DIOPHANTINE APPROXIMATIONS AND DIRECTIONAL DISCREPANCY OF ROTATED LATTICES.

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Abstract. In this paper we study the following question related to Diophantine approximations and geometric measure theory: for a given set Ω find α such that $\alpha-\theta$ has bad Diophantine properties simultaneously for all $\theta \in \Omega$. How do the arising Diophantine inequalities depend on the geometry of the set Ω? We provide several methods which yield different answers in terms of the metric entropy of Ω and consider various examples.

Furthermore, we apply these results to explore the asymptotic behavior of the *directional discrepancy*, i.e. the discrepancy with respect to rectangles rotated in certain sets of directions. It is well known that the extremal cases of this problem (fixed direction vs. all possible rotations) yield completely different bounds. We use rotated lattices to obtain directional discrepancy estimates for general rotation sets and investigate the sharpness of these methods.

1. Introduction

In the present paper we study an interesting problem which lies at the interface of Diophantine approximations and geometric measure theory and apply our results to problems in geometric discrepancy theory.

1.1. **Diophantine approximation.** The central question of this investigation is the following:

Given a set $\Omega \subset [0,1)$, find a point $\alpha \in [0,1)$ so that its distances to all points of Ω have simultaneously bad Diophantine approximation properties, i.e. for each $\theta \in \Omega$

$$\left|(\alpha-\theta)-\frac{p}{q}\right| > \frac{1}{q^2\cdot\psi(q)}, \tag{1.1}$$

where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and ψ is a non-decreasing function. What is the optimal relation between the geometry of the set Ω and the function ψ?

We briefly comment on the history and the background of the problem. Of course, if $\Omega = \{\theta_0\}$ is a singleton, one can choose α so that $\omega = \alpha - \theta_0$ has a countinued fraction with bounded partial quotients

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(see §4.2 for proper definitions) and hence is a *badly approximable* number, i.e. $\left|\omega - \frac{p}{q}\right| \geq \frac{c}{q^2}$, in which case ψ is a constant. This inequality is best possible due to Dirichlet's theorem.

In 1947 Hall [14] proved that any real number can be represented as a sum of two continued fractions with partial quotients bounded by 4. This easily implies that for any two-element set $\Omega = \{\theta_1, \theta_2\}$ there exists α such that $\left|(\alpha - \theta_1) - \frac{p}{q}\right| \geq \frac{c}{q^2}$ and $\left|(\alpha - \theta_2) - \frac{p}{q}\right| \geq \frac{c}{q^2}$, where $c > 0$ is an absolute constant.

This result was extended to all finite sets Ω by Cassels [8] in 1956 (with some generalizations by Davenport [10] in 1964): *there exists a constant $c = c(N) > 0$ such that for any $\Omega = \{\theta_1, ..., \theta_N\}$ there exists $\alpha \in [0, 1)$ such that for all $j = 1, ..., N$ we have*

$$\left|(\alpha - \theta_j) - \frac{p}{q}\right| \geq \frac{c}{q^2} \tag{1.2}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$. The constant in this inequality behaves like $c(N) \approx 1/N^2$.

In their previous work [6] the authors of the present paper had made an attempt to understand this question in the case of infinite sets Ω. Obviously, it is too optimistic to expect the same estimates in this situation, hence $\psi(q)$ in (1.1) has to be an increasing function whose nature depends on the geometry of Ω. Generalizing the methods of Cassels and Davenport, we have considered several particular classes of sets: lacunary sequences, lacunary sets of finite order (see §3.3 for the definition), and sets with small upper Minkowski dimension. In this article, we continue this line of investigation by introducing new methods, extending and sharpening our previous results.

We consider generic sets $\Omega \subset [0, 1)$ and obtain estimates of the type (1.1) in terms of the entropy properties of Ω. More precisely, if $N(\delta)$ denotes the covering number of Ω, i.e. the cardinality of the smallest covering of Ω by open intervals of length δ, we define $F(\delta) = \delta \cdot N(\delta)$ (see Definitions 2.1 and 2.2 for more details). In §2 we prove the following results:

There exists $\alpha \in [0, 1)$ such that for all $\theta \in \Omega$, all $p \in \mathbb{Z}$, and all $q \in \mathbb{N}$:

- Theorem 2.1:

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \geq cF^{-1}\left(F^{-1}\left(\frac{\gamma}{q^2}\right)\right) \tag{1.3}$$

for some absolute constants c, $\gamma > 0$.

- Theorem 2.2:

$$\left|(\alpha-\theta)-\frac{p}{q}\right| \geq F^{-1}\left(\frac{\gamma}{q^2\cdot h(q)}\right), \tag{1.4}$$

where $h(q)$ *satisfies* $\sum \frac{1}{q\cdot h(q)} < \infty$ *and* $\gamma > 0$ *depends on* h.

- Theorem 2.3:

$$\left|(\alpha-\theta)-\frac{p}{q}\right| \geq \delta(q), \tag{1.5}$$

where $\delta(q)$ *satisfies* $\sum_{q=Q}^{\lceil 1/\delta(Q)\rceil} q\cdot F(\delta(q)) \leq c_0$ *for a small constant* $c_0 > 0$ *and all sufficiently large* $Q \in \mathbb{N}$.

The estimates above are obtained using different methods: (1.3) is a generalization of the Cassels–Davenport approach and the authors' prior work, (1.4) comes from a rather trivial counting argument, and (1.5) is based on the dyadic constructions of Peres and Schlag [21].

The range of applicability of these estimates also varies: while (1.3) gives better estimates for "sparse" sets Ω (finite, "superlacunary", see §3.5), (1.4) and (1.5) yield better results for "thicker" sets (lacunary of finite order, positive Minkowski dimension, etc.).

Both main methods exploited in this paper are iterative schemes which construct systems of nested intervals by avoiding the "bad" sets. The difference between the estimates they yield may be heuristically explained by the fact that the method of Cassels and Davenport is "local" – it carefully builds a single sequence of nested intervals (see §2.1 for details), while the Peres–Schlag approach is "global" – it produces a dyadic Cantor-like set (the details are described in §2.3). Hence the former technique is better suited for "thin" sets, while the latter is better adjusted to "denser" ones.

The boundary between these methods seems to happen when Ω is a lacunary sequence, in which case both (1.3) and (1.5) provide the right-hand side of the same order $\frac{c}{q^2\log^2 q}$. These issues are explored and compared in §3.

It is not clear whether the estimates above are optimal. It may be reasonable to conjecture that the correct lower bound is perhaps $F^{-1}\big(\frac{\gamma}{q^2}\big)$. In particular, in the lacunary case this would yield $\frac{c}{q^2\log q}$, i.e. an improvement of our bound by $\log q$. This resonates with the fact that in the problem of Erdős (see the beginning of §2.3) the result of Peres and Schlag is generally believed to be worse than the sharp bound by a logarithm.

1.2. **Directional discrepancy.** The second part of the paper is devoted to investigating the *directional discrepancy* of finite point-sets in two dimensions. For a set of directions $\Omega \subset [0,\pi/2)$, we consider the family of rectangles pointing in directions determined by Ω (or, equivalently, axis-parallel rectangles rotated by angles from Ω):

$$\mathcal{A}_\Omega = \{\text{rectangles } R: \text{ a side of } R \text{ makes angle } \phi \in \Omega \text{ with the } x\text{-axis}\}. \tag{1.6}$$

We will interchangeably refer to Ω as the *set of directions* or *rotation set*. For a finite point set $\mathcal{P}_N \subset [0,1]^2$ of N points, its directional discrepancy with respect to Ω is defined as

$$D_\Omega(\mathcal{P}_N) = \sup_{R \in \mathcal{A}_\Omega,\, R \subset [0,1]^2} |D_\Omega(\mathcal{P}_N, R)| = \sup_{R \in \mathcal{A}_\Omega,\, R \subset [0,1]^2} \Big| \#\mathcal{P}_N \cap R - N \cdot |R| \Big|$$

and shows how well the discrete set $\mathcal{P}_N$ approximates the Lebesgue measure with respect to rectangles $R \in \mathcal{A}_\Omega$. We shall be interested in the relations between the asymptotic behavior of the quantity

$$D_\Omega(N) = \inf_{\mathcal{P}_N \subset [0,1]^2} D_\Omega(\mathcal{P}_N).$$

for large values of N and the geometric properties of the set Ω.

The interest in this question comes from the following classical results in discrepancy theory:

- When the direction is fixed, e.g. the case of axis-parallel rectangles ($\Omega = \{0\}$),
$$D_\Omega(N) \approx \log N. \tag{1.7}$$
The lower bound was proved by Schmidt in 1972 [25], and the upper bound goes back to at least 1904, Lerch [18]. This estimate continues to hold when Ω is finite [9].
- When the rectangles are allowed to rotate in arbitrary directions, i.e. $\Omega = [0, \pi/2)$, the relevant discrepancy estimates become polynomial in N, see Beck [3, 4]:
$$N^{\frac{1}{4}} \lesssim D_\Omega(N) \lesssim N^{\frac{1}{4}} \sqrt{\log N}. \tag{1.8}$$

These well-known results already make it obvious that the behavior of directional discrepancy depends radically on the geometry of the set of rotations Ω. It is therefore interesting to understand the exact nature of the relation between $D_\Omega(N)$ and the geometry of the set Ω. Several intermediate situations, such as lacunary sets (and their generalizations) and sets with small upper Minkowski dimension, have been considered in the prior work of the authors of this paper [6], where some upper discrepancy bounds for these partial cases have been obtained. The present paper generalizes previous results and provides a general method of obtaining upper bounds on directional discrepancy based on the geometric (entropy) properties of the set Ω.

The Diophantine approximation question described in §1.1 enters the picture in the following way. In the case of axis-parallel rectangles (when $\Omega = \{0\}$ is a singleton), one of the most classical ways to construct a low-discrepancy point distribution is to take a standard (appropriately rescaled) integer lattice $\frac{1}{\sqrt{N}}\mathbb{Z} \times \frac{1}{\sqrt{N}}\mathbb{Z}$ and rotate it by an angle α whose tangent has bad Diophantine approximation properties. The resulting distribution $\mathcal{P}_N^\alpha$ In particular, it is well known that, if $\tan\alpha$ is *badly approximable*, i.e. $\left|\tan\alpha - \frac{p}{q}\right| \geq \frac{c}{q^2}$ for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, then $D_\Omega(\mathcal{P}_N) \approx \log N$, which according to (1.7) is asymptotically best possible. Other Diophantine estimates can also be translated into discrepancy bounds, see §4–§5.

