The L^2 regularity problem for elliptic equations satisfying a Carleson measure condition

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Abstract

We prove that the L^2 regularity problem is solvable for the elliptic equation $\sum_{j,k=1}^n \partial_j(a_{jk}\partial_k u) = 0$ when $\sum_{i,j,k} |\partial_i a_{jk}(x)|^2 x_n dx$ is a Carleson measure with a sufficiently small constant, $\sum_{i,j,k} |\partial_i a_{jk}(x)| \leq C/x_n$ and the bottom row of the coefficient matrix has the particular form $(0, 0, \dots, 0, 1)$. This is done in any dimension n. This was proved in the case $n = 2$ earlier in [9] without the assumption on the bottom row of (a_{jk}).

1 Introduction

Consider the equation

$$L(u) := \operatorname{div}(A\nabla u) = \sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) = 0, \tag{1}$$

in a domain $\Omega \subset \mathbf{R}^n$ which is either a bounded Lipschitz domain or the domain above the graph of a Lipschitz function. The matrix A is assumed to be uniformly elliptic, that is, there exist constants λ and Λ such that

$$\lambda|\xi|^2 \leq A\xi \cdot \xi \leq \Lambda|\xi|^2. \tag{2}$$

There have been many works, going back to the work of Dahlberg [3, 6], dedicated to proving estimates such as

$$\begin{aligned} \|N(u)\|_{L^p(\partial\Omega)} &\leq C\|u\|_{L^p(\partial\Omega)}, && \text{(3a)}\\ \|N(\nabla u)\|_{L^p(\partial\Omega)} &\leq C\|\nu \cdot A\nabla u\|_{L^p(\partial\Omega)}, && \text{(3b)}\\ \text{or} \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} &\leq C\|\nabla_\tau u\|_{L^p(\partial\Omega)}, && \text{(3c)} \end{aligned}$$

where ν is the outward unit normal to Ω and ∇_τ is the projection of the gradient ∇ onto the tangent plane of Ω. The function $N(u)$ is the *non-tangential maximal function* of u and is defined by

$$N(u)(q) = \sup_{\Gamma(q)} |u|,$$

where $\Gamma(q)$ is a non-tangential cone at the point $q \in \partial\Omega$. When the estimate (3a) exists we say the *L^p Dirichlet problem* is solvable. Respectively, when estimates (3b) or (3c) exist we say the *L^p Neumann problem* or the *L^p regularity problem* is solvable. In practice it is often necessary to replace the non-tangential maximal function N in estimates (3b) and (3c) with a variant denoted by $\tilde{N}$ which takes the supremum of averages over balls in the cone $\Gamma(q)$ with radii comparable to the distance to the boundary. This is because the gradient of weak solutions to (1) need not be locally finite so $N(\nabla u)$ may not be well-defined. The results presented here will be proved in the particular case that the data and coefficients are smooth, although the estimates obtained will be independent of this smoothness. As a consequence, it makes sense to consider $N(\nabla u)$. We refer the reader to the beginning of Section 4 for the precise assumptions we make and the method by which we can then pass to the general case.

Such estimates allow one to formulate boundary value problems for solutions to (1) with data specifying either u (Dirichlet data) or $\nu \cdot A\nabla u$ (Neumann data) on $\partial\Omega$ merely in L^p. They also allow one to prove non-tangential convergence of the solution to the data. For arbitrary measurable matrices A satisfying (2), however, it is not always possible to prove these estimates [1]. Nevertheless, under certain conditions on A, they can be proved.

The new result we wish to present here is that the L^2 regularity problem is solvable when

$$\sum_{i,j,k} |\partial_i a_{jk}(x)|^2 x_n dx \tag{4}$$

is a Carleson measure with small constant (see Definition 2.1), $\sum_{i,j,k} |\partial_i a_{jk}(x)| \leq C/x_n$ and the last row of the coefficient matrix has the special form $a_{nk} = 0$ for $k < n$ and $a_{nn} = 1$. We will do this for $\Omega = \mathbf{R}^n_+$, the upper half-plane, although the case of a domain above the graph of a Lipschitz function may be easily reduced to this case. This is formulated as Theorem 3.1. We do not expect the condition on the bottom row of A to be necessary, although we require it in our proof. We note here that we eventually want to replace the condition (4) by an averaging condition on coefficients of A as was done in [8] for the Dirichlet problem. In this paper, a solvability result was first established with a condition similar to (4). Then this condition was replaced by a weaker averaging condition by the use of perturbation theory that is know for the Dirichlet problem. What we present here should therefore be considered as a first step on the road that will bring our knowledge of the Neumann problem to the same level as the Dirichlet problem.

