half space, to theorems such as those in [20] which specifically connect absolute continuity of the representing measures associated to second order divergence form operators to Carleson measure conditions, led us to a conjecture concerning solvability of the Dirichlet problem with data in BMO.

Specifically, we are interested in properties of the elliptic measure of an operator $L = \mathrm{div}A\nabla$ which determine that it belongs to the Muckenhoupt A_∞ class with respect to the surface measure on the boundary of the domain of solvability. On the one hand, A_∞ is a "perturbable" condition, in the sense that $A_\infty = \bigcup A_p = \bigcup B_p$. And when the density of harmonic measure with respect to surface measure belongs to B_p, it turns out that the Dirichlet problem is solvable with data in L^q, where $1/q + 1/p = 1$. (Again, see section 2 for the definitions.) On the other hand, a boundary value problem which is equivalent to A_∞ would have to be "perturbable" as well: solving it would have to imply solvability of the Dirichlet problem in some L^q. Clearly L^∞ cannot be such a perturbable endpoint space: all solutions satisfy a maximum principle, a precise version of the L^∞ Dirichlet problem. In the end, perturbing from a BMO problem seems quite natural.

We will use a variety of properties of solutions to divergence form elliptic operators with bounded measurable coefficients. The De Giorgi-Nash-Moser theory of the late 1950s and early 1960s assures us that weak solutions to these equations are in fact Holder continuous. Further properties of solutions, of the elliptic measure whose existence is guaranteed by the maximum principle and the Riesz representation theorem, and of the relationship of this measure to the Green's function were developed in the 1970s and 1980s. For the basic properties of solutions to divergence form operators with bounded measurable coefficients, as in [24] or [1], one can consult the introduction of [20] where many primary references are cited, and where the issues for the non-symmetric situation are discussed.

2 Definitions and Statements of Main Theorems

Let us begin by defining introducing Carleson measures and square functions on domains which are locally given by the graph of a function. We shall assume that our domains are Lipschitz, even though it is possible to formulate and prove these results with less stringent geometric conditions on the domain. Most likely, the minimal geometric conditions required would be chord-arc and nontangentially accessible. [1]

Definition 2.1. *$\mathbb{Z} \subset \mathbb{R}^n$ is an M-cylinder of diameter d if there exists a coordinate system (x,t) such that*

$$\mathbb{Z} = \{(x,t) \;:\; |x| \le d,\; -2Md \le t \le 2Md\}$$

and for $s > 0$,

$$s\mathbb{Z} = \{(x,t) \;:\; |x| < sd, -2Md \le t \le 2Md\}.$$

Definition 2.2. *$\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with Lipschitz 'character' (M, N, C_0) if there exists a positive scale r_0 and at most N cylinders $\{Z_j\}_{j=1}^N$ of diameter d, with $\frac{r_0}{c_0} \le d \le c_0 r_0$ such that*
(i) $8Z_j \cap \partial\Omega$ is the graph of a Lipschitz function ϕ_j,

$$\|\phi_j\|_\infty \le M,\; \phi_j(0) = 0,$$

(ii)

$$\partial\Omega = \bigcup_j (Z_j \cap \partial\Omega)$$

(iii)

$$Z_j \cap \Omega \supset \left\{(x,t) \;:\; |x| < d,\; \mathrm{dist}\,((x,t), \partial\Omega) \le \frac{d}{2}\right\}.$$

If $Q \in \partial\Omega$ and

$$B_r(Q) = \{x : |x - Q| \le r\}$$

then $\Delta_r(Q)$ denotes the surface ball $B_r(Q) \cap \partial\Omega$ and $T(\Delta_r) = \Omega \cap B_r(Q)$ is the called the Carleson region above $\Delta_r(Q)$.

Definition 2.3. *Let $T(\Delta_r)$ be a Carleson region associated to a surface ball Δ_r in $\partial\Omega$. A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \le r_0$,*

$$\mu(T(\Delta_r)) \le C\sigma(\Delta_r).$$

For such measure μ we denote by $\|\mu\|_{Car}$ the number

$$\|\mu\|_{Car} = \sup_{\Delta \subset \partial\Omega} \left(\sigma(\Delta)^{-1}\mu(T(\Delta))\right)^{1/2}.$$

Definition 2.4. *A cone of aperture a is a non-tangential approach region for $Q \in \partial\Omega$ of the form*

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \le a \;\; dist(X, \partial\Omega)\}.$$

Sometimes it is necessary to truncate the height of Γ by h. Then $\Gamma_h(Q) = \Gamma(Q) \cap B_h(Q)$.

[1]This was pointed out to us by M. Badger.

We remind the reader that L will stand for $L = \mathrm{div} A\nabla$ where the matrix A has bounded measurable coefficients $a_{i,j}$ and is strongly elliptic: there exists λ such that uch that for all $\xi \in \mathbb{R}^n \backslash \{0\}$,

$$\lambda|\xi|^2 \leq \sum a_{i,j}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2.$$

Definition 2.5. *If $\Omega \subset \mathbb{R}^n$, and u is a solution to L, the square function in $Q \in \partial\Omega$ relative to a family of cones Γ is*

$$Su(Q) = \left(\int_{\Gamma(Q)} |\nabla u(X)|^2 \delta(X)^{2-n} dX\right)^{1/2}.$$

and the non-tangential maximal function at Q relative to Γ is

$$Nu(Q) = \sup\{|u(X)| : X \in \Gamma(Q)\}.$$

Here $\delta(X) = dist(X, \partial\Omega)$. We also consider truncated versions of these operators which we denote by $S_h u(Q)$ and $N_h(Q)$, respectively; the only difference in the definition is that the nontangential cone $\Gamma(Q)$ is replaced by the truncated cone $\Gamma_h(Q)$.