Therefore, for a general set Ω, in order to produce a set with low directional discrepancy, one may attempt to find a rotation α which is 'bad' with respect to *all* directions in Ω at the same time, in other words, such that $\tan(\alpha-\theta)$ has bad Diophantine properties for all $\theta \in \Omega$ simultaneously. Then, just as in the axis-parallel case, for each fixed $\theta \in \Omega$, the rotated lattice $\mathcal{P}_N^\alpha$ will have low discrepancy with respect to rectangles pointing in the direction of θ. But since the estimates hold for all $\theta \in \Omega$, the directional discrepancy $D_\Omega(\mathcal{P}_N^\alpha)$ will also be small.

For finite sets Ω the existence of such a rotation is precisely the Cassels–Davenport lemma (1.2) with $(\alpha - \theta_j)$ replaced by $\tan(\alpha - \theta_j)$ (the fact that one can replace $(\alpha - \theta)$ by more general $f(\alpha - \theta)$ was observed by Davenport [10], see §2.4). This fact has been applied in the discrepancy context by Beck and Chen [5], Chen and Travaglini [9].

For infinite rotation sets Ω one should anticipate the right-hand side to be somewhat smaller than $1/q^2$, which in turn would lead to larger discrepancy bounds depending on the geometry of Ω. Results of this type have been obtained in [6] for several particular examples of rotation sets.

The results of our current work (1.3)–(1.5) with $(\alpha - \theta)$ replaced by $\tan(\alpha - \theta)$ provide the rotation angle α with bad Diophantine properties relative to Ω for an arbitrary set Ω in terms of its covering function. These Diophantine inequalities can then be translated into one-dimensional discrepancy estimates using either the Erdős-Turan inequality or the asymptotics of the partial quotients (the second method works slightly better in the most delicate situations, see §4).

Finally, using the approach described in [5, 9, 6], these estimates are applied to directional discrepancy in §5. Using this machinery one can obtain directional discrepancy estimates for *any* rotation set Ω; however the optimization required along the way prevents one from being able to write a generic formula for $D_\Omega(N)$ in terms of the covering function of Ω. In the specific partial cases that we have considered this algorithm gives the following bounds:

- If Ω is a lacunary sequence, we have
$$D_\Omega(N) \lesssim \log^3 N.$$
- If Ω is a lacunary set of order M, we have
$$D_\Omega(N) \lesssim \log^{M+2} N.$$

- If Ω has upper Minkowski dimension $d \in [0,1)$, we have

$$D_\Omega(N) \lesssim N^{\frac{d}{d+1}+\varepsilon}. \tag{1.9}$$

- If Ω is a "superlacunary" sequence, we have

$$D_\Omega(N) \lesssim \log N \cdot (\log\log N)^2.$$

In §6 we approach the question of sharpness of the discrepancy estimates obtained in this fashion in §5 and obtain some results, which are conditional in the following sense: as long the Diophantine bounds of §2 are sharp, we can prove lower bounds for the directional discrepancy, which almost match the upper bounds obtained in §5. In particular, in the case when Ω has upper Minkowski dimension $0 < d < 1$, we show that the directional discrepancy estimate (1.9) essentially cannot be improved, provided that the rotation α produced in (1.5) is best possible.

Throughout the paper we use the notation $A \lesssim B$, which means that there exists an absolute constant C, independent of N, such that $A \le CB$, and write $A \approx B$ if $A \lesssim B \lesssim A$. For a finite set $\mathcal{F}$, we denote its cardinality by $\#\mathcal{F}$.

2. Main results on simultaneous Diophantine approximations

In this section we shall describe three different approaches to the main Diophantine approximation question formulated in the introduction. The first approach (§2.1) is based on the methods of the aforementioned lemma of Cassels and Davenport and is a direct generalization of the results obtained in [6]. The second method (§2.2) is a rather simple measure-theoretic counting argument. The approach presented in §2.3 involves Cantor-type constructions based on the ideas of Peres and Schlag [21] and somewhat refines the result of §2.2. While the methods provide different answers, they are not redundant – in various geometric situations better estimates are given by different approaches. This will be discussed in detail in §3.

Definition 2.1. *Let $\Omega \subset \mathbb{R}$. The* covering function *of Ω is defined as*

$$N(\delta) = \min\left\{ N \in \mathbb{N} : \exists\, J_1,\, J_2, ..., J_N \text{ with } |J_k| = \delta \text{ and } \Omega \subset \bigcup_{k=1}^{N} J_k \right\}, \tag{2.10}$$

where $J_1,\, J_2, ..., J_N$ are open intervals, i.e. $N(\delta)$ is the size of the smallest covering of Ω by open intervals of length δ. The logarithm of this function (and sometimes the function itself) is often referred to as the metric entropy *of Ω.*

In particular, in the specific cases considered in [6], one has the following relations.

- $N(\delta)$ is constant, if Ω is finite;
- $N(\delta) \lesssim \log \frac{1}{\delta}$, if Ω is lacunary;
- $N(\delta) \lesssim \left(\log \frac{1}{\delta} \right)^M$, if Ω is lacunary of order M;
- $N(\delta) \leq C_\varepsilon \left(\frac{1}{\delta}\right)^{d+\varepsilon}$, if Ω has upper Minkowski dimension d.

Definition 2.2. *Let $N(x)$ be the covering function of Ω as defined in (2.10). Define the function*

$$F(x) = x \cdot N(x).$$

This function can be viewed as the total mass of the most economical covering of Ω by intervals of length x.

We briefly discuss some simple technical properties of this function. In most situations of interest $N(x) \ll 1/x$, therefore, $F(x) \to 0$ as $x \to 0$. Since F is not monotone, we define

$$F^{-1}(y) = \sup\{x > 0 : \; F(x) < y\}.$$

Obviously F^{-1} is an increasing function. Since $F(x) \geq x$, we have $F^{-1}(y) \leq y$. Since F is piecewise linear and only has downward jumps, one can easily see that $F\big(F^{-1}(y)\big) = y$, i.e. F^{-1} is the right inverse of F. Take any $0 < y^* \leq y < 1$ and set $x = F^{-1}(y)$ and $x^* = F^{-1}(y^*)$. We obtain that $x^* \leq x$ and

$$\frac{F^{-1}(y)}{y} = \frac{x}{F(x)} = \frac{1}{N(x)} \geq \frac{1}{N(x^*)} = \frac{x^*}{F(x^*)} = \frac{F^{-1}(y^*)}{y^*}.$$

Since F^{-1} is increasing, writing $\dfrac{F^{-1}\big(F^{-1}(y)\big)}{y} = \dfrac{F^{-1}\big(F^{-1}(y)\big)}{F^{-1}(y)} \cdot \dfrac{F^{-1}(y)}{y}$ and applying the above inequality twice, we find that

$$\frac{F^{-1}\big(F^{-1}(y)\big)}{y} \geq \frac{F^{-1}\big(F^{-1}(y^*)\big)}{y^*} \tag{2.11}$$

whenever $0 < y^* \leq y < 1$. We shall make use of this monotonicity in (2.16).

2.1. Generalized Cassels–Davenport lemma.

We start by describing the general idea of the argument which generalizes the ideas of Cassels and Davenport. This approach was initiated in [6]. Assume that for a certain choice of parameters R_n, $|I_n|$, c_n, depending on the set Ω we can obtain the following statement:

Let $\Omega \subset [0,1)$. There exists a sequence of nested closed intervals $I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$ in $[0,1)$ with $|I_n| \to 0$ such that for all $\alpha \in I_n$ and *all $p, q \in \mathbb{Z}$ with $R_n \le q < R_{n+1}$ we have, for all $\theta \in \Omega$:*

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \ge \frac{c_n}{q^2}. \tag{2.12}$$

This would immediately imply that there exist $\alpha \in [0,1)$ and $C > 0$ such that for all $\theta \in \Omega$, all $p \in \mathbb{Z}$, $q \in \mathbb{N}$ we have

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \ge \frac{C}{q^2\,\psi(q)},$$

where the function $\psi(q)$ is determined by the relation between c_n and R_n.

To prove (2.12), one proceeds inductively. At the n^{th} step, the set Ω is covered by at most N_n open intervals $\mathcal{J}_{n,k}$ of length δ_n, $k = 1, ..., N_n$. The dependence between N_n and δ_n is governed by the geometry of the set Ω, namely $N_n = N(\delta_n)$.

Next, one has to choose parameters R_n, $|I_n|$, c_n, δ_n, N_n so that they satisfy two inequalities:

$$\frac{2c_n}{R_n^2} + |I_{n-1}| + \delta_n \le \frac{1}{R_{n+1}^2} \quad \text{and} \tag{2.13}$$

$$|I_{n-1}| - N_n\left(\frac{2c_n}{R_n^2} + \delta_n\right) \ge (N_n + 1)|I_n|. \tag{2.14}$$

Indeed, assuming that I_{n-1} is constructed, fix one of the chosen intervals $\mathcal{J}_{n,k}$ of length δ_n. Suppose inequality (2.12) fails for two sets of numbers α', $\alpha'' \in I_n$, θ', $\theta'' \in \mathcal{J}_{n,k}$, p', $p'' \in \mathbb{Z}$, $R_n \le q', q'' < R_{n+1}$. Then by (2.13)

$$\begin{aligned}
\left|\frac{p'}{q'} - \frac{p''}{q''}\right| &\le \left|\frac{p'}{q'} - (\alpha' - \theta')\right| + \left|\frac{p''}{q''} - (\alpha'' - \theta'')\right| + |(\alpha' - \theta') - (\alpha'' - \theta'')| \\
&< \frac{2c_n}{R_n^2} + |\alpha' - \alpha''| + |\theta' - \theta''| \\
&\le \frac{2c_n}{R_n^2} + |I_{n-1}| + \delta_n \le \frac{1}{R_{n+1}^2},
\end{aligned}$$

which shows that $p'/q' = p''/q''$ (for otherwise they would have to differ by at least $1/R_{n+1}^2$). In other words, for each interval $\mathcal{J}_{n,k}$ there is at most one fraction p_k/q_k with $R_n \le q_k < R_{n+1}$ such that inequality (2.12) is violated.

This implies that the inequality (2.12) is true for all α away from the set

$$S_n = \bigcup_{k=1}^{N_n}\left\{\left(\frac{p_k}{q_k} - \frac{c_n}{R_n^2}, \frac{p_k}{q_k} + \frac{c_n}{R_n^2}\right) + \mathcal{J}_{n,k}\right\}.$$

Obviously, $|S_n| \le N_n \left(\frac{2c_n}{R_n^2} + \delta_n\right)$ and $I_{n-1} \backslash S_n$ consists of at most N_n+1 closed intervals. Thus the validity of (2.14) proves that $I_{n-1} \setminus S_n$ contains at least one interval of length $|I_n|$.

In particular when the set Ω is finite ($\#\Omega = N$), to prove the Cassels–Davenport's lemma (1.2), one can choose the parameters $R_n = R^n$, $c_n = c/N^2$ (for some absolute constants $R, c > 0$), $\delta_n = 0$, $N_n = N$ – this is essentially the argument of Cassels and Davenport. The task of proving similar lemmas for more general sets Ω is hence reduced to the proper choice of these parameters.