The paper is organised as follows. In Section 2 we fix some notation, state various definitions and recall some well-known results that we will use later. In Section 3 we motivate the hypotheses and state the main result, Theorem 3.1. In the Section 4 we prove Theorem 3.1, using three different tools. The first tool is that of distributional inequalities, which have been used before in this context (for example, [14]). This is combined with the use of an auxilary inhomogeneous equation, which was introduced in [9], although the idea is applied in a different manner here. Finally, we adapt the use of Rellich identities (see for example, [12, 13, 17]) to complete the proof.

2 Preliminaries

For a point $x \in \mathbf{R}^n$ we will write $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$ is the projection onto the first $n-1$ components of x. Given a Lipshtiz function ϕ, we define the domain $\Omega_\phi = \{x \,|\, x_n > \phi(x')\}$, so $\Omega_0 = \mathbf{R}^n_+$ is the upper half-plane. The *non-tangential maximal function* (of a function $u\colon \Omega_\phi \to \mathbf{R}$) adapted to this domain with aperture a is

$$N_{\phi,a}(u)(q) = \sup_{\Gamma_a(q)} |u|$$

where $\Gamma_a(q) = \{x \,|\, |x' - q'| < a(x_n - q_n)\}$ is a cone with vertex $q \in \partial\Omega_\phi$ and aperture a. Of course, it makes sense to require $a < \|\phi'\|^{-1}_{L^\infty(\mathbf{R})}$, so that $\Gamma_a(q) \subset \Omega_\phi$ for each $q \in \partial\Omega_\phi$. In a similar fashion we can adapt the *square function* to such a domain by defining

$$S_{\phi,a}(u)(q) = \left(\iint_{\Gamma_a(q)} |u(x)|^2 \,\frac{dx}{(x_n - \phi(q'))^{n-2}} \right)^{\frac{1}{2}}.$$

This notation differs slightly from common usage, in that we have written u on the right-hand side, rather than ∇u. However, our calculations will be clearer if we use this convention. With this definition the usual square function is $S_{0,a}(\nabla u)$. When u is vector-valued, $N_{\phi,a}(u)$ and $S_{\phi,a}(u)$ are defined in the same way, but with $|u|$ meaning the usual vector norm. We will also need truncated versions of the non-tangential maximal function and square function. These will be denoted by $S^d_{\phi,a}(u)$ and $N^d_{\phi,a}(u)$ respectively, where $\Gamma_a(q)$ is replaced with $\Gamma^d_a(q) = \{x \,|\, |x' - q'| < a(x_n - q_n),\, 0 < x_n - q_n < d\}$.

Definition 2.1 *Given a function* $u\colon \Omega_\phi \to \mathbf{R}$*, the measure* $x \mapsto (x_n - \phi(x'))|u(x)|^2 dx$ *is called a* Carleson measure *on* Ω_ϕ *if*

$$q \mapsto \mathscr{C}_{\phi,a}(u)(q) := \sup_{Q \ni q'} \left(\frac{1}{|Q|} \iint_{t_a^\phi(Q)} (x_n - \phi(x'))|u(x)|^2 \, dx \right)^{\frac{1}{2}}$$

is in $L^\infty(\partial\Omega_\phi)$*, where the supremum is taken over all cubes* Q *in* $\mathbf{R}^{n-1}$ *containing* q' *and*

$$t_a^\phi(Q) = \{x \,|\, x' \in Q \text{ and } x_n \in (\phi(x'), \phi(x') + diam(Q)/a)\}$$

is a tent over Q*. The number* $\|\mathscr{C}(u)\|_{L^\infty(\partial\Omega_\phi)}$ *is the Carleson measure constant.*

We will also need a localised dyadic version of $\mathscr{C}_{\phi,a}(u)$. For a cube R, we will write $\mathscr{C}^R_{\phi,a}(u)(q)$ to mean

$$\mathscr{C}^R_{\phi,a}(u)(q) := \sup_{q' \in Q \subseteq R} \left(\frac{1}{|16Q|} \iint_{t_a^\phi(16Q)} (x_n - \phi(x'))|u(x)|^2 \, dx \right)^{\frac{1}{2}},$$

where the supremum is taken over dyadic cubes Q which contain q' and are contained in R. The cube $16Q$ is the concentric enlargement of Q by a factor of 16 (and similarly for factors different from 16). Obviously, if $R' \subseteq R$, then $\mathscr{C}^{R'}_{\phi,a}(u)(q) \le \mathscr{C}^R_{\phi,a}(u)(q)$.