Definition 2.6. *The Dirichlet problem with the $L^p(\partial\Omega, d\sigma)$ data is solvable for L if the solution u for continuous boundary data f satisfies the estimate*

$$\|N(u)\|_{L^p(\partial\Omega,d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{2.1}$$

where the implied constant does not depend on the given function.

Definition 2.7. *If $d\mu$ and $d\nu$ are finite measures on the boundary of Ω, then $d\mu$ belongs to A_∞ with respect to $d\nu$ if for all ϵ there exists an η such that, for every surface ball Δ and subset $E \subset \Delta$, whenever $\nu(E)/\nu(\Delta) < \eta$, then $\mu(E)/\mu(\Delta) < \epsilon$.*

This space was investigated in [2], where various equivalent definitions were given. In particular, $d\mu \in A_\infty(d\nu)$ if and only if $d\nu \in A_\infty(d\mu)$.

Let us specialize this definition to the domain Ω, to surface measure $d\sigma$ and to the elliptic measure $d\omega_L$ associated to some divergence form operator L. We are assuming that $d\omega_L$ is evaluated at some fixed point P in the interior of Ω so that a solution to L with continuous data f at the point P is represented by this measure: this means that $u(P) = \int_{\partial\Omega} f(y) d\omega_L(y)$. If $d\omega$ belongs to $A_\infty(d\sigma)$, then there is a density function: $d\omega_L(y) = k(y)d\sigma$. The apriori estimate of definition 2.6 turns out to be equivalent to the fact that the density $k(y)$ satisfies a reverse Hölder estimate $B_{p'}$. For general $q > 1$, the density k is said to belong to $B_q(d\sigma)$ if there exists a constant C such that for every surface ball Δ, $((\sigma(\Delta))^{-1} \int_\Delta k^q d\sigma)^{1/q} < C\sigma(\Delta))^{-1} \int_\Delta k d\sigma$. The relationship between the reverse Hölder classes and A_∞ is ([15] and [2])

$$A_\infty(\partial\Omega, d\sigma) = \bigcup_{p>1} B_q(\partial\Omega, d\sigma),$$

Definition 2.8. *We say that a function $f : \partial\Omega \to \mathbb{R}$ belongs to BMO with respect to the surface measure $d\sigma$ if*

$$\sup_{I\subset\partial\Omega} \sigma(I)^{-1} \int_I |f - f_I|^2 d\sigma < \infty.$$

Here $f_I = \sigma(I)^{-1} \int_I f d\sigma$. We denote by $\|f\|_{BMO(p)}$ the number

$$\|f\|_{BMO(p)} = \sup_{I\subset\partial\Omega} \left(\sigma(I)^{-1} \int_I |f - f_I|^p d\sigma\right)^{1/p}.$$

It can be shown for any $1 \le p < \infty$ that $\|f\|_{BMO(2)} < \infty$ if and only if $\|f\|_{BMO(p)} < \infty$. Moreover, $\|.\|_{BMO(p)}$ and $\|.\|_{BMO(2)}$ are equivalent in the sense that there is a constant $C > 0$ such that the inequality

$$C^{-1}\|f\|_{BMO(p)} \le \|f\|_{BMO(2)} \le C\|f\|_{BMO(p)} \tag{2.2}$$

holds for any BMO function f.

This definition can be modified further. Instead of using the difference $f - f_I$ in the definition of the BMO norm one can take

$$\|f\|_{BMO^*(p)} = \sup_{I\subset\partial\Omega} \inf_{c_I\in\mathbb{R}} \left(\sigma(I)^{-1} \int_I |f - c_I|^p d\sigma\right)^{1/p}. \tag{2.3}$$

Again, it can be shown that this gives an equivalent norm, i.e., there is $C > 0$ such that

$$C^{-1}\|f\|_{BMO^*(p)} \le \|f\|_{BMO(2)} \le C\|f\|_{BMO^*(p)}.$$

Definition 2.9. *The BMO-Dirichlet problem is solvable for L if the solution u for continuous boundary data f satisfies*

$$\||\nabla u|^2\delta(X)dX\|_{Car} \lesssim \|f\|^2_{BMO(2)}.$$

Equivalently, there exists a constant C such that for all continuous f,

$$\sup_{\Delta\subset\partial\Omega} \sigma(\Delta)^{-1} \iint_{T(\Delta)} |\nabla u|^2\delta(X)dX \le C \sup_{I\subset\partial\Omega} \sigma(I)^{-1} \int_I |f - f_I|^2 d\sigma. \tag{2.4}$$

Remark. It follows from our results that even though we define BMO-solvability in the Definition 2.9 only for *continuous* boundary data, the solution can be defined for any BMO function $f : \partial\Omega \to \mathbb{R}$ and moreover the estimate (2.4) will hold. In addition, such a solution u will have a well-defined nontangential maximal function $N(u)$ for almost every point $Q \in \partial\Omega$ and in the nontangential sense

$$f(Q) = \lim_{X\to Q,\, X\in\Gamma(Q)} u(X), \qquad \text{for a.e. } Q \in \partial\Omega.$$

Indeed, by Theorem 2.2 given that (2.4) holds, the L^p Dirichlet boundary value problem is solvable for some large $p < \infty$. Consider now an arbitrary BMO function $f : \partial\Omega \to \mathbb{R}$. As we argue in (3.18), there exists a sequence of *continuous* functions $f_n : \partial\Omega \to \mathbb{R}$ such that $f_n \to f$ in $L^p(\partial\Omega)$ and $\|f_n\|_{BMO} \leq C\|f\|_{BMO}$ for some $C > 0$ independent of n.