We now turn to the first main result of this subsection.

Theorem 2.1. *For any set $\Omega \in [0,1)$ there exists $\alpha \in \mathbb{R}$ such that for all $\theta \in \Omega$, all $p \in \mathbb{Z}$, and all $q \in \mathbb{N}$ we have*

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \ge cF^{-1}\left(F^{-1}\left(\frac{\gamma}{q^2}\right)\right)$$

for some absolute constants c, $\gamma > 0$.

Proof. For technical reasons instead of $F(x)$ we shall use the function $\widetilde{F}(x) = x \cdot \big(4N(x) + 1\big)$. Since $F(x) \le \widetilde{F}(x) \le 5F(x)$ this would only affect the constants. Obviously $\widetilde{F}$ enjoys essentially the same properties as F. In particular, similarly to (2.11) we have the relation

$$\frac{\widetilde{F}^{-1}\big(\widetilde{F}^{-1}(y)\big)}{y} \ge \frac{\widetilde{F}^{-1}\big(\widetilde{F}^{-1}(y^*)\big)}{y^*} \tag{2.15}$$

for each $0 < y^* \le y < 1$.

We follow the approach sketched in the beginning of this subsection and construct a sequence of nested closed intervals $I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$ in $[0,1)$ with $|I_n| \to 0$ such that for all $\alpha \in I_n$, all $\theta \in \Omega$, and all $p, q \in \mathbb{Z}$ with $R_n \le q < R_{n+1}$, we have

$$\left|(\alpha - \theta) - \frac{p}{q}\right| > \frac{c_n}{q^2},$$

where c_n, $|I_n|$, R_n, δ_n will be chosen so as to satisfy the inequalities (2.13) and (2.14). The exact choice of parameters is the following:

(1) The lengths of the intervals I_n are chosen so that

$$|I_n| = \widetilde{F}^{-1}(|I_{n-1}|),$$

which implies $\widetilde{F}(|I_n|) = |I_{n-1}|$. We also set $\delta_n = |I_n|$. Notice that $4|I_n| \le \widetilde{F}(|I_n|) \le |I_{n-1}|$; therefore, $|I_n| \le \frac{1}{4}|I_{n-1}|$ and $|I_n| \to 0$.

(2) We define the sequence R_n by the relation
$$|I_{n-1}| = \frac{1}{2R_{n+1}^2}.$$
(3) The sequence c_n is defined as
$$\begin{aligned} c_n &= |I_n| \cdot R_n^2 = \widetilde{F}^{-1}(\widetilde{F}^{-1}(|I_{n-2}|)) \cdot R_n^2 \\ &= \widetilde{F}^{-1}\big(\widetilde{F}^{-1}(1/2R_n^2)\big) \cdot R_n^2. \end{aligned}$$

In order to prove (2.13) we note that
$$\begin{aligned} \frac{2c_n}{R_n^2} + |I_{n-1}| + \delta_n &= 3|I_n| + |I_{n-1}| \leq \frac{7}{4}|I_{n-1}| \\ &= \frac{7}{8R_{n+1}^2} < \frac{1}{R_{n+1}^2}. \end{aligned}$$
In order to establish (2.14) we observe that
$$\begin{aligned} |I_{n-1}| - N_n\left(\frac{2c_n}{R_n^2} + \delta_n\right) &= |I_{n-1}| - 3N_n|I_n| = \widetilde{F}(|I_n|) - 3N_n|I_n| \\ &= (4N_n+1)|I_n| - 3N_n|I_n| = (N_n+1)|I_n|. \end{aligned}$$
We conclude that there exists an interval I_n such that for all $\alpha \in I_n$, and all $p, q \in \mathbb{Z}$ with $R_n \leq q < R_{n+1}$,
(2.16)
$$\left|(\alpha-\theta) - \frac{p}{q}\right| \geq \frac{c(n)}{q^2} = \frac{\widetilde{F}^{-1}(\widetilde{F}^{-1}(1/2R_n^2)) \cdot R_n^2}{q^2} \geq \widetilde{F}^{-1}\left(\widetilde{F}^{-1}\left(\frac{1}{2q^2}\right)\right),$$
where we have used the monotonicity of $\frac{\widetilde{F}^{-1}(\widetilde{F}^{-1}(x))}{x}$ (2.15).

□

2.2. **Trivial measure-counting argument.** While Theorem 2.1 provides an estimate of the order $F^{-1}\big(F^{-1}(1/q^2)\big)$, there is a quick argument which allows one to obtain only one iteration of F^{-1} at the expense of a small loss. Let $h(q)$ be an increasing function of q such that $\sum \frac{1}{q \cdot h(q)} < \infty$. (Naturally, typical choices are $h(q) = q^{\varepsilon}$, $h(q) = \log^{1+\varepsilon} q$, etc.)

Theorem 2.2. *Fix $h(q)$ as above. For any $\Omega \subset [0,1)$ there exists $\alpha \in [0,1)$ such that for all $\theta \in \Omega$*
$$\left|(\alpha-\theta) - \frac{p}{q}\right| \geq F^{-1}\left(\frac{\gamma}{q^2 \cdot h(q)}\right)$$
for some constant $\gamma > 0$ depending on h.

Remark 2.1. *Since $F^{-1}(x) \leq x$ we see that this result is better than Theorem 2.1 as long as the factor $h(q)$ can be "shoved under the rug".*

Proof. Fix $q \in \mathbb{N}$ and cover Ω by $N(\delta(q))$ intervals $\mathcal{J}_{q,j}$, $j = 1, ..., N(\delta(q))$, of length $\delta(q)$ to be chosen later. Since $\alpha - \theta \in [-1, 1]$, we only need to consider fractions p/q with $-q \le p \le q$. For each such fraction let us remove intervals of the form $I_j(p, q) = \left(\frac{p}{q} - f(q), \frac{p}{q} + f(q)\right) + \mathcal{J}_{q,j}$, which have length $2f(q) + \delta(q)$. Obviously, if α lies outside all of these intervals, then it satisfies $\left|(\alpha - \theta) - \frac{p}{q}\right| > f(q)$ for this fixed value of q.

We set $f(q) = \delta(q)$. Then the length of each $I_j(p, q)$ is $3\delta(q)$. We choose $\delta(q)$ so that $F(\delta(q)) \le \dfrac{\gamma}{q^2 \cdot h(q)}$. Then the total measure of all the removed intervals for all $q \in \mathbb{N}$ is at most

$$\begin{aligned}\sum_{q=1}^{\infty} \sum_{|p| \le q} \sum_{j=1}^{N(\delta(q))} |I_j(p, q)| &= \sum_{q=1}^{\infty} \sum_{|p| \le q} 3\delta(q) \cdot N(\delta(q)) \\ &\le 9 \sum_{q=1}^{\infty} q \cdot F(\delta(q)) \le 9\gamma \sum_{q=1}^{\infty} \frac{1}{q \cdot h(q)} < 1\end{aligned}$$

if γ is small enough. Therefore, there are points in the interval $[0, 1)$ which have not been removed. For such points α we have $\big|(\alpha - \theta) - p/q\big| > f(q) = \delta(q)$ for all $q \in \mathbb{N}$. ☐

This argument is rather crude as it completely disregards possible intersections of the removed intervals. However, in some situation it provides a satisfactory answer, see §3.4. The shortcoming of this approach is circumvented by the method that we describe next.

While the result of Theorem 2.2 is superseded by Theorem 2.3 in the next subsection, we chose to still include it since it provides only a slightly weaker estimate using a much simpler method.

2.3. **The Peres–Schlag method.** The method that we describe here originates from the beautiful paper of Peres and Schlag [21] dealing with a Diophantine approximation question which is somewhat similar in spirit to ours. In particular, they prove that, given a lacunary sequence of positive integers $\{n_j\}$ with $\frac{n_{j+1}}{n_j} > 1 + \varepsilon$, there exists $\theta \in [0, 1)$ such that for all $j \in \mathbb{N}$

$$\|n_j \theta\| > \frac{c\varepsilon}{\log \frac{1}{\varepsilon}}, \text{ i.e. } \left|\theta - \frac{p}{n_j}\right| > \frac{c\varepsilon}{n_j \cdot \log \frac{1}{\varepsilon}}, \tag{2.17}$$

where $\|x\|$ stands for the distance from x to the nearest integer.

The question has interesting history: in 1975 [12] Erdős asked whether for any lacunary sequence of integers $\{n_j\}$ there exists $\theta \in [0, 1)$ such that the fractional parts $n_j \theta \pmod 1$ are not dense in the unit interval. However, it turned out that already in 1926 Khintchine [16] proved that for any lacunary sequence $\{n_j\}$ there exists $\theta \in [0, 1)$ and $\gamma > 0$ such that $\|n_j \theta\| > \gamma$ which of course gives an affirmative answer to the question of Erdős. However, since Khintchine's result had not been rediscovered until very recently, Erdős' problem has been independently solved by de Mathan [19] (1980) and Pollington [22] (1979).

Quantitative bounds similar to (2.17) are interesting due to their relations to chromatic numbers, intersective sets, and lacunary Fourier series (see [15], [21]). While Khintchine was not concerned with the dependence of γ on the lacunarity constant, his proof may be traced to yield $c\varepsilon^2/\log^2 \frac{1}{\varepsilon}$ which was only improved by Katznelson [15] and Peres and Schlag [21] in the early 2000's.

In a nutshell, the method of Peres and Schlag consists of two main ideas. First of all, one may observe that for each fixed $n \in \mathbb{N}$ the set where $\|n\theta\|$ is large is periodic:

$$G_n = \big\{\theta \in \mathbb{R} : \|n\theta\| > \gamma\big\} = \bigcup_{k\in\mathbb{Z}} \left(\frac{k}{n} + \frac{\gamma}{n}, \frac{k+1}{n} - \frac{\gamma}{n}\right),$$

i.e., it consists of equal-length intervals repeated with step $\frac{1}{n}$. Therefore, for a sparse sequence $\{n_j\}$, the intersection of such sets closely resembles a Cantor set construction. The main objective now is to show that the arising Cantor-type set is non-empty.

The second important idea is the use of dyadic rather than arbitrary intervals – this approximation allows one to better handle intersections, since any two dyadic intervals are either disjoint or one is contained in the other.

The original paper of Peres and Schlag uses a variant of the local Lovász lemma to demonstrate that at each step the intersection is not void. The heuristic explanation is simple: if the sets G_{n_j} are independent events with positive probabilities, then $\cap_{j=1}^{N} G_{n_j}$ also has positive probability and is thus non-empty. Various conditional probability arguments (e.g., the local Lovász lemma) allow one to draw the same conclusion in the presence of some weak dependence between the events.