It will often be convenient to write $S_{\phi,a}(u)(q') := S_{\phi,a}(u)(q)$, $N_{\phi,a}(u)(q') := N_{\phi,a}(u)(q)$, $\mathscr{C}^R_{\phi,a}(u)(q') := \mathscr{C}^R_{\phi,a}(u)(q)$, etc., but this need not cause confusion.

We now recall a fact proved by Coifman, Meyer and Stein [2, (4.1)] that we will use later. We have

$$\iint_{\Omega_\phi} |u(x)v(x)|(x_n - \phi(x'))dx \le C(a,b) \int_{\mathbf{R}^{n-1}} \mathscr{C}_{\phi,b}(u)(x')S_{0,a}(v)(x')dx'. \quad (5)$$

Definition 2.2 *Given a non-empty open proper subset* $D \subset \mathbf{R}^n$ *a* Whitney decomposition *of the set* D *is a family of closed dyadic cubes* $\{Q_j\}_j$ *such that:*

(a) $\cup_j Q_j = D$ *and the* Q_j *have disjoint interiors;*

(b) $diam(Q_j) \le dist(Q_j, D^c) \le 4 diam(Q_j)$*; and*

(c) If the boundaries of Q_j *and* Q_k *touch then*

$$\frac{1}{4} \le \frac{diam(Q_j)}{diam(Q_k)} \le 4.$$

The existence of Whitney decompositions is proved in, for example, [11, pA-34]. We remark that an examination of the proof therein reveals if $\{Q_j^0\}_j$ is the Whitney decomposition of the set $D^0 \supseteq D$, then when $Q_j \cap Q_k^0 \neq \emptyset$ we have $Q_j \subseteq Q_k^0$. Throughout the paper we will denote the dilation of a cube Q by a factor 5 by Q^* and $Q^{**} = (Q^*)^* = 25Q$.

3 Motivation and the Main Result

The motivation for the conditions placed upon the matrix A come from the following example due to Kenig and Pipher [15]. Consider the Dirichlet problem for the Laplacian $\Delta = \sum_{i=1}^n \partial_i^2$:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega_\phi; \\ u = f_0, & \text{on } \partial\Omega_\phi. \end{cases}$$

It is well-known [4] that the L^2 Dirichlet problem is solvable when ϕ is Lipschitz, that is, we have the estimate

$$\|N_{\phi,a}(u)\|_{L^2(\partial\Omega)} \leq C\|f_0\|_{L^2(\partial\Omega_\phi)}. \tag{6}$$

We now consider the transformation $\Phi\colon \mathbf{R}^n_+ \to \Omega_\phi$, first introduced by Dahlberg, Kenig and Stein (see [5] and [7]), defined as

$$\Phi(x) = (x', c_0 x_n + (\theta_{x_n} * \phi)(x')), \tag{7}$$

where $\{\theta_t\}_{t>0}$ is a smooth compactly supported approximate identity and c_0 can be chosen large enough, depending only on $\|\nabla\phi\|_{L^\infty(\mathbf{R}^{n-1})}$, so that Φ is one-to-one. One may compute that the function Φ enjoys the properties

- $|\partial\Phi(x)| \leq C$,
- $|\partial^2\Phi(x)| \leq C/x_n$, and
- $x \mapsto x_n|\partial^2\Phi(x)|^2 dx$ is a Carleson measure.