For each f_n we can solve the continuous Dirichlet boundary value problem which will give us solutions u_n such that

$$\|N(u_n)\|_{L^p(\partial\Omega)} \leq C\|f_n\|_{L^p(\partial\Omega)} \leq C\|f\|_{BMO}.$$

In addition, also

$$\|N(u_n - u_m)\|_{L^p(\partial\Omega)} \leq C\|f_n - f_m\|_{L^p(\partial\Omega)} \to 0, \qquad \text{as } n, m \to \infty,$$

since $f_n \to f$ in L^p and (2.1) holds. This implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is locally uniformly Cauchy in $L^\infty_{loc}(\Omega)$, hence

$$u(X) = \lim_{n\to\infty} u_n(X), \qquad \text{for } X \in \Omega$$

is pointwise well defined.

We claim that this u is a weak solution to $Lu = 0$. That is,

$$\int_\Omega A(X)\nabla u(X).\nabla\psi(X)\, dX = 0, \qquad \text{for all } \psi \in C_0^\infty(\Omega), \tag{2.5}$$

To see this, fix a compact set $K \subset \Omega$. By the dominated convergence theorem we know that

$$u_n \to u, \qquad \text{in any } L^p(K), \quad p < \infty.$$

Hence for any $K' \subset\subset K$ by Cacciopoli we have that

$$\int_{K'} |\nabla(u_n - u_m)(X)|^2\, dX \leq C_{K,K'} \int_K |(u_n - u_m)(X)|^2\, dX \to 0, \quad \text{as } n, m \to \infty.$$

It follows that ∇u_n converges locally uniformly in L^2, from which we get that u belongs to $W^{1,2}_{loc}(\Omega)$ and $\nabla u_n \to \nabla u$ in $L^2_{loc}(\Omega)$. Therefore (2.5) follows as we already now that (2.5) holds for every u_n and we can pass to the limit $n \to \infty$.

Hence with the use of Fatou's lemma (see Appendix B of [9] for details) we get that $N(u - u_n) \to 0$ in $L^p(\partial\Omega)$ as $n \to \infty$. This implies that $\|N(u)\|_{L^p} < \infty$, so $N(u)(Q) < \infty$ a.e. for $Q \in \partial\Omega$ and also one has existence of nontangential limits a.e.: $\lim_{X\to Q,\, X\in\Gamma(Q)} u(X)$.

Finally, we also get that (2.4) will also hold for u by the limiting argument, since it holds for each u_n:

$$\sup_{\Delta\subset\partial\Omega} \sigma(\Delta)^{-1} \iint_{T(\Delta)} |\nabla u_n|^2 \delta(X) dX \lesssim \sup_{I\subset\partial\Omega} \sigma(I)^{-1} \int_I |f_n - f_{n,I}|^2 d\sigma. \tag{2.6}$$

Notice that taking the limsup on the right-hand side of (2.6) yields just a multiple of BMO norm of f, as $\|f_n\|_{BMO} \leq C\|f\|_{BMO}$. On the left-hand side we may take the limit

$$\sigma(\Delta)^{-1} \iint_{T(\Delta)\setminus\mathcal{C}_\varepsilon} |\nabla u_n|^2 \delta(X)dX \to \sigma(\Delta)^{-1} \iint_{T(\Delta)\setminus\mathcal{C}_\varepsilon} |\nabla u|^2 \delta(X)dX, \quad n \to \infty,$$

since $\nabla u_n \to \nabla u$ in $L^2_{loc}(\Omega)$. Here $\mathcal{C}_\varepsilon = \{X \in \Omega : \operatorname{dist}(X, \partial\Omega) < \varepsilon\}$. It follows that for any $\varepsilon > 0$

$$\sup_{\Delta\subset\partial\Omega} \sigma(\Delta)^{-1} \iint_{T(\Delta)\setminus\mathcal{C}_\varepsilon} |\nabla u|^2 \delta(X)dX \lesssim \|f\|^2_{BMO}. \tag{2.7}$$

As the constant in (2.7) does not depend on ε we get the required estimate on the whole $T(\Delta)$. In fact, it can be shown that *equivalence* holds between the two quantities in (2.4).

We now state our main results.

Theorem 2.1. *Let Ω be a Lipschitz domain and L be a divergence form elliptic operator with bounded coefficients satisfying the strong ellipticity hypothesis.*

If the elliptic measure $d\omega_L$ associated with L is in $A_\infty(\partial\Omega, d\sigma)$ then the BMO-Dirichlet problem is solvable for L, with in fact equivalence of the two norms in the estimate (2.4).

Conversely, if the estimate (2.4) holds for all continuous functions f with constants only depending on the Lipschitz character of the domain Ω and the ellipticity constant of L, then the elliptic measure $d\omega_L$ belongs to $A_\infty(\partial\Omega, d\sigma)$.

Remark. The closure of continuous functions in BMO norm is the VMO class ([25]). From the proof of the theorem, we will see that A_∞ is actually equivalent to solvability of a VMO-Dirichlet problem.

Recall that if a Dirichlet problem for an elliptic operator L is L^p solvable for some $p \in (1, \infty)$, then it is solvable for all L^q $p - \varepsilon < q < \infty$, which shows that the "solvability" is stable under small perturbations.

Theorem 2.1 implies the same kind of stability result for the end-point BMO problem on the L^p interpolation scale.

Theorem 2.2. *(Stability of BMO solvability) Let Ω be a Lipschitz domain and L be a divergence form elliptic operator with bounded coefficients satisfying the strong ellipticity hypothesis.*

Assume that the BMO-Dirichlet problem is solvable for L and the estimate (2.4) holds. Then there exist $p_0 > 1$ such that the L^p Dirichlet problem for L is solvable for all $p_0 < p < \infty$.