This method has an additional advantage which lies in the fact that one can easily estimate the Hausdorff dimension of the constructed exceptional sets due to their Cantor-type structure, see e.g. [7]. However, we do not pursue this issue in the current paper.

Subsequently, this approach has been successfully applied to a number of problems in Diophantine approximation (in particular, to questions related to the celebrated Littlewood conjecture) by Bugeaud and Moshchevitin [7], Moshchevitin [20], Rochev [23], etc., which required significant refinements and extensions of the original method. Our arguments strongly resonate with [7]. The first author would like to express his deep and sincere gratitude to Nikolay Moshchevitin for pointing out the Peres–Schlag method and for his interesting and fruitful comments.

We prove the following theorem:

Theorem 2.3. *Let $\Omega \subset [0,1)$. Assume that a decreasing function $\delta : \mathbb{N} \to [0,1)$ with the property that $\delta(q) \lesssim q^{-2}$ satisfies the inequality*

$$\sum_{q=Q}^{\lceil 1/\delta(Q)\rceil} q \cdot F(\delta(q)) \leq \frac{1}{64} \tag{2.18}$$

for all $Q \geq q_0$. Then there exists $\alpha \in [0,1)$ such that we have

$$\left|(\alpha - \theta) - \frac{p}{q}\right| \geq \delta(q) \tag{2.19}$$

for all $\theta \in \Omega$ and for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \geq q_0$.

Remark 2.2. *First of all, we notice that this theorem implies the result of Theorem 2.2. Indeed, set $\delta(q) = F^{-1}\left(\frac{\gamma}{q^2 \cdot h(q)}\right)$. Then $q \cdot F(\delta(q)) = \frac{\gamma}{q \cdot h(q)}$ and, since $\sum \frac{1}{q \cdot h(q)} < \infty$, condition (2.18) is automatically satisfied for Q large enough.*

We also want to remark that the precise form of condition (2.18) is a technicality – in the concrete cases that we shall consider it can be easily rescaled, see e.g. §3.3, §3.4, however in this general form the relation does not scale nicely.

Proof. Following the notation of Theorem 2.2, for each fixed $q \in \mathbb{N}$ we cover the set Ω by $N(\delta(q))$ open intervals $\mathcal{J}_{q,j}$, $j = 1, \ldots, N(\delta(q))$ of length $\delta(q)$. Denote by

$$I_j(p,q) = \left(\frac{p}{q} - \delta(q), \frac{p}{q} + \delta(q)\right) + \mathcal{J}_{q,j}$$

(this interval has length $3\delta(q)$) and consider the problematic sets

$$E_{q,j} = \bigcup_{p=-q}^{q} I_j(p,q), \qquad E_q = \bigcup_{j=1}^{N(\delta(q))} E_{q,j}.$$

Thus E_q is the set on which (2.19) may fail for the given value of q. We shall prove that the intersection of E_q^c over $q \in \mathbb{N}$ is non-empty.

Rather than working with the sets E_q directly, we shall look at their dyadic approximations. Open dyadic intervals are intervals of the form $\left(m2^{-l}, (m+1)2^{-l}\right)$, where m and l are integers. We set the scale $l(q) = \lfloor \log_2(1/3\delta(q)) \rfloor$. Each of the intervals $I_j(p,q)$ comprising the set E_q can be then covered by either one or two open dyadic interval of length $2^{-l(q)}$. More precisely, in the latter case we take an open interval of the form $\left(m2^{-l(q)}, (m+2)2^{-l(q)}\right)$.

We denote by $A_{q,j}$ the union of such dyadic intervals which cover $E_{q,j}$ and

$$A_q = \bigcup_{j=1}^{N(\delta(q))} A_{q,j}.$$

Notice that $A_q^c = [0,1] \setminus A_q$, the complement of A_q, is a union of *closed* dyadic intervals of length $2^{-l(q)}$.

Define the sequence of denominators $\{q_k\}$ inductively by setting $q_{k+1} = \lceil 1/\delta(q_k) \rceil$ and set $L_k = l(q_k)$. Consider the sets

$$B_q = \bigcap_{n=q_0}^{q} A_n^c.$$

We shall prove that for each $k \in \mathbb{N}$ the set B_{q_k} is not empty. Since the sets B_q are nested and closed, this will imply the existence of a point $\alpha \in \bigcap_{q=q_0}^{\infty} A_q^c \subset \bigcap_{q=q_0}^{\infty} E_q^c$.

We claim that B_{q_k} (which consists of closed dyadic intervals of length 2^{-L_k}) contains $2^{L_{k+1}-(k+3)}$ dyadic intervals of length $2^{-L_{k+1}}$ which are a part of $B_{q_{k+1}}$. We call the union of these intervals $C_{q_{k+1}}$. This statement means that the measure $\mu\big(B_{q_{k+1}}\big) \geq \mu\big(C_{q_{k+1}}\big) = 2^{-(k+3)}$ and hence $B_{q_{k+1}}$ is not empty.

The proof of the claim is inductive and produces a nested sequence of sets $\{C_{q_k}\}$ with $\mu\big(C_{q_k}\big) = 2^{-(k+2)}$. Assume that C_{q_k} and $C_{q_{k+1}}$ are already constructed. We shall show that we can construct $C_{q_{k+2}} \subset C_{q_{k+1}}$ with the above properties. It is easy to see that

$$C_{q_{k+1}} \cap B_{q_{k+2}} = C_{q_{k+1}} \setminus \bigcup_{q=q_{k+1}+1}^{q_{k+2}} \big(C_{q_{k+1}} \cap A_q\big). \tag{2.20}$$

Going one level back we write $C_{q_k} = \bigcup_\nu J_\nu$ as a union of closed dyadic intervals J_ν of length 2^{-L_k}. We observe that $E_{q,j}$ consists of open intervals of length $3\delta(q)$ whose centers are equally spaced with step $\frac{1}{q}$ and $A_{q,j}$ inherits roughly the same structure. If $q \geq q_{k+1}$, then

$$|J_\nu| = 2^{-L_k} = 2^{-l(q_k)} \geq 3\delta(q_k) \geq \frac{3}{q_{k+1}} \geq \frac{3}{q}, \qquad \text{i.e.} \quad q|J_\nu| \geq 3.$$

We then obtain

$$\begin{aligned} \mu\big(J_\nu \cap A_{q,j}\big) &\leq \left\lceil \frac{|J_\nu|}{(1/q)} \right\rceil \cdot 2 \cdot 2^{-l(q)} \leq \big(1 + q|J_\nu|\big)\, 2^{-l(q)+1} \\ &\leq \frac{4}{3} q|J_\nu| \cdot 4 \cdot 3\delta(q) = 16\, q\, \delta(q) \cdot |J_\nu|. \end{aligned} \tag{2.21}$$

Therefore, for $q_{k+1} \leq q \leq q_{k+2}$

$$\mu\big(C_{q_k} \cap A_{q,j}\big) \leq 16\, q\, \delta(q)\, \cdot \mu\big(C_{q_k}\big)$$

and hence

$$\mu\big(C_{q_k} \cap A_q\big) \leq 16\, q \cdot N(\delta(q)) \cdot \delta(q) \cdot \mu\big(C_{q_k}\big) = 16\, q \cdot F(\delta(q)) \cdot \mu\big(C_{q_k}\big).$$

Since $\mu\big(C_{q_k}\big) = 2\mu\big(C_{q_{k+1}}\big)$ and $C_{q_{k+1}} \subset C_{q_k}$, we have

$$\mu\big(C_{q_{k+1}} \cap A_q\big) \leq \mu\big(C_{q_k} \cap A_q\big) \leq 32\, q \cdot F(\delta(q)) \cdot \mu\big(C_{q_{k+1}}\big).$$

Using (2.20) we finally arrive at

$$\mu\big(C_{q_{k+1}} \cap B_{q_{k+2}}\big) \geq \mu\big(C_{q_{k+1}}\big) - \sum_{q=q_{k+1}+1}^{q_{k+2}} \mu\big(C_{q_{k+1}} \cap A_q\big)$$

$$\geq \Bigg(1 - \sum_{q=q_{k+1}}^{\lceil 1/\delta(q_{k+1})\rceil} 32\, qF(\delta(q))\Bigg)\mu\big(C_{q_{k+1}}\big) \geq \frac{1}{2}\mu\big(C_{q_{k+1}}\big), \tag{2.22}$$

where we have used (2.18). Thus at least a half of $C_{q_{k+1}}$ is contained in $B_{q_{k+2}}$, and we can choose $C_{q_{k+2}} \subset C_{q_{k+1}} \cap B_{q_{k+2}}$ which consists of dyadic intervals of length $2^{-L_{k+2}}$ and has total measure $\mu\big(C_{q_{k+2}}\big) = \frac{1}{2}\mu\big(C_{q_{k+1}}\big) = 2^{-(k+4)}$.

To prove the claim it remains to establish the base case of the induction. Namely we need to show that we can choose $q_0 \in \mathbb{N}$ and construct C_{q_0} and C_{q_1}. Similarly to (2.21) we establish that $\mu\big([0,1] \cap A_{q,j}\big) \leq (q+1) \cdot 2^{-l(q)+1} \leq 12(q+1)\delta(q)$, and hence

$$\mu\big(B_{q_0}\big) = \mu\big(A_{q_0}^c\big) \geq 1 - 12(q_0+1)F\big(\delta(q_0)\big) \geq 1 - \frac{24}{64} \geq \frac{1}{2},$$

where we have used the condition (2.18). Analogously to (2.22) we can estimate

$$\mu\big(B_{q_1}\big) \geq \mu\big(B_{q_0}\big) - \sum_{q=q_0+1}^{q_1} \mu\big([0,1] \cap A_q\big)$$

$$\geq \frac{1}{2} - \sum_{q=q_0}^{\lceil 1/\delta(q_0)\rceil} 24\, q\, F\big(\delta(q)\big) \geq \frac{1}{2} - \frac{24}{64} = \frac{1}{8}.$$

Since $B_{q_1} \subset B_{q_0}$, we can construct C_{q_0} and C_{q_1} such that $C_{q_0} \subset B_{q_0}$, $C_{q_1} \subset C_{q_0} \cap B_{q_1}$, and C_{q_k} consists of $2^{L_k-(k+2)}$ closed dyadic intervals of length 2^{-L_k} for $k = 0, 1$. □

2.4. **Remarks.** We would like to note that the statements of Theorems 2.1, 2.2, and 2.3 remain true if we replace $(\alpha - \theta)$ by $f(\alpha - \theta)$ where f is some function whose derivative is bounded above and below. Indeed, instead of removing the exceptional sets directly we would have to remove their preimages under f – this generalization was made by Davenport [10]. In particular, for applications to directional discrepancy we would need the existence of α such that $\tan(\alpha-\theta)$ has certain Diophantine properties for each $\theta \in \Omega \subset [0, \pi/2)$. In this setup (see e.g. [6]) one has to initially restrict the range of α to $\big(\alpha_0, \frac{\pi}{2} - \alpha_0\big)$ so that $\tan'(\alpha-\theta) = \frac{1}{\cos^2(\alpha-\theta)}$ stays bounded. All the arguments are then repeated verbatim and we shall not make the distinction between these formulations in the text.