Moreover, it is straightforward to see that the composition $w = u \circ \Phi$ is such that $\operatorname{div}(A\nabla w) = 0$ in $\mathbf{R}^n_+$, where $A = (\det \Phi')((\Phi')^{-1})^t(\Phi')^{-1}$. Therefore A inherits from Φ the properties

(i) $|\partial A(x)| \leq \mathcal{C}_0/x_n$ and

(ii) $x_n|\partial A(x)|^2 dx$ is a Carleson measure with constant $\mathcal{C}_0$, for some constant $\mathcal{C}_0$, and from (6) we can see the corresponding non-tangential estimate

$$\|N_{0,a'}(w)\|_{L^2(\mathbf{R}^{n-1})} \leq C\|f_0(\cdot,\phi(\cdot))\|_{L^2(\mathbf{R}^{n-1})} \tag{8}$$

holds for some $a' < a$. So the natural question is: 'Are the conditions (i) and (ii) sufficient to conclude estimate (8) for an arbitrary solution to a divergence form equation?' Kenig and Pipher [15] prove an L^p version of such an estimate holds under closely related conditions, where $p > 1$ may possibly be large and Dindoš, Petermichl and Pipher [8] show that the L^p version holds for any given $1 < p < \infty$ under the same assumptions, but when $\mathcal{C}_0$ is sufficiently small (depending on p). In fact, both of these results are proved for bounded Lipschitz domains for an elliptic equation which has lower-order drift terms satisfying a similar Carleson measure condition.

Given [13], the same motivation can be used to justify posing the same question regarding the regularity and Neumann problems. This was studied for domains in the plane in [9] and is also the subject of this paper. We now state our main result.

Theorem 3.1 *Suppose u solves $L(u) = 0$ in $\mathbf{R}^n_+$, with A satisfying (i) and (ii) above, and $a_{nk} = 0$ for $k < n$ and $a_{nn} = 1$. If $\mathcal{C}_0$ is sufficiently small (determined in terms of n, λ, Λ and a), then there exists a constant C, which depends only on n, $\mathcal{C}_0$, λ, Λ and a, such that*

$$\|N_{0,a}(\nabla u)\|_{L^2(\partial\mathbf{R}^n_+)} \leq C\|\nabla_\tau u\|_{L^p(\partial\mathbf{R}^n_+)}. \tag{9}$$

We prove this in the following section.

4 The Proof of Theorem 3.1

First, we need to define precisely how we obtain solutions to (1). We produce a unique weak solution u to (1), whose gradient is in $L^2(\mathbf{R}^n_+)$, via the method presented in [16, Lem 1.1] (or equally, [18, Lem 1.1]). The proof there is carried out when $n = 2$, but the same method yields an analogous result for any dimension n. All our results will be proved under the a priori assumptions that the coefficients are smooth, A is the identity matrix outside a ball centred at the origin, and the Dirichlet data $u|_{\partial\mathbf{R}^n_+}$ is smooth and compactly supported. Under the a priori smoothness assumptions, the solution u is smooth and $u(x) = O(x^{\delta-n+1})$ as $x \to \infty$ for each $\delta > 0$ (see [18, Thm B.1]). From this we can further conclude $\partial^\alpha u(x) = O(x^{\delta-|\alpha|-n+1})$ as $x \to \infty$. Consequently all the integrals we consider converge. Once our results are proved, we may then drop the smoothness assumptions, passing to the general case using the method used to prove Theorem B.2 in [18]. This is where we are forced to abandon an estimate on $N(\nabla u)$ and introduce the averaged version $\tilde{N}(\nabla u)$ mentioned in Section 1. As this is a repeat of well-known arguments, we leave the details to the interested reader.

We now introducing the auxilary inhomogeneous equation. Simply differentiating equation (1) in the ith direction we obtain

$$\operatorname{div}(A\nabla v_i) = -\operatorname{div}((\partial_i A)\mathbf{v}) \quad \text{for } i = 1, 2, \ldots, n, \tag{10}$$

where $v_i = \partial_i u$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)^t = \nabla u$. The idea now is to view (10) as an equation in v_i and use distributional inequality techniques to prove

$$\|N_{0,a}(\mathbf{v})\|_{L^2(\partial \mathbf{R}^n_+)} \leq C\|\mathbf{v}\|_{L^2(\partial \mathbf{R}^n_+)} \tag{11}$$

We then have a separate argument, inspired by the Rellich identity techniques of [12, 13], but implimented in a different fashion, to prove

$$\|\nabla u\|_{L^2(\partial \mathbf{R}^n_+)} \leq C\|\nabla_\tau u\|_{L^2(\partial \mathbf{R}^n_+)}. \tag{12}$$

Of course, these two estimates combine to prove (9). The more technically challenging estimate will be (11) and the proof of this will not be completed until after Theorem 4.8, once we have proved a string of estimates. The proof of (12) is carried out after this, at the end of the paper.