Proof. By Theorem 2.1 if follows that $d\omega_L \in A_\infty(\partial\Omega, d\sigma)$. Since

$$A_\infty(\partial\Omega, d\sigma) = \bigcup_{p>1} B_p(\partial\Omega, d\sigma),$$

we see that $d\omega_L \in B_p(\partial\Omega, d\sigma)$ for some $p > 1$. From this the claim follows since $d\omega_L \in B_p(\partial\Omega, d\sigma)$ implies the solvability of the $L^{p'}$ Dirichlet problem. The range of solvability (p_0, ∞) can be then obtained by realizing that $B_p(\partial\Omega, d\sigma) \subset B_q(\partial\Omega, d\sigma)$ for $q < p$. □

3 Proof of Theorem 2.1.

We establish the A_∞ property of $d\omega_L$ by assuming the estimate (2.4) holds uniformly for continuous data.

The elliptic measure for L will be abbreviated $d\omega$ and is evaluated at a fixed interior point, P_0, of the domain Ω.

Let Δ be a surface ball on the boundary of Ω of radius r. Let Δ' be another surface ball of radius r separated from Δ by a distance of r. By assumption, if $Lu = 0$ and $u = f$ on the boundary, we have

$$\sigma(\Delta')^{-1} \iint_{T(\Delta')} |\nabla u|^2 \delta(X) dX \lesssim \|f\|_{BMO}. \tag{3.8}$$

Let us now assume that f is a positive and continuous function supported in Δ.

Recall that $S_r u(Q)$ denotes the square function defined using cones truncated at height r. We claim that there exists a constant C such that for all $Q \in \Delta'$,

$$\omega(\Delta)^{-1} \int_\Delta f d\omega \leq C S_r u(Q) \tag{3.9}$$

To establish this claim, we introduce a little more notation.

For $Q \in \Delta'$, set $\Gamma_j(Q) = \Gamma(Q) \cap B_{2^{-j}r}(Q) \backslash B_{2^{-j-1}r}(Q)$, a slice of the cone $\Gamma(Q)$ at height $2^{-j}r$.

By Lemma 5.8 (see also 5.13) of [21], we have the following Poincare type estimate, which was established using Sobolev embedding and boundary Cacciopoli to exploit the fact that u vanishes on Δ':

$$(2^j r)^{-2} \int_{\Gamma_j(Q)} u^2 dX \lesssim \iint_{\Gamma_j(Q)} |\nabla u|^2 \delta(X) dX \tag{3.10}$$

Let A' denote a point in $T(\Delta')$ whose distance to the boundary of Ω is approximately r. By the comparison theorem for solutions which vanish at the boundary, and with $G(X)$ denoting the Green's function for L with pole at P_0 in Ω,

$$\frac{u(X)}{G(X)} \approx \frac{u(A')}{G(A')}, \tag{3.11}$$

for all $X \in \Gamma(Q) \cap T(\Delta')$.

We use this to estimate the square function:

$$S_r^2 u(Q) \geq \sum_{j=0}^{\infty} \int_{\Gamma_j} \delta(X)^{2-n} |\nabla u|^2 dX \tag{3.12}$$

$$\geq \frac{u^2(A')}{G^2(A')} \sum_{j}^{\infty} (2^{-j} r)^n \int_{\Gamma_j} G^2(X) dX. \tag{3.13}$$

Now let A_j be a nontangential point in Γ_j, so that $|A_j - Q| \approx 2^{-j} r$. By Harnack, $G(X) \approx G(A_j)$ for all $X \in \Gamma_j(Q)$. Moreover, again by Harnack, there is constant $C > 1$ for which $G(A_{j-1}) < CG(A_j)$. Thus,

$$\frac{u^2(A')}{G^2(A')} \sum_{j=0}^{\infty} G^2(A_j) \lesssim S_r^2 u(Q), \tag{3.14}$$

and now since $G(A') \leq C^j G(A_j)$, we can sum this series and we find that

$$u^2(A') \lesssim S_r^2 u(Q). \tag{3.15}$$

Since, by properties of harmonic measure, we also know that $u(A') \approx \omega(\Delta)^{-1} \int_\Delta f d\omega$, this proves 3.9.

For any such f, positive, continuous and supported in Δ, the estimate in 3.8 implies that, for some constant C_0,

$$(\omega(\Delta)^{-1} \int_\Delta f d\omega)^2 \leq C_0^2 \|f\|_{BMO}^2. \tag{3.16}$$

We now establish absolute continuity of the elliptic measure. Suppose that $\sigma(\Delta) = r$ and that ϵ is given. Let $E \subset \Delta$ be an open set. We shall find an η such that $\sigma(E)/\sigma(\Delta) < \eta$ implies that $\omega(E)/\omega(\Delta) < \epsilon$.

Let $h = \chi_E$, the characteristic function of E. If $M(h)$ is the Hardy-Littlewood maximal function of h with respect to surface measure on the boundary of Ω, define (as in [18]) the BMO function

$$f = \max\{0, 1 + \delta \log M(h)\}, \tag{3.17}$$

where δ is to be determined. The function f has a structure which is typical of BMO functions: see [3] for this characterization. Also, this particular choice of BMO function was exploited in [18] in their proof of weak convergence in H^1. It has the following properties:

- $f \geq 0$
- $\|f\|_{BMO} \leq \delta$
- $f = 1$ on E

Observe that if $x \notin 2\Delta$, then $M(h)(x) < \sigma(E)/\sigma(\Delta) < \eta$. For any δ, if we choose η sufficiently small, the function $1 + \delta \log M(h)$ will be negative, and thus $f = 0$ outside 2Δ.

Using a standard mollification process (as in [25]) we can find a family f_t of continuous functions, $t > 0$ such that:

- $f_t \to f$ in L^p,
- For all t, there exists a C such that $\|f_t\|_{BMO} \le C\|f\|_{BMO}$,
- support of f_t is contained in 3Δ.