3. Examples and comparison of the methods

We now turn to considering various specific classes of sets Ω. As pointed out earlier, in different cases different methods would produce better results.

We shall start by revisiting the case studied by Cassels and Davenport – finite sets §3.1. We shall then look at more interesting infinite examples: lacunary sequences §3.2, a slightly thicker example – lacunary sets of order $M \geq 1$ §3.3, and a substantially thicker case – sets with upper Minkowski dimension $0 \leq d < 1$ §3.4. These three examples have been considered in the previous work of the authors [6]; however we now obtain better results in the latter two cases. We also consider one new example – "superlacunary" sequences §3.5 which are substantially sparser than lacunary sequences. In §3.6 we summarize and compare our methods.

3.1. **Finite sets.** If Ω is finite, then $N(x) = \#\Omega$ is constant for small x, in which case $F(x) = Nx$. Thus Theorem 2.1 gives the original lemma of Cassels and Davenport even with the same constant

$$F^{-1}(F^{-1}(\gamma/q^2)) \approx \frac{1}{N^2q^2},$$

which is not surprising since the method of §2.1 is a direct generalization of their result.

The two other methods do not give the same result (which is best possible). Theorem 2.2 would only give the right-hand side of the order $\dfrac{1}{Nq^2h(q)}$ with e.g. $h(q) = \log^{1+\varepsilon} q$. Theorem 2.3 removes ε in this estimate: since $\displaystyle\sum_{q=a}^{b} \frac{1}{q \cdot \log q} \approx \log\log b - \log\log a$, it is easy to see that condition (2.18) is satisfied by $\delta(q) = \dfrac{c}{q^2 \log q}$.

3.2. **Lacunary sequences.** A sequence $\Omega = \{\omega_n\}_{n=1}^{\infty} \subset [0, 1)$ is called *lacunary* if for some $\omega \in [0, 1]$ and some $\lambda < 1$ we have $0 < \dfrac{\omega_{n+1} - \omega}{\omega_n - \omega} < \lambda$. A typical example is the sequence $\Omega = \{2^{-n}\}_{n\in\mathbb{N}}$. In this case, $N(x) \approx \log \frac{1}{x}$, and thus $F(x) \approx x \cdot \log \frac{1}{x}$ and $F^{-1}(x) \approx \dfrac{x}{\log(1/x)}$. Hence we find that Theorem 2.1 yields

$$F^{-1}(F^{-1}(1/q^2)) \approx \frac{1}{q^2 \log^2 q}. \tag{3.23}$$

At the same time, the trivial method of Theorem 2.2 with e.g. $h(q) = \log^{1+\varepsilon} q$ provides $F^{-1}\left(\dfrac{1}{q^2 \cdot h(q)}\right) \approx \dfrac{1}{q^2 \log^{1+\varepsilon} q} \cdot \dfrac{1}{\log q} = \dfrac{1}{q^2 \log^{2+\varepsilon} q}$, i.e. falling just short of $\log^2 q$ in the denominator.

On the other hand, Theorem 2.3 with $\delta(q) = \dfrac{c}{q^2 \log^2 q}$ yields the same right-hand side as (3.23). Indeed, then $F\big(\delta(q)\big) \le \frac{c'}{q^2 \log q}$, and we have

$$\sum_{q=Q}^{\lceil 1/\delta(Q)\rceil} q \cdot F\big(\delta(q)\big) \le c' \sum_{q=Q}^{Q^3} \frac{1}{q \log q} \approx \log\log Q^3 - \log\log Q \lesssim 1, \tag{3.24}$$

i.e. condition (2.18) is satisfied if Q is large and c is small.

We see that in this case the Cassels–Davenport and the Peres–Schlag methods give exactly the same answer.

3.3. **Lacunary sets of finite order.** We now take a look at some classes of slightly denser sets. Let $E \subset E'$ be closed subsets of $\mathbb{R}$ of measure 0. In our case we restrict our attention to subsets of some bounded interval, $[0,1]$ or $[0,\pi/2]$. We say that E' is a *successor* of E if the following holds: there exists a constant $c > 0$ such that for each x, $y \in E'$ with $x \ne y$, we have

$$|x - y| \ge c d(x, E),$$

where $d(x,E)$ is the distance from x to the closed set E. We now define lacunary sets of finite order.

Definition 3.1. *A lacunary set of order zero is a singleton. A set is a lacunary set of order M if it is a successor of a lacunary set of order $M-1$.*

It is quite easy to see that a lacunary set of order 1 consists of at most two lacunary sequences converging to the original singleton from different sides with lacunarity constant $\lambda \ge 1+c$, e.g., $\{0\} \cup \{\pm 2^{-n}\}_{n\in\mathbb{N}}$. In general, it is not hard to see that the set $\{2^{-n_1} + 2^{-n_2}\}_{n_i \in \mathbb{N}\cup\{\infty\}}$ is a lacunary set of order 2, and one can construct examples of lacunary sets of this type for any finite order, namely

$$\Omega = \{2^{-j_1} + 2^{-j_2} + \dots + 2^{-j_M}\}_{j_1,\dots,j_M \in \mathbb{N}\cup\{\infty\}}.$$

If $\Omega \subset [0,1]$ is a lacunary set of order M, it is not hard to establish the covering function estimate

$$N(\varepsilon) \lesssim \log^M \frac{1}{\varepsilon}, \tag{3.25}$$

where the implicit constant would depend on the 'successor' constants c used in each step of the definition.

Indeed, let $\Gamma \subset [0,1]$ be the lacunary set of order $M-1$, which is a *predecessor* of Ω . Let us cover Γ by N_{M-1} intervals of length ε. Consider an interval (α,β) between two consecutive intervals from the given cover of F. For any x, $y \in \Omega \cap (\alpha,\beta)$, we have $|x-y| \ge cd(x,\Gamma) \ge c\min\{|x-\alpha|, |x-\beta|\}$. Hence, $\Omega \cap (\alpha,\beta)$ is a successor of the two-point set $\{\alpha,\beta\}$ and thus consists of at most of two lacunary sequences: one converging to α, one to β. Therefore $\Omega \cap (\alpha, \beta)$ can be covered by $\mathcal{O}(\log \frac{1}{\varepsilon})$ intervals of length ε, which implies that one can cover Ω by

$$N_M \lesssim (N_{M-1} + 1) \cdot \log \frac{1}{\varepsilon}$$

intervals of length ε, which proves (3.25).

Remark. Such sets are natural objects in analysis. In particular, Bateman [2] proved that the directional maximal function

$$\mathcal{M}_\Omega f(x) = \sup_{R \in \mathcal{A}_\Omega \colon x \in R} \frac{1}{|R|} \int_R |f(x)|\, dx,$$

where $\mathcal{A}_\Omega$ is the set of rectangles pointing in directions of Ω as defined in (1.6), is bounded on $L^p(\mathbb{R}^2)$, $1 < p < \infty$, if and only if Ω is covered by a finite union of lacunary sets of finite order. This condition is also equivalent to the fact that Ω does not "admit Kakeya sets" (see [2, 26]). Unfortunately, despite the apparent similarity of definitions, we do not know any direct connections between the directional maximal function and directional discrepancy.

We would also like to point out that the original definition of lacunary sets of order M (a union of a lacunary set of order $M-1$ and lacunary sequences converging to its points), which was given in [2] and used in [6], is actually strictly weaker than Definition 3.1. We refer the reader to [13] for a discussion of this issue. This inaccuracy, which propagated into several papers, is quite important: in particular, the directional maximal function theorem, as well as the covering number estimate (3.25), both fail to hold under the original definition.

Since for lacunary sets of order M we have $N(x) \lesssim \log^M(1/x)$, it follows that $F(x) \lesssim x \cdot \log^M(1/x)$, and $F^{-1}(x) \gtrsim \dfrac{x}{\log^M(1/x)}$. Theorem 2.3 then provides the estimate with the right-hand side of the form

$$\delta(q) = \frac{c}{q^2 \log^{M+1} q}.$$

The calculation is identical to (3.24) since $q \cdot F\big(\delta(q)\big) \lesssim \dfrac{1}{q \log q}$. (The trivial bound of Theorem 2.2 would be of the order $\dfrac{1}{q^2 \log^{M+1+\varepsilon} q}$.)

The Cassels–Davenport method, however, would give us a worse bound

$$F^{-1}(F^{-1}(1/q^2)) \gtrsim \frac{1}{q^2 \log^{2M} q}.$$

This estimate has been obtained in [6]. We see that in this situation a better answer is provided by the Peres–Schlag method.

3.4. **Positive upper Minkowski dimension.** We now take a look at a class of substantially thicker sets. The upper Minkowski dimension of a set $\Omega \subset \mathbb{R}$ is defined as the infimum of exponents d such that for any $0 < \delta \ll 1$ the set E can be covered by $\mathcal{O}(\delta^{-d})$ intervals of length δ. Assume that Ω has upper Minkowski dimension $0 \leq d < 1$. In this case $N(x) \lesssim \left(\frac{1}{x}\right)^{d+\epsilon}$, and therefore, $F(x) \lesssim x^{1-d-\epsilon}$ and $F^{-1}(x) \gtrsim x^{\frac{1}{1-d}+\epsilon}$ for every $\epsilon > 0$. Theorem 2.1 then yields

$$F^{-1}(F^{-1}(1/q^2)) \approx q^{-\frac{2}{(1-d)^2}-\epsilon}, \tag{3.26}$$

which was proved in [6].

However already the simple argument of Theorem 2.2 provides a significantly larger bound. Taking $h(q) = q^{\epsilon'}$ we obtain

$$F^{-1}(1/q^{2+\epsilon}) \gtrsim q^{-\frac{2}{1-d}-\epsilon}, \tag{3.27}$$

for arbitrary $\epsilon > 0$, which is obviously much better than (3.26). Theorem 2.3 would not provide us with a stronger answer in this situation since the exponents are only determined up to ϵ. However, if more delicate information were available, e.g. $N(x) \lesssim \left(\frac{1}{x}\right)^d$ as in the case of Cantor-type sets, then the Peres–Schlag method would have given a better result, namely, $q^{-\frac{2}{1-d}}$. Anyway, the Cassels–Davenport approach again loses the battle in this situation.