We now begin the task of proving (11) with an important lemma. This is the essential tool to control the non-tangential maximal function by the square function.

Lemma 4.1 *Let u solve* (1) *and set $\mathbf{v} = \nabla u$ so $\mathbf{v}$ solves system* (10)*. Let ϕ and ϕ_+ be two non-negative Lipschitz functions and let Q be a cube in $\mathbf{R}^{n-1}$ with $r := diam(Q)$. Suppose $\phi_+ \geq \phi$ and $\phi_+(x') - \phi(x') \leq 12 diam(Q)/a$ for $x' \in Q^*$. Then for sufficiently small a, which can be determined in terms $\|\nabla\phi_+\|_{L^\infty(\mathbf{R}^{n-1})}$, there exists a constant C, depending only on λ, Λ, $\mathcal{C}_0$, $\|\nabla\phi_+\|_{L^\infty(\mathbf{R}^{n-1})}$ and a, such that*

$$\|\mathbf{v}(\cdot, \phi_+(\cdot))\|^2_{L^2(Q)} \leq C(r^{n-1}\min_{Q^*}(\mathscr{C}^{Q^*}_{\phi,a}(\nabla\mathbf{v}))^2 + \mathcal{C}_0\|N_{\phi,a}(\mathbf{v})\|^2_{L^2(Q^*)} + \|N_{\phi,a}(\mathbf{v})\|_{L^2(Q^*)}\|S^{3r/a}_{\phi,a}(\nabla\mathbf{v})\|_{L^2(Q^*)} + r^{n-1}|\mathbf{v}(x_r)|^2),$$

where x_r is any point in $\{x \,|\, x' \in Q^*, \phi_+(x') + r/2 \leq x_n \leq \phi_+(x') + 6r/a\}$.

Proof. In order to facilitate the use of integration by parts, we will use the mapping $\Phi\colon \mathbf{R}^2_+ \to \Omega_{\phi_+}$ defined by (7) with ϕ_+ replacing ϕ. This pulls back $\mathbf{v}$ in Ω_{ϕ_+} to a solution $\mathbf{w} = \mathbf{v} \circ \Phi$ in $\mathbf{R}^n_+$ of the equation

$$\operatorname{div}(\tilde{A}\nabla w_i) = \operatorname{div}(F_i\mathbf{w}) \quad \text{for } i = 1, 2, \ldots, n, \tag{13}$$

where

$$\tilde{A} = (\det \Phi')(\Phi'^{-1})^t (A \circ \Phi)(\Phi'^{-1})$$

and

$$F_i = (\det \Phi')(\Phi'^{-1})^t (\partial_i A \circ \Phi) \tag{14}$$

The coefficient matrix $\tilde{A}$ satifies the ellipticity condition (2) with the constants λ and Λ being replaced by multiples of λ and Λ, respectively, which can be determined in terms of $\|\nabla\phi_+\|_{L^\infty(\mathbf{R}^{n-1})}$. For each i, the matrix F_i satisfies the properties (i) and (ii) with ∂A replaced by F and $\mathcal{C}_0$ replaced by a multiple of $\mathcal{C}_0$, which again can be determined in terms of $\|\nabla\phi_+\|_{L^\infty(\mathbf{R}^{n-1})}$.

We choose a smooth function $\xi_1 \colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that $\xi_1(x') = 1$ for $x' \in Q$, $|\xi_1'| \le 16/r$, with support contained in the concentric dilation $(9/8)Q$. Choose another function $\xi_2 \colon [0,\infty) \to \mathbf{R}$ such that $\xi_2(x_n) = 1$ for $x_n \in [0,r]$, $|\xi_1'| \le 5/r$ and support contained in $[0,2r]$. Now define $\xi(x',x_n) = \xi_1(x')\xi_2(x_n)$.