Because $f \ge 1$ on E, (3.16) implies that

$$\begin{aligned} \frac{\omega(E)}{\omega(3\Delta)} &\le \omega(3\Delta)^{-1} \int_{3\Delta} f\, d\omega = \omega(3\Delta)^{-1} \lim_{t \to 0+} \int_{3\Delta} f_t\, d\omega \\ &\le C_0 \limsup_{t \to 0+} \|f_t\|_{BMO}. \end{aligned} \tag{3.18}$$

Hence by (3.18)

$$\frac{\omega(E)}{\omega(3\Delta)} \le C_1 \|f\|_{BMO}.$$

Now we choose δ so that $2C_1\delta < \varepsilon$, where C_1 is the constant in the estimate above and this gives that

$$\frac{\omega(E)}{\omega(\Delta)} < M\epsilon \tag{3.19}$$

where M depends on the doubling constant of the measure ω.

Now that absolute continuity is established, the exact same argument gives A_∞. The function f, constructed in (3.17), will have the same properties as before, except that, for general sets E, $f \ge 1$ a.e. $d\sigma$ on E, and hence a.e. $d\omega$ on E by absolute continuity.

Before turning to the proof of the converse, we note the following corollary of this argument.

Suppose that the Dirichlet problem for L with data in L^p is solvable in the sense that an apriori estimate in terms of square functions holds:

$$\|S(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)}.$$

Then the argument above shows that also

$$\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)}.$$

This can be derived from 3.9 as follows. Let f be positive and supported in a surface ball Δ of radius r, and let Δ' be as above. Then

$$(\omega(\Delta)^{-1}\int_{\Delta} f d\omega)^p \leq C\sigma(\Delta)^{-1}\int_{\Delta'} S_r^p(u) d\sigma \leq C\sigma(\Delta)^{-1}\int_{\Delta'} f^p d\sigma \tag{3.20}$$

shows that $d\omega$ is absolutely continuous with respect to $d\sigma$ and the density belongs to B_q, where $1/p + 1/q = 1$.

Proof of the Converse. This part of the proof of Theorem 2.1 uses ideas in in Fabes-Neri [12], where the authors showed that the BMO Dirichlet problem was solvable for the Laplacian in Lipschitz domains.

By assumption, since $d\omega_L \in A_\infty(\partial\Omega, d\sigma)$, there is $p_0 > 1$ such that the Dirichlet problem (D_p) for L is solvable for all $p_0 < p \leq \infty$.

Consider $f \in BMO(\partial\Omega)$. We will establish that

$$\iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX \leq C\sigma(\Delta)\|f\|^2_{BMO}. \tag{3.21}$$

Consider any $\Delta \subset \partial\Omega$ a surface ball of radius r. Let us denote by $\widetilde{\Delta}$ and enlargement of Δ such that $3\Delta \subset \widetilde{\Delta} \subset 5\Delta$. We will write the solution u of the Dirichlet problem for boundary data f as $u_1 + u_2 + u_3$, where u_1, u_2 solve

$$\begin{aligned} Lu_1 &= 0, \quad u_1\big|_{\partial\Omega} = (f - f_{\widetilde{\Delta}})\chi_{\widetilde{\Delta}}, \\ Lu_2 &= 0, \quad u_2\big|_{\partial\Omega} = (f - f_{\widetilde{\Delta}})\chi_{\partial\Omega\setminus\widetilde{\Delta}}, \\ u_3 &= f_{\widetilde{\Delta}} \text{ in } \Omega. \end{aligned}$$

Here $f_{\widetilde{\Delta}}$ denotes, as before the average of f over the set $\widetilde{\Delta}$ and $\chi_{\widetilde{\Delta}}$ is the characteristic function of the set $\widetilde{\Delta}$.

We first estimate u_1. We claim that

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX \leq C\int_{\widetilde{\Delta}} S_r^2(u_1)\, d\sigma. \tag{3.22}$$

Let us denote by Δ_X the set $\{Q \in \partial\Omega; X \in \Gamma(Q)\}$. It follows that $\sigma(\Delta_X \cap \widetilde{\Delta}) \approx \delta(X)^{n-1}$. Hence

$$\begin{aligned} \iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX &\leq C\iint_{T(\Delta)} \delta(X)^{2-n}|\nabla u_1|^2 \sigma(\Delta_X \cap \widetilde{\Delta}) dX \\ &\leq C\int_{Q\in\widetilde{\Delta}}\int_{\Gamma_r(Q)} \delta(X)^{2-n}|\nabla u_1|^2\, dX\, d\sigma \\ &\leq C\int_{\widetilde{\Delta}} S_r^2(u_1)\, d\sigma. \end{aligned} \tag{3.23}$$

By Hölder inequality for sufficiently large p (such that the L^p Dirichlet problem is solvable on Ω)

$$\begin{aligned}\int_{\widetilde{\Delta}} S_r^2(u_1)\,d\sigma &\leq \sigma(\widetilde{\Delta})^{\frac{p-2}{p}}\left(\int_{\widetilde{\Delta}} S^p(u_1)\,d\sigma\right)^{2/p}\\ &\leq C\sigma(\Delta)^{\frac{p-2}{p}}\left(\int_{\widetilde{\Delta}} |u_1|^p\,d\sigma\right)^{2/p}. \end{aligned}\tag{3.24}$$

The last inequality uses solvability of the Dirichlet problem in L^p, which implies that the L^p norm the square function is comparable to the L^p norm of the boundary data. We put (3.22) and (3.24) together to obtain an estimate

$$\iint_{T(\Delta)} |\nabla u_1|^2\delta(X)dX \leq C\sigma(\Delta)^{\frac{p-2}{p}}\left(\int_{\widetilde{\Delta}} |f-f_{\widetilde{\Delta}}|^p\,d\sigma\right)^{2/p} \leq C\sigma(\Delta)\|f\|^2_{BMO(p)}. \tag{3.25}$$