3.5. **"Superlacunary" sequences.** We now look at a substantially less dense sets. Consider sequences that converge at a doubly exponential rate, e.g. $\Omega = \left\{2^{-2^n}\right\}$. In this case we have $F(x) \approx x \cdot N(x) \approx x \cdot \log\log\frac{1}{x}$ and $F^{-1}(x) \approx \dfrac{x}{\log\log(1/x)}$. Theorem 2.1 then yields

$$F^{-1}\left(F^{-1}\left(\frac{c}{q^2}\right)\right) \gtrsim \frac{1}{q^2(\log\log q)^2}. \tag{3.28}$$

Theorem 2.2 with $h(q) = \log^{1+\epsilon} q$ gives

$$F^{-1}\left(\frac{1}{q^2 \cdot h(q)}\right) \gtrsim \frac{1}{q^2 \log^{1+\epsilon} q} \cdot \frac{1}{\log\log q} \gtrsim \frac{1}{q^2 \log^{1+\epsilon'} q},$$

which is much smaller. (Theorem 2.3 would only allow us to get rid of the ϵ). Therefore, in this case a better estimate comes from the Cassels–Davenport approach.

3.6. **Summary.** The discussion of this section suggests that the Cassels–Davenport method yields better results for "thinner" sets (finite, "superlacunary"), while the Peres–Schlag approach is more fruitful for "thicker" sets (lacunary of finite order, positive Minkowski dimension). Lacunary sequences seem to be the natural boundary between the ranges of applicability of the two methods.

The examples considered in this section are summarized in the following theorem.

Theorem 3.1. *(i) Let $\Omega \subset [0,1)$ be a lacunary sequence, then there exists $\alpha \in [0,1)$ such that for all $\theta \in \Omega$*

$$\left|(\alpha-\theta)-\frac{p}{q}\right| \gtrsim \frac{1}{q^2\log^2 q}$$

(ii) Let Ω be a lacunary set of order M, then there exists $\alpha \in [0,1)$ such that for all $\theta \in \Omega$

$$\left|(\alpha-\theta)-\frac{p}{q}\right| \gtrsim \frac{1}{q^2\log^{M+1} q}$$

(iii) Let Ω be a "superlacunary" set, then there exists $\alpha \in [0,1)$ such that for all $\theta \in \Omega$

$$\left|(\alpha-\theta)-\frac{p}{q}\right| \gtrsim \frac{1}{q^2(\log\log q)^2}$$

(iv) Assume Ω has upper Minkowski dimension $0 \le d < 1$, in this case there exists $\alpha \in [0,1)$ such that for all $\theta \in \Omega$

$$\left|(\alpha-\theta)-\frac{p}{q}\right| \gtrsim \frac{1}{q^{\frac{2}{1-d}+\varepsilon}}. \tag{3.29}$$

The theorem continues to hold for $\Omega \subset [0,\pi/2)$ with $(\alpha-\theta)$ replaced by $\tan(\alpha-\theta)$.

3.7. **Further examples.** The methods exploited in this section can provide an answer to the Diophantine question posed in §1.1 for an arbitrary set Ω as long as its metric entropy estimates are available, and the list of examples could be endless. We briefly include one more example, which underpins the delicate difference in the performance of our methods.

Consider a *weakly* lacunary sequence $\Omega = \{\omega_k\} \subset [0,1]$ converging to 0 satisfying $0 \le \dfrac{\omega_{k+1}}{\omega_k} \le 1 - ck^a$, for some for $-1 < a \le 0$, see e.g. [1]. We then have $F(x) \approx x \cdot \log^{1/(1+a)}(1/x)$ and and $F^{-1}(x) \approx x \cdot \log^{-1/(1+a)}(1/x)$. Running this through the methods described in the previous section we obtain the following results.

Cassels–Davenport method yields the right-hand side of the order $\dfrac{1}{q^2 \cdot \log^{\frac{2}{1+a}} q}$. At the same time, the trivial method gives the bound $\dfrac{1}{q^2 \cdot \log^{1+\frac{1}{1+a}+\varepsilon} q}$, while the Peres–Schlag approach slightly refines this estimate: $\dfrac{1}{q^2 \cdot \log^{1+\frac{1}{1+a}} q}$.

Therefore, in this situation, the Peres–Schlag method outperforms the Cassels–Davenport approach, except for breaking even when $a = 0$, i.e., when Ω is a lacunary sequence.

4. One-dimensional discrepancy estimates

There are at least two standard ways to obtain discrepancy estimates for the Kronecker sequence $(\{n\beta\})_{n=1}^{\infty}$ in terms of the Diophantine properties of β: using the Erdős–Turan inequality or by exploring the behavior of the partial quotients of the continued fraction of β. In this section we shall explore and compare these approaches in the present context.

Let us denote by $\|\beta\|$ the distance from β to the nearest integer, and by $\{\beta\}$ its fractional part. We say that a real number β is of type $< \psi$ for some non-decreasing function ψ on $\mathbb{R}_+$ if for all natural q we have $q\|q\beta\| > 1/\psi(q)$. In other words, for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, we have

$$\left|\beta - \frac{p}{q}\right| > \frac{1}{q^2 \cdot \psi(q)}. \tag{4.30}$$

In these terms the results of the previous section state that the numbers $\beta = \tan(\alpha - \theta)$ are of type $< \psi$ with $\psi(q) = \dfrac{1}{q^2 \cdot f(q)}$, where $f(q)$ is the right-hand side in the estimates of Theorems 2.1, 2.2, or 2.3, respectively.

The discrepancy of a one-dimensional infinite sequence $\omega = \{\omega_n\}_{n=1}^{\infty} \subset [0,1]$ is defined as

$$D_N(\omega) = \sup_{x \in [0,1]} \Big|\#\big\{\{\omega_1, ..., \omega_N\} \cap [0,x)\big\} - Nx\Big|.$$

4.1. Erdős–Turan inequality.

A simplified form of the Erdős–Turan inequality (see e.g. [17]) states that, for any sequence $\omega \subset [0,1]$,

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left|\sum_{n=1}^{N} e^{2\pi i h \omega_n}\right|$$

for all natural numbers m. It is well adapted to sequences of the form $\{n\beta\}$, since

$$\left|\sum_{n=1}^{N} e^{2\pi i h n \beta}\right| \le \frac{2}{|e^{2\pi i h\beta} - 1|} = \frac{1}{|\sin(\pi h \beta)|} \le \frac{1}{2\|h\beta\|}.$$

For a number β of type $< \psi$, one can bound the arising sum as follows (see e.g., Exercise 3.12, page 131, [17])

$$\sum_{h=1}^{m} \frac{1}{h\|h\beta\|} \lesssim \log^2 m + \psi(m) + \sum_{h=1}^{m} \frac{\psi(h)}{h}. \tag{4.31}$$

Therefore, one obtains the estimate

$$D_N(\{n\beta\}) \lesssim \frac{N}{m} + \log^2 m + \psi(m) + \sum_{h=1}^{m} \frac{\psi(h)}{h}. \tag{4.32}$$

To deduce the final estimate in terms of N, the standard line of reasoning is to optimize the right-hand side in m.

While for some particular examples this was done by the authors in [6], we consider these examples again since the Diophantine estimates of Theorems 2.3 yield better bounds:

- Lacunary sequence: in view of (3.23) we have $\psi(q) = C\,\log^2 q$. Then
$$\sum_{h=1}^{m} \frac{1}{h\|h\beta\|} \lesssim \log^2 m + \sum_{h=1}^{m} \frac{\log^2 h}{h} \approx \log^3 m.$$
Inequality (4.32) with $m \approx N$ then yields
$$D_N(\{n\beta\}) \lesssim \log^3 N. \tag{4.33}$$
- For lacunary sets of order M, since $\psi(q) = C\,\log^{M+1} q$, we similarly obtain
$$D_N(\{n\beta\}) \lesssim \log^{M+2} N.$$
- For sets with upper Minkowski dimension d, according to (3.27) we have $\psi(q) = C\,q^{\frac{2}{1-d}-2+\varepsilon}$. Therefore, denoting $\tau = \frac{2}{1-d} - 2 + \varepsilon = \frac{2d}{1-d} + \varepsilon$, we obtain from (4.31)
$$\sum_{h=1}^{m} \frac{1}{h\|h\beta\|} \lesssim m^\tau + \sum_{h=1}^{m} h^{\tau-1} \approx m^\tau.$$
Inequality (4.32) with $m \approx N^{\frac{1}{\tau+1}}$ yields
$$D_N(\{n\beta\}) \lesssim N^{\frac{\tau}{\tau+1}} = N^{\frac{2d}{d+1}+\varepsilon'}.$$

4.2. **Discrepancy in terms of the partial quotients.** While the Erdős–Turan inequality is a powerful tool for converting Diophantine inequalities into discrepancy estimates, it is well known that in some cases it is not delicate enough and misses the correct bounds by one power of the logarithm. In particular, for a badly approximable number β, the Erdős–Turan inequality only yields $D_N(\{n\beta\}) \lesssim \log^2 N$, while the sharp bound is $\log N$. This phenomenon will be further discussed in the next subsection.

An alternative way to estimate the discrepancy of the sequence $\{n\beta\}$ involves the partial quotients of the number β. Let β be represented by its continued fraction

$$\beta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0;\, a_1,\, a_2,\, a_3, \ldots],$$

with partial quotients $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ $(k > 1)$. The convergents $\frac{p_n}{q_n} = [a_0; a_1, a_2, ..., a_n]$ satisfy the recurrence relations $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$. The convergent provides the best approximation to β by a rational number with denominator not exceeding q_n, in particular

$$\left|\beta - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

If we assume additionally that β is of type $< \psi$ and apply (4.30) with $p/q = p_n/q_n$, we find that $q_{n+1} < q_n \cdot \psi(q_n)$. Since $q_{n+1} \geq a_{n+1} q_n$ we arrive at

$$a_{n+1} \leq \psi(q_n). \tag{4.34}$$

Fix an integer $N > 0$ and choose m so that $q_m \leq N < q_{m+1}$. The discrepancy of $\{n\beta\}$ may be bounded in terms of the partial quotients of β as follows

$$D_N(\{n\beta\}) \lesssim \sum_{j=1}^{m+1} a_j, \tag{4.35}$$

which is essentially sharp (for details see e.g. Corollary 1.64 in [11]). Inequalities (4.34) and (4.35) together with the recurrence formulas for q_n may be used to bound discrepancy in terms of N.