We now calculate as in [14], [15], [17] and [9]. First of all, for each $i = 1,2,\dots,n$,

$$\begin{aligned} \int w_i(x',0)^2 \xi_1(x')\,dx' &= -\iint_{\mathbf{R}^n_+} \partial_n(w_i^2\xi)(x)\,dx \\ &= -\iint_{\mathbf{R}^n_+} 2w_i(\partial_n w_i)\xi - \iint_{\mathbf{R}^n_+} w_i^2\xi_1\xi_2' \end{aligned} \tag{15}$$

The second term on the right-hand side of (15) is controlled by $r^{-1}\iint_K w_i^2$ where $K = \{x = (x',x_n) \mid x' \in Q^*, r/3 \le x_n \le 7r/a\}$. Let x_r be any point in K and choose K' and K'' to be appropriate concentric enlargements of K. We set $c = \frac{1}{|K'|}\iint_{K'} w_i$. Using [10, Thm 8.17] and Poincaré's inequality, we may further estimate this term by

$$\begin{aligned} & r^{-1}\iint_K (w_i - w_i(x_r))^2\,dx + r^{-1}\iint_K w_i^2(x_r)\,dx \\ & \le Cr^{n-1}\operatorname{osc}_K(w_i)^2 + Cr^{n-1}|w_i(x_r)|^2 \\ & \le Cr^{n-1}\sup_K |w_i - c|^2 + Cr^{n-1}|w_i(x_r)|^2 \\ & \le Cr^{-1}\iint_{K'} |w_i - c|^2\,dx + Cr^{n-1+2(1-n/q)}\|F\mathbf{w}\|^2_{L^q(K')} + Cr^{n-1}|w_i(x_r)|^2 \\ & \le Cr\iint_{K''} |\nabla(w_i)|^2\,dx + C\mathcal{C}_0\|N_{\phi,a}(\mathbf{w})\|^2_{L^2(Q^*)} + Cr^{n-1}|w_i(x_r)|^2 \\ & \le C|Q^*|\min_{Q^*}\mathscr{C}^{Q^*}_{\phi,a}(\nabla w_i)^2 + C\mathcal{C}_0\|N_{\phi,a}(\mathbf{w})\|^2_{L^2(Q^*)} + Cr^{n-1}|w_i(x_r)|^2. \end{aligned}$$

The first term on the right-hand side of (15) is

$$
\begin{aligned}
&-\iint_{\mathbf{R}^n_+} 2w_i(\partial_n w_i)\xi\, dx \\
&= -\iint_{\mathbf{R}^n_+} 2w_i(\partial_n w_i)\xi[\partial_n(x_n)]\, dx = 2\iint_{\mathbf{R}^n_+} [\partial_n(w_i(\partial_n w_i)\xi)]x_n\, dx \\
&= 2\iint_{\mathbf{R}^n_+} (\partial_n w_i)^2\xi x_n\, dx + 2\iint_{\mathbf{R}^n_+} w(\partial_n^2 w_i)\xi x_n\, dx \\
&\quad + 2\iint_{\mathbf{R}^n_+} w_i(\partial_n w_i)\xi_1\xi_2' x_n\, dx \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Observe that we are free to assume $\tilde{A}$ is upper triangular, provided we introduce lower order terms. Indeed, writing $\tilde{A} = (\tilde{a}_{jk})_{jk}$,

$$
\begin{aligned}
\operatorname{div}(\tilde{A}\nabla w_i) &= \sum_{j,k} \partial_j(\tilde{a}_{jk}\partial_k w_i) \\
&= \sum_{j=k} \partial_j(\tilde{a}_{jk}\partial_k w_i) + \sum_{j<k}(\partial_j(\tilde{a}_{jk}\partial_k w_i) + \partial_k(\tilde{a}_{kj}\partial_j w_i)) \\
&= \sum_{j=k} \partial_j(\tilde{a}_{jk}\partial_k w_i) + \sum_{j<k} \partial_j((\tilde{a}_{jk} + \tilde{a}_{kj})\partial_k w_i) \\
&\quad + \sum_{j<k}((\partial_k\tilde{a}_{kj})\partial_j w_i - (\partial_j\tilde{a}_{kj})\partial_k w_i) \\
&= \operatorname{div}(\bar{A}\nabla w_i) + B\cdot\nabla w_i,
\end{aligned}
$$

Using the fact that w_i solves (13) and writing $\bar{A} = \{\bar{a}_{jk}\}_{jk}$, we see that II is equal to

$$
\begin{aligned}
&-2\iint_{\mathbf{R}^n_+} w_i \operatorname{div}(F_i\mathbf{w})\xi x_n\, dx - 2\iint_{\mathbf{R}^n_+} w_i \sum_{j\le k,\, j<n} \partial_j(\bar{a}_{jk}\partial_k w_i)\xi x_n\, dx \\
&-2\iint_{\mathbf{R}^n_+} w_i\, B\cdot\nabla w_i \xi x_n\, dx =: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
\mathrm{II}_2 = 2\iint_{\mathbf{R}^n_+} &\sum_{j\le k,\, j<n} (\partial_j w_i)\bar{a}_{jk}(\partial_k w_i)\xi x_n\, dx \\
+ 2\iint_{\mathbf{R}^n_+} &\sum_{j\le k,\, j<n} w_i\bar{a}_{jk}(\partial_k w_i)(\partial_j\xi)x_n\, dx.
\end{aligned}
\tag{16}
$$