This is the desired estimate for u_1. Now we handle u_2. This function is a solution of the equation $Lu_2 = 0$ with Dirichlet boundary data $f_2 := f - (f - f_{\widetilde{\Delta}})\chi_{\widetilde{\Delta}}$. Let us call by f_2^+ and f_2^- the positive and negative part of the function f_2, that is $f_2 = f_2^+ - f_2^-$ and $f_2^+, f_2^- \geq 0$. We denote by $u_2^\pm$ the solution of the Dirichlet problem

$$Lu_2^\pm = 0 \quad \text{in } \Omega, \quad u_2^\pm\big|_{\partial\Omega} = f_2^\pm.$$

Hence $u_2^\pm \geq 0$ and $u_2 = u_2^+ - u_2^-$. We claim the following

Lemma 3.1. *There exist $C > 0$ depending only on the ellipticity of the operator L such that for any $X \in \Omega$*

$$\left(\delta(X)^{-n}\int_{B(X,\,\delta(X)/2)} |\nabla u_2^\pm(Y)|^2 dY\right)^{1/2} \leq \frac{C}{\delta(X)}\int_{\partial\Omega} f_2^\pm(Q)\,d\omega^X(Q). \tag{3.26}$$

Here ω^X is the elliptic measure for the operator L at the point X.

This statement is a consequence of the Poincaré inequality that allows to estimate the integral of a gradient by an average of $(u_2^\pm - u_2^\pm(X))^2$ over slightly larger ball and by Harnack inequality that implies $u_2^\pm(Y) \approx u_2^\pm(X)$ for $Y \in B(X,\,\delta(X)/2)$. Notice that the integral $\int_{\partial\Omega} f_2^\pm(Q)\,d\omega^X(Q)$ equals to the value of $u_2^\pm$ at the point X.

Let us set

$$v_2(X) = \int_{\partial\Omega} |f_2(Q)|\,d\omega^X(Q) = \int_{\partial\Omega} (f_2^+(Q) + f_2^-(Q))\,d\omega^X(Q). \tag{3.27}$$

It follows that $v^2(X) = u_2^+(X) + u_2^-(X)$.

Lemma 3.2. *There exist $C, \varepsilon > 0$ depending only on the ellipticity constant of the operator L such that for all $x \in T(\Delta)$:*

- $v_2(X) \leq C\|f\|_{BMO}$

- $v_2(X) \le C\|f\|_{BMO}\left(\frac{\delta(X)}{r}\right)^{\varepsilon}$. *Here r is the radius of the surface ball Δ.*

We postpone the proof of this lemma until we show how it gives us the desired estimate.

To to that we consider a standard 'dyadic' decomposition of the Carleson region $T(\Delta)$. What this means is that $T(\Delta)$ can be written as a union of disjoint regions I_n, $n = 1, 2, 3, \dots$ such that for each region I_n the diameter of the region $d = \mathrm{diam}(I_n)$ is comparable to the distance $\mathrm{dist}(I_n, \partial\Omega)$ and the volume of the region is comparable to d^n. For each region I_n we denote by x_n a point inside I_n. It follows that

$$\begin{aligned}\iint_{T(\Delta)} |\nabla u_2^{\pm}|^2 \delta(X) dX &\le \sum_n \iint_{I_n} |\nabla u_2^{\pm}|^2 \delta(X) dX \le C \sum_n \delta(x_n) \frac{|u_2^{\pm}(x_n)|}{\delta(x_n)^2} \delta(x_n)^n \\ \le \; C \int_{T(\Delta)} \frac{|u_2^{\pm}(X)|^2}{\delta(X)} dX &\le C \left(r^{-2\varepsilon} \int_{T(\Delta)} \delta(X)^{2\varepsilon - 1} dX \right) \|f\|^2_{BMO}. \end{aligned} \tag{3.28}$$

Here we used Lemma 3.1 for the last estimate in the first line of (3.28) and Lemma 3.2 for the last estimate in the second line (clearly $u_2^{\pm}(X) \le v_2(X)$).

Since $r^{-2\varepsilon} \int_{T(\Delta)} \delta(X)^{2\varepsilon-1} dX \le C r^{n-1} \approx \sigma(\Delta)$ we see that (3.25) and (3.28) together implies the estimate (3.21) we sought (function u_3 is constant, hence the required estimate hold trivially).

Proof of Lemma 3.2. The first estimate of the lemma, namely that $v_2(X) \le C\|f\|_{BMO}$, essentially follows from Lemma on p.35 in [12]. As stated there

$$v_2(X) = \int_{\partial\Omega \setminus \widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| K(X, Q) d\sigma(Q),$$

for some kernel $K(X, Q)$ (a Radon-Nykodim derivative of the elliptic measure ω^X). Fabes and Neri then use then fact that $K \in B^2(d\sigma)$[2] to establish the estimate. By looking at their proof we see that it is enough to have $K \in B^q$ for some $q > 1$. This holds, as we assume that $\omega^X \in A_\infty(d\sigma) = \bigcup_{q>1} B_q(d\sigma)$.

The further improvement in the estimate $v_2(X) \le C\|f\|_{BMO}\left(\frac{\delta(X)}{r}\right)^{\varepsilon}$ is a consequence of Di Giorgi-Nash-Moser theory. Nonnegative solutions u of L in the region $T(\widetilde{\Delta})$ which vanish on 2Δ satisfy

$$u(X) \le C \left(\frac{|X - Q|}{r} \right)^{\varepsilon} \sup_{T(2\Delta)} u, \qquad \text{for any } X \in T(\Delta).$$

Here ε only depends on the ellipticity constant of the operator L and Q is the center of the ball Δ. (See for example (1.9) in [20] for reference). From this the estimate follows as we can move point Q around (within Δ) as our function vanishes on $\widetilde{\Delta} \supset 3\Delta$.