4.3. **Comparison of the methods.** Let us consider an example not previously treated in [6] – a "superlacunary" sequence of directions as defined in §3.5, e.g. $\Omega = \left\{2^{-2^n}\right\}$. In this case, $F(x) \approx x \cdot \log\log\frac{1}{x}$, hence $F^{-1}(x) \approx \frac{x}{\log\log(1/x)}$. As shown in (3.28), Theorem 2.1 guarantees the existence of α such that

$$\left|(\alpha - \theta) - \frac{p}{q}\right| > F^{-1}\left(F^{-1}\left(\frac{c}{q^2}\right)\right) \approx \frac{1}{q^2(\log\log q)^2}$$

for all $\theta \in \Omega$, i.e. $(\alpha - \theta)$ is of type $< \psi$ uniformly in $\theta \in \Omega$ with $\psi(q) \approx (\log\log q)^2$. We now try to estimate $D_N(\{n\beta\})$ with $\beta = \alpha - \theta$ using both methods described above.

It is easy to see that, for any β, $\{q_k\}$ grows at least exponentially; therefore $m \lesssim \log N$. Since $a_{n+1} \leq \psi(q_n) \approx (\log\log q_n)^2$, we easily obtain

$$D_N(\{n\beta\}) \lesssim \sum_{k=1}^{m+1} a_k \lesssim \sum_{k=1}^{m} (\log\log q_k)^2 \lesssim m\cdot(\log\log q_m)^2 \lesssim \log N\cdot(\log\log N)^2.$$

At the same time, the Erdős-Turan inequality would only yield $D_N(\{n\beta\}) \lesssim \log^2 N$.

On the other hand, if we use partial quotients to bound the discrepancy when $\psi(q) \approx \log^2 q$ (lacunary directions), we similarly find that

$$D_N(\{n\beta\}) \lesssim \sum_{k=1}^{m+1} a_k \lesssim \sum_{k=1}^{m} \log^2 q_k \lesssim m \cdot \log^2 q_m \lesssim \log^3 N,$$

which is the same answer as (4.33) provided by the Erdős–Turan inequality.

Hence we see that the two methods yield different bounds only in the most delicate situations when the true bounds are close to the optimal $\log N$, in particular, less than $\log^2 N$ (inequality (4.32) suggests that Erdős–Turan cannot produce estimates below $\log^2 N$).

We stress that the methods we used are rather general and not restricted to the particular examples considered here – they give answers for arbitrary rotation sets Ω based on their covering function. One-dimensional discrepancy estimates may be obtained using specific forms of the function ψ provided by Theorems 2.1, 2.2, or 2.3, although unfortunately one cannot write down a general estimate for $D_N(\{n\beta\})$ in terms of ψ or in terms of the covering function. The one-dimensional discrepancy estimates can be then converted into bounds for directional discrepancy as explained in the next section.

5. Discrepancy with respect to rotated rectangles

In the present section we demonstrate how one can translate the one-dimensional discrepancy estimates into the estimates for $D_\Omega(N)$. These ideas are classical and go back to Roth [24]. In the present context they were first applied by Beck and Chen [5] and Chen and Travaglini [9]. Since full details have been developed and presented in our prior work [6], the exposition of this section will be rather condensed.

As announced in the introduction, sets with low directional discrepancy will be constructed as rotations of the appropriately scaled integer lattice $(N^{-1/2}\mathbb{Z})^2$ by an angle α provided by Theorems 2.1, 2.2, or 2.3, such that $\tan(\alpha - \theta)$ has bad Diophantine properties for all $\theta \in \Omega$. For technical reasons, it will be more convenient to rescale and rotate the unit square and the rectangles and keep the integer lattice $\mathbb{Z}^2$.

Let V the square $[0, N^{1/2})$ rotated clockwise by α and let $\mathcal{A}_{\Omega,\alpha}$ be the family of all rectangles $R \subset V$ which have a side that makes angle $\theta - \alpha$ with the x-axis for some $\theta \in \Omega$. For each point $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, consider a square of area one centered around it

$$S(\mathbf{n}) = \left[n_1 - \frac{1}{2}, n_1 + \frac{1}{2}\right) \times \left[n_2 - \frac{1}{2}, n_2 + \frac{1}{2}\right).$$

For any set $U \subset V$, we shall denote by $D(U)$ the discrepancy of the integer lattice $\mathbb{Z}^2$ with respect to U. The main point of the transference between the directional discrepancy and the discrepancy of one-dimensional sequences may be formulated in the following claim.

Proposition 5.1. *We have the following relation*

(5.36)
$$D_\Omega(N) \lesssim \sup_{R\in\mathcal{A}_{\Omega,\alpha}} |D(R)| \lesssim \sup_{\theta\in\Omega} \sup_{1\le M\le 2\sqrt{N}} D_M\big(\{n\cdot\tan(\alpha-\theta)\}\big).$$

Rather than giving a complete proof of this claim, we explain the heuristics behind it. An interested reader is referred to Section 4 of [6] for the details. The first estimate above is rather obvious and follows from rescaling.

The heart of the matter is the second estimate. It is obtained in the following way. We have

$$D(R) = \sum_{\mathbf{n}\in\mathbb{Z}^2} D(R\cap S(\mathbf{n})).$$

Obviously, the squares $S(\mathbf{n})$ which lie completely inside or completely outside of R do not contribute to the sum. Therefore, the discrepancy comes from the squares which intersect the boundary, i.e. from lattice points which lie close to some side of the rectangle R.

For $j = 1,\ldots,4$, let $\mathcal{N}^j = \{\mathbf{n} : S(\mathbf{n})$ intersects the j^{th} side of $R\}$ and denote by $I_j = \{n_1 \in \mathbf{Z} : (n_1,n_2)\in\mathcal{N}^j$ for some $n_2\in\mathbb{Z}\}$ the projection of $\mathcal{N}^j$ onto the x-axis. It is then standard to show that

$$\sum_{\mathbf{n}\in\mathcal{N}^j} D(R\cap S(\mathbf{n})) = \pm\sum_{n\in I_j}\psi(c-n\tan\phi), \tag{5.37}$$

where ψ the "sawtooth" function, $\psi(x) = x-[x]-\frac{1}{2} = \{x\}-\frac{1}{2}$, and c is the y-intercept of the line containing the j^{th} side of R.

The sawtooth function arises naturally in relation to the one-dimensional discrepancy. In particular, it is possible to show that for a sequence $\omega = \{\omega_n\}$ and all $c\in\mathbb{R}$ we have

$$\left|\sum_{n=1}\psi(c-\omega_n)\right| \lesssim D_N(\omega). \tag{5.38}$$

Therefore

$$\left|\sum_{\mathbf{n}\in\mathcal{N}^j} D(R\cap S(\mathbf{n}))\right| \lesssim D_{\#I_j}\big(\{n\cdot\tan(\alpha-\theta)\}\big). \tag{5.39}$$

Since the cardinality of I_j satisfies $\#I_j \lesssim N^{\frac{1}{2}}$, inequalities (5.38) and (5.39) prove (5.36).

The argument may be summarized as follows: discrepancy arises from the boundary of R, and the contribution of each side can be bounded by the one-dimensional discrepancy $D_L\big(\{n\beta\}\big)$, where β is the slope of the side and L is its length, which is bounded by $2\sqrt{N}$.

Therefore, to obtain the estimates for the directional discrepancy roughly speaking all one has to do is use the bounds for the one-dimensional discrepancy obtained by the methods of §4 with N replaced by $cN^{1/2}$. In particular, for the concrete examples considered in §3 we obtain the following results.

Theorem 5.2. *We have the following estimates for the directional discrepancy*

(i) *Let Ω be a lacunary sequence, then*

$$D_\Omega(N) \lesssim \log^3 N.$$

(ii) *Let Ω be a lacunary set of order M, then*

$$D_\Omega(N) \lesssim \log^{M+2} N. \tag{5.40}$$

(iii) *Let Ω be a "superlacunary" set, then*

$$D_\Omega(N) \lesssim \log N \cdot (\log\log N)^2.$$

(iv) *Assume Ω has upper Minkowski dimension $0 \le d < 1$. In this case for every $\varepsilon > 0$*

$$D_\Omega(N) \lesssim N^{\frac{d}{d+1}+\varepsilon}, \tag{5.41}$$

where the implicit constant depends on ε.

We note that estimates (5.40) and (5.41) are better than the corresponding bounds obtained in [6].

Remark 5.1. *We close this section with an interesting remark about the critical dimension. Due to inequality (1.8), for any Ω we have the bound $D_\Omega(N) \lesssim N^{1/4}\log^{1/2} N$. The bound arising from (5.41) provides a better answer only if $\frac{d}{d+1} \le \frac{1}{4}$. Thus estimate (5.41) is interesting only if the set of rotations has low Minkowski dimension:*

$$d \le \frac{1}{3}.$$

6. Lower bounds for the directional discrepancy of rotated lattices

In this section we shall complement some of the upper bounds for the directional discrepancy of rotated lattices derived in this paper and in [6] by lower bounds depending on the Diophantine properties of the rotation angle with respect to the direction set Ω. In particular, we consider the case when the direction set has upper Minkowski dimension $0 < d < 1$ and show that, if the Diophantine estimate (3.29) cannot be improved, then the directional discrepancy bound (5.41) is best possible up to ε.

The discussion of the previous sections leads to the following natural definition.

Definition 6.1. *Let $\Omega \subset [0, \pi/2)$ be an arbitrary set of directions. For any $\alpha \in \mathbb{R}$, we say that η is the Ω-type of α if*

$$\eta = \inf\left\{ r \geq 1 : \exists c > 0 \text{ such that } \forall \theta \in \Omega, \forall q \in \mathbb{N},\ q^r \| \tan(\alpha - \theta) q \| > c \right\},$$

where $\|\cdot\|$ stands for the distance to the nearest integer. The condition $r \geq 1$ above follows from Dirichlet's theorem.

We note that if Ω is a singleton $\Omega = \{0\}$, then this definition just yields the standard notion of type from number theory (see e.g. Definition 3.4 in Chapter 2 of [17]) for the number $\tan \alpha$. Notice that relation (3.29) of Theorem 3.1 in these terms says that for a set Ω of upper Minkowski dimension $d < 1$ there exists α such that its Ω-type η satisfies

$$\eta \leq \frac{2}{1-d} - 1 = \frac{1+d}{1-d}.$$

As before, let $\mathcal{P}_\alpha^N$ (or simply $\mathcal{P}_\alpha$) denote the intersection of the rescaled lattice $(N^{-1/2}\mathbb{Z})^2$ rotated by α with the unit square $[0,1]^2$, and $\mathcal{D}_\Omega(\mathcal{P}_\alpha^N)$ is the extremal discrepancy of $\mathcal{P}_\alpha^N$ with respect to rectangles $R \in \mathcal{A}_\Omega$. Let us observe that the proof of relation (5.41) of Theorem 5.2 essentially establishes the following fact:

Lemma 6.1. *Let $\Omega \subset [0, \pi/2)$ and assume that $\alpha \in \mathbb{R}$ has Ω-type at most η. Then*

$$\mathcal{D}_\Omega\big(\mathcal{P}_\alpha^N\big) \leq C_\varepsilon N^{\frac{1}{2}\left(1 - \frac{1}{\eta} + \varepsilon\right)} \tag{6.42}$$

for any $\varepsilon > 0$.