Now we prove the reverse estimate to (3.21). We want to show that

$$\|f\|^2_{BMO^*(d\sigma)} \le C \sup_{\Delta \subset \partial\Omega} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) \frac{dX}{\sigma(\Delta)}. \tag{3.29}$$

[2]We denote by B^q the class of Gehring weights. The weights in this class satisfy the reverse Hölder inequality with exponent q.

In this case it is more convenient to use (2.3) to define BMO norm. We first prove the following

$$\sup_{\Delta\subset\partial\Omega}\inf_{c_\Delta}\sigma(\Delta)^{-1}\int_\Delta |f-c_\Delta|d\sigma \le C\sup_{\Delta\subset\partial\Omega}\left(\inf_{c_\Delta}\omega(\Delta)^{-1}\int_\Delta |f-c_\Delta|^p d\omega\right)^{1/p}. \tag{3.30}$$

Here $\omega=\omega^{X_0}$ is the elliptic measure for the operator L at some (fixed) interior point X_0. This inequality implies that a BMO function with respect to the surface measure σ is also a BMO function with respect to the elliptic measure ω. Indeed, Let $d\sigma = kd\omega$. The fact $\omega\in A_\infty(d\sigma)$ implies that $\sigma\in A_\infty(d\omega)=\bigcup_{q>1}B_q(d\omega)$. Hence there exists $q>1$ such that k satisfies the reverse Hölder inequality

$$\left(\omega(\Delta)^{-1}\int_\Delta k^q\,d\omega\right)^{1/q}\le C\omega(\Delta)^{-1}\int_\Delta k\,d\omega \quad \text{for all } \Delta\subset\partial\Omega. \tag{3.31}$$

It follows

$$\begin{aligned}
&\sigma(\Delta)^{-1}\int_\Delta |f-c_\Delta|d\sigma = \sigma(\Delta)^{-1}\int_\Delta |f-c_\Delta|k\,d\omega\\
\le\ & \sigma(\Delta)^{-1}\left(\int_\Delta k^q\,d\omega\right)^{1/q}\left(\int_\Delta |f-c_\Delta|^p d\omega\right)^{1/p} \qquad (3.32)\\
\le\ & C\sigma(\Delta)^{-1}\omega(\Delta)^{1/q-1}\left(\int_\Delta k\,d\omega\right)\left(\int_\Delta |f-c_\Delta|^p d\omega\right)^{1/p}.\\
=\ & C\left(\omega(\Delta)^{-1}\int_\Delta |f-c_\Delta|^p d\omega\right)^{1/p}. \qquad (3.33)
\end{aligned}$$

This gives (3.30). It also follows that it suffices to prove (3.29) with $d\omega$ measure on the left-hand side instead of $d\sigma$.

In what follows we use the following lemma from [19].

Lemma 3.3. *Let X_0 be a fixed point inside a Lipschitz domain Ω, ω^{X_0} the elliptic measure for an operator L at X_0 and $G(.,.)$ the Green's function for L. Then for any open surface ball $\Delta_r\subset\partial\Omega$ or radius r such that $\delta(X_0)\ge 2r$ and*

$$G(X_0,Y)r^{n-2}\approx\omega(\Delta_r), \tag{3.34}$$

where $Y\in\Omega$ such that dist$(Y,\Delta_r)\approx\delta(Y)=r$. The precise constants in the estimate (3.34) only depends on the ellipticity of L and Lipschitz character of domain Ω.

The following lemma is crucial for the proof.

Lemma 3.4. *There exists $C>0$ such that for all $f\in BMO(d\omega)$*

$$\|f\|_{BMO^*(d\omega)}\le C\sup_{\Delta\subset\partial\Omega}\left(\iint_{T(\Delta)}|\nabla u|^2 G(X_0,X)\frac{dX}{\omega(\Delta)}\right)^{1/2}. \tag{3.35}$$

Assume for the moment the Lemma is true. By using Lemma 3.3 we get that

$$\iint_{T(\Delta)} |\nabla u|^2 G(X_0, X)dX \leq C \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \omega(\Delta_X)dX, \tag{3.36}$$

where Δ_X is as before the set $\{Q \in \partial\Omega; X \in \Gamma(Q)\}$. By changing the order of integration we get that

$$\iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \omega(\Delta_X)dX \leq \int_{\widetilde{\Delta}} S_r^2 u(Q)\, d\omega(Q). \tag{3.37}$$

Combining (3.35)-(3.37) we get that

$$\|f\|_{BMO^*(d\omega)} \leq \sup_{\Delta \subset \partial\Omega} \left(\int_{\Delta} S_r^2 u(Q)\, \frac{d\omega(Q)}{\omega(\Delta)} \right)^{1/2}. \tag{3.38}$$

Now we use the same trick as above to change measure back from ω to σ. Again using reverse Hölder inequality (now for k^{-1}) we get that

$$\sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^2 u(Q)\, \frac{d\omega(Q)}{\omega(\Delta_r)} \right)^{1/2} \leq C \sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^q u(Q)\, \frac{d\sigma(Q)}{\sigma(\Delta_r)} \right)^{1/q} \quad \text{for some } q > 2.$$

Finally, there exists $C > 0$

$$\begin{aligned} \sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^q u(Q)\, \frac{d\sigma(Q)}{\sigma(\Delta_r)} \right)^{1/q} &\leq C \sup_{\Delta_r \subset \partial\Omega} \left(\int_{\Delta_r} S_r^2 u(Q)\, \frac{d\sigma(Q)}{\sigma(\Delta_r)} \right)^{1/2} \\ &= C \sup_{\Delta \subset \partial\Omega} \left(\iint_{T(\Delta)} |\nabla u|^2 \delta(X) \frac{dX}{\sigma(\Delta)} \right)^{1/2}. \end{aligned} \tag{3.39}$$

The first estimate in (3.39) follows from the BMO John-Nirenberg argument (same way as (2.2) is established). This concludes the proof of Theorem 2.1 (modulo Lemma 3.4).