Indeed, the Erdős–Turan approach of §4.1 proves that $D_N\big(\{n \cdot \tan(\alpha - \theta)\}\big) \lesssim N^{1 - \frac{1}{\eta} + \varepsilon}$ for all $\theta \in \Omega$. In turn, the transference principle of Proposition 5.1 proves (6.42).

We shall show that Lemma 6.1 cannot be improved, i.e. in a certain sense our methods are optimal.

Theorem 6.2. *Let $\Omega \subset [0, \pi/2)$. Assume that $\alpha \in \mathbb{R}$ has Ω-type η and for all $\theta \in \Omega$ we have $\big|\tan(\alpha - \theta)\big| \leq C$. Then for any $\varepsilon > 0$ there are infinitely many values of N for which*

$$\mathcal{D}_\Omega\big(\mathcal{P}_\alpha^N\big) > c_\varepsilon N^{\frac{1}{2}(1 - \frac{1}{\eta} - \varepsilon)}. \tag{6.43}$$

Remark 6.1. *In the case of Ω with upper Minkowski dimension $0 \leq d < 1$ we know that $\eta \leq \frac{1+d}{1-d}$ which yields $\mathcal{D}_\Omega\big(\mathcal{P}_\alpha^N\big) \lesssim N^{\frac{d}{d+1} + \varepsilon}$. Theorem 6.2 implies that, if in fact the Ω-type of α satisfies $\eta = \frac{1+d}{1-d}$, in other words if the rotation provided by Theorem 3.1 is optimal, then for infinitely many values of N we have $\mathcal{D}_\Omega\big(\mathcal{P}_\alpha^N\big) \gtrsim N^{\frac{d}{d+1} - \varepsilon}$. Therefore, we obtain conditional sharpness of the upper bound (5.41) in Theorem 5.2:*

if the Ω-type of α provided by Theorem 3.1 cannot be improved, then the exponent in the discrepancy bound (5.41) is best possible.

The condition $\big|\tan(\alpha-\theta)\big| \le C$ in the statement of Theorem 6.2 is technical and means that $\alpha-\theta$ stays away from $\pm\frac{\pi}{2}$. This assumption is satisfied by the angles constructed in Theorems 2.1, 2.2, 2.3 in view of the remark made in §2.4. We now turn to the proof of Theorem 6.2.

Proof. The beginning of the argument closely follows the proof of Theorem 3.3 in Chapter 2 of [17]. Let $\varepsilon > 0$ and choose $\delta > 0$ so that $\frac{1}{\eta-\delta} = \frac{1}{\eta} + \varepsilon$. There exist $q_i \to \infty$ and $\theta_i \in \Omega$ such that

$$\lim_{i\to\infty} q_i^{\eta-\delta/2} \, \|q_i \tan(\alpha-\theta_i)\| = 0.$$

In particular, for some $p_i \in \mathbb{Z}$, $\big|\tan(\alpha-\theta_i) - \frac{p_i}{q_i}\big| \le q_i^{-1-\eta+\delta/2}$. Take $N_i = \big[q_i^{\eta-\delta}\big]$, the integer part of $q_i^{\eta-\delta}$. Write $\beta_i = \tan(\alpha-\theta_i)$. Then $\beta_i = \frac{p_i}{q_i} + c q_i^{-1-\eta+\delta/2}$ with $|c| < 1$. For $1 \le n \le N_i$, we have $n\beta_i = \frac{np_i}{q_i} + c_n$, with $|c_n| = \big|cnq_i^{-1-\eta+\delta/2}\big| < q_i^{-1-\delta/2}$. It then follows that none of the fractional parts $\{n\beta_i\}$, $1 \le n \le N_i$, lie in the interval $I = \big[q_i^{-1-\frac{\delta}{2}}, q_i^{-1} - q_i^{-1-\delta/2}\big]$ since $\{\frac{np_i}{q_i}\}$ is either 0 or is greater than or equal to $\frac{1}{q_i}$.

We now consider the one-dimensional discrepancy

$$D^*_{N_i} = D_{N_i}\big(\{n\beta_i\}\big) = \sup_{0\le\gamma\le1} D_{N_i}(\gamma) = \sup_{0\le\gamma\le1} \big|\#\big\{[0,\gamma)\cap\{n\beta_i\}_{n=1}^{N_i}\big\} - N_i\gamma\big|.$$

Since there is an interval of length $q_i^{-1} - 2q_i^{-1-\delta/2}$ which contains no points from the sequence, it follows that $D^*_{N_i} \ge \frac{1}{2}(q_i^{-1} - 2q_i^{-1-\delta/2})N_i$. For sufficiently large q_i we have $D^*_{N_i} \gtrsim N_i/q_i$, and $N_i = [q_i^{(\eta-\delta)}]$ implies that

$$D^*_{N_i} \gtrsim N_i^{1-\frac{1}{\eta-\delta}} = N_i^{1-1/\eta-\epsilon}.$$

We now translate this estimate into the the lower bound for $\mathcal{D}_\Omega\big(\mathcal{P}_\alpha^N\big)$ in a manner outlined in §5. As before, rather than rotating the lattice, we rotate and rescale the unit square and use the integer lattice $\mathbb{Z}^2$. Let V denote the rectangle $[0, M_i^{1/2}] \times [0, M_i^{1/2}]$ rotated clockwise by α, where $M_i \approx N_i^2$ is to be determined later. The number of integer points in V is $N \approx M_i$.

Consider a rectangle $R \subset V$ pointing in the direction given by $\alpha-\theta_i$. According to equation (5.37) the contribution of its j^{th} side to the discrepancy is given by $\sum_{n\in\mathcal{N}_j} D(R\cap S(n)) = \pm\sum_{n\in I_j} \psi(c - n\tan\phi)$, where $\psi(x) = \{x\} - 1/2$ is the sawtooth function and c is the y-intercept of the line containing the j^{th} side of R. Observe that $\psi(c-n\tan\phi) = \psi(\{c\} - n\tan\phi)$.

For each N_i, since $D^*_{N_i} \geq cN_i^{1-1/\eta-\epsilon}$, one can choose $\gamma \in [0,1]$ with $\left|D^*_{N_i}(\gamma)\right| \geq cN^{1-1/\eta-\epsilon}$. Let us pick a line T_1 satisfying the following properties:

i) The slope of T_1 is $\tan(\theta_i - \alpha)$.
ii) The intersection of T_1 with the y-axis, denoted by c_1, is an integer.
iii) The line T_1 intersects V in a segment of length at least $\frac{1}{3}M_i^{1/2}$.

Choose T_3 parallel to T_1 so that c_3 – the y-intercept of T_3 – has integer part equal to c_1 and fractional part γ to be determined later. Let R be the biggest rectangle in V with two sides along T_1 and T_3. We will now let R_1 and R_3 refer to the sides of R lying on T_1 and T_3 and denote the other two sides of R by R_2 and R_4. Therefore, according to (5.37) $|D(R)|$, the discrepancy of $\mathbb{Z}^2$ with respect to R, up to a bounded error can be written as

$$\Bigg| \sum_{n\in I_1} \psi(c_1 - n\tan\phi) - \sum_{n\in I_3} \psi(c_3 - n\tan\phi) + \\ + \sum_{n\in I_2} \psi(c_2 - n\tan(\phi+\pi/2)) - \sum_{n\in I_4} \psi(c_4 - n\tan(\phi+\pi/2)) \Bigg|,$$

where $I_j, j = 1,2,3,4$, are as in §5 and $\phi = \theta_i - \alpha$. By construction, $|R_2| \leq 1$ and $|R_4| \leq 1$, hence $\#I_2 \leq 1$ and $\#I_4 \leq 1$. Since ψ is bounded, we have

$$\left| \sum_{n\in I_2} \psi(c_2 - n\tan\phi) - \sum_{n\in I_4} \psi(c_4 - n\tan\phi) \right| \leq 1.$$

The focus will be on the expression $\sum_{n\in I_1} \psi(c_1 - n\tan\phi) - \sum_{n\in I_3} \psi(c_3 - n\tan\phi)$.

Let $c^* = \inf\{\cos(\alpha-\theta) : \theta \in \Omega\}$. Since α is chosen so that $\alpha - \theta$ stays away from $\pm\pi/2$, we have $c^* > 0$. Set also $I^* = I_1 \cap I_3$, then $I_1 = I^* \cup (I_1 \setminus I^*)$ and $I_3 = I^* \cup (I_3 \setminus I^*)$. One can see that $|I_1 \setminus I^*| \leq 1$ and $|I_3 \setminus I^*| \leq 1$; hence $|I_1 \cap I_3| \geq \frac{c^*}{4} M_i^{1/2}$. Thus $D(R)$ differs by at most a constant from the quantity

$$\sum_{n\in I^*} \big(\psi(c_1 - n\tan\phi) - \psi(c_3 - n\tan\phi)\big). \tag{6.44}$$

Write $n_0 = \min I_1$ (note that this parameter does not depend on the choice of $c_3 = c_1 + \gamma$). If $M_i^{1/2} > \frac{4}{c*} N_i$, by possibly reducing the length of the rectangle R we may guarantee that $I^* = \{n_0+1, n_0+2, \ldots, n_0+N_i\}$, i.e. $\#I^* = N_i$.

In order to estimate (6.44), we utilize the one-dimensional discrepancy D^*_N. One can easily check that for any sequence ω

$$D_N(\omega, x) = \sum_{n=1}^{N} \big(\psi(\omega_n - x) - \psi(\omega_n)\big).$$

Therefore

$$\sum_{n\in I^*}\left(\psi(c_1 - n\tan\phi) - \psi(c_3 - n\tan\phi)\right)$$

$$= \sum_{n=1}^{\#I^*}(\psi((n+n_0)\tan\phi - \gamma) - \psi((n+n_0)\tan\phi)) = D_{N_i}\big(\omega,\gamma\big),$$

where the sequence $\omega = \{(n+n_0)\tan\phi\}$. It is obvious that this sequence satisfies the same discrepancy estimates as $\{n\tan\phi\}$. Therefore, there exists $\gamma \in [0,1]$ such that we have $\big|D_{N_i}\big(\omega,\gamma\big)\big| \geq cN_i^{(1-\frac{1}{\eta}-\epsilon)}$ and hence $D(R) \geq cN_i^{(1-\frac{1}{\eta}-\epsilon)}$. Since $N = \#(V\cap Z^2) \leq M_i \leq cN_i^2$, we obtain

$$D(R) \geq cN^{\frac{1}{2}(1-\frac{1}{\eta}-\epsilon)},$$

which proves (6.43).

□

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