Proof of Lemma 3.4. We fix a surface ball $\Delta \subset \partial\Omega$ or radius r and center Q. As before we consider a point X_0 inside Ω such that $\delta(X_0) \geq 5r$. Finally, let us denote by $\mathcal{D}$ the domain $\Omega \cap B(Q, 4r)$. We pick a point $X \in \mathcal{D}$ such that $\mathrm{dist}(X, \partial\mathcal{D}) \approx 2r$. We denote by ν the elliptic measure for operator L on the domain $\mathcal{D}$ with pole at X.

We study relations between measures ω and ν. The following Lemma holds

Lemma 3.5. *For any measurable set $E \subset \Delta$*

$$\frac{\omega(E)}{\omega(\Delta)} \leq C\nu(E), \tag{3.40}$$

where the constant $C > 0$ only depends on the ellipticity constant and Lipschitz character of the domain Ω.

It suffices to establish (3.40) for all balls $\Delta' \subset \Delta$, as the general statement for all measurable sets E follows by a covering lemma. For both balls Δ' and Δ we find points Y' and Y, respectively such that $\text{dist}(Y', \partial\Delta') \approx \delta(Y') = r'$ and $\text{dist}(Y, \partial\Delta) \approx \delta(Y) = r$, where r' and r are radii of these balls. According to Lemma 3.3

$$\omega(\Delta') \approx G_\Omega(X_0, Y')(r')^{n-2}, \quad \text{and} \quad \nu(\Delta') \approx G_{\mathcal{D}}(X, Y')(r')^{n-2}.$$

Hence

$$\frac{\omega(\Delta')}{\nu(\Delta')} \approx \frac{G_\Omega(X_0, Y')}{G_{\mathcal{D}}(X, Y')} \approx \frac{G_\Omega(X_0, Y)}{G_{\mathcal{D}}(X, Y)}.$$

The last relation comes form the comparison principle for two positive solutions $v(.) = G_\Omega(X_0, .)$ and $w(.) = G_{\mathcal{D}}(X, .)$ that vanish at the boundary. Finally,

$$\frac{\omega(\Delta')}{\nu(\Delta')} \approx \frac{G_\Omega(X_0, Y)}{G_{\mathcal{D}}(X, Y)} \approx \frac{\omega(\Delta) r^{n-2}}{\nu(\Delta) r^{n-2}},$$

again by using Lemma 3.3. However, $\nu(\Delta) = O(1)$, since the measure ν is doubling, and $\nu(\partial\mathcal{D}) = 1$. Hence Lemma 3.5 follows.

By Lemma 3.5 we see that for any $c_\Delta \in \mathbb{R}$

$$\int_\Delta |f - c_\Delta|^2 \frac{d\omega}{\omega(\Delta)} \leq C \int_\Delta |f - c_\Delta|^2 d\nu \leq C \int_{\partial\mathcal{D}} |u - c_\Delta|^2 d\nu. \tag{3.41}$$

Since ν is the natural (elliptic) measure for the domain $\mathcal{D}$ it follows that the $L^2(d\nu)$ Dirichlet problem is always solvable in this domain. This implies the the $L^2(d\nu)$ norm of the square function is comparable with the $L^2(d\nu)$ of the (normalized) boundary data, i.e.,

$$\inf_{c_\Delta \in \mathbb{R}} \int_{\partial\mathcal{D}} |u - c_\Delta|^2 d\nu \approx \int_{\partial\mathcal{D}} S^2 u \, d\nu \approx \iint_{\Omega \setminus B_{r/8}(X)} |\nabla u(Y)|^2 G_{\mathcal{D}}(X, Y) dY. \tag{3.42}$$

Finally, we claim that

$$G_{\mathcal{D}}(X, Y) \leq G_\Omega(X, Y) \approx \frac{G_\Omega(X_0, Y)}{\omega(\Delta)}, \quad \text{for all } Y \in \Omega \setminus B_{r/8}(X). \tag{3.43}$$

Combining the estimates (3.41)-(3.43) we obtain Lemma 3.4. The first estimate of (3.43) is simply a maximum principle, as $G_{\mathcal{D}}(X, Y)$ vanishes on the whole $\partial\mathcal{D}$, and $G_\Omega(X, Y)$ is positive at the portion of this boundary. Both functions have same pole at X. The relation $G_\Omega(X, Y) \approx \frac{G_\Omega(X_0, Y)}{\omega(\Delta)}$ can be established as follows. For $Y \in \Omega \setminus B_{r/8}(X)$ such that $\delta(Y) \geq r$ Lemma 3.3 implies that $G_\Omega(X_0, Y) \approx r^{n-2}\omega(\Delta)$. On the other hand $G_\Omega(X, Y) \approx r^{n-2}$ as Y is of distance r from the pole and also r away from the boundary. For Y near the boundary we use the comparison principle (since both function vanish at $\partial\Omega$. This gives

$$\frac{G_\Omega(X, Y)}{G_\Omega(X_0, Y)} \approx \frac{G_\Omega(X, Y')}{G_\Omega(X_0, Y')}$$

for all $Y, Y' \in \Omega \setminus B_{r/8}(X)$. This establishes (3.43) and concludes the proof of Theorem 2.1. □

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