The first integral may be combined with I to produce

$$\iint_{\mathbf{R}^n_+} \nabla w_i \cdot \bar{A} \nabla w_i \xi 2x_n \, dx \leq C|Q^*| \min_{Q^*} \mathscr{C}^{Q^*}_{\phi,a}(\nabla w_i)^2,$$

using the ellipicity condition (2), which $\bar{A}$ also satisfies, and the assumption $\phi_+ - \phi \leq 11\mathrm{diam}(Q)/a$. We now use (5) to control the second integral in (16) by

$$\begin{aligned} C \iint_{\mathbf{R}^n_+} |\mathbf{w}(\nabla \mathbf{w})(\nabla \xi)| x_n \, dx &\leq \iint_{\Omega_\phi} |\mathbf{v}(\nabla \mathbf{v})(\nabla \xi) \circ \Phi^{-1}| y_n \, dy \\ &\leq C\mathscr{C}_0(|\nabla \xi|) \int_{Q^*} S^{3r/a}_{\phi,a}(\mathbf{v}(\nabla \mathbf{v}))(y') dy' \\ &\leq C\|N_{\phi,a}(\mathbf{v})\|_{L^2(Q^*)} \|S^{3r/a}_{\phi,a}(\chi_{Q,a} \nabla \mathbf{v})\|_{L^2(Q^*)}, \end{aligned}$$

where we have extended ξ to $\mathbf{R}^n$ by zero and used that $\mathscr{C}_0(|\nabla \xi|) \leq C$, which may be checked directly, and the simple observation

$$S^{3r/a}_{\phi,a}(\mathbf{v}(\nabla \mathbf{v}))(x') \leq N_{\phi,a}(\mathbf{v})(x') S^{3r/a}_{\phi,a}(\nabla \mathbf{v})(x').$$

Thus we obtain

$$|\mathrm{II}_2| \leq C(|Q^*| \min_{Q^*} \mathscr{C}^{Q^*}_{\phi,a}(\nabla w_i)^2 + \|N_{\phi,a}(\mathbf{v})\|_{L^2(Q^*)} \|S^{3r/a}_{\phi,a}(\nabla \mathbf{v})\|_{L^2(Q^*)}).$$

Again, integrating by parts we see that

$$\begin{aligned} \mathrm{II}_1 = 2 \iint_{\mathbf{R}^n_+} \nabla w_i \cdot F_i \mathbf{w} \xi x_n \, dx + 2 \iint_{\mathbf{R}^n_+} w_i F_i \mathbf{w} \cdot \nabla \xi x_n \, dx \\ + 2 \iint_{\mathbf{R}^n_+} w_i \xi F_i \mathbf{w} \cdot e_n \, dx, \end{aligned}$$

where $e_n = (0, 0, \ldots, 0, 1)^t$.

Now we claim that $F_i \mathbf{w} \cdot e_n = 0$. To see this, first observe that suffices to show the n-th entry of the vector $F_i \mathbf{w}$ is zero, and consequently, it is enough to show that the bottom row of the matrix F_i is zero. Now, the bottom row of $\partial_i A$ is identically zero as we are assuming that $a_{nk} = 0$ for $k < n$ and $a_{nn} = 1$. Moreover, when $j < n$, $\partial_n \Phi_j = 0$, as may be seen from (7), so the jn-th entries of the matrix Φ'^{-1} are zero when $j < n$. Therefore, using formula (14), we see that the bottom row of the matrix F_i is indeed zero.

Consequently the third term of II_1 is zero. The remaining terms can be controlled using (5), as above. Indeed,

$$\left|\iint_{\mathbf{R}^n_+} \nabla w_i \cdot F_i \mathbf{w} \xi x_n \, dx\right| \le C \int_{Q^*} \mathcal{C}_0 S^{3r/a}_{\phi,a}(\nabla \mathbf{v}) N_{\phi,a}(\mathbf{v}) \, dx, \quad \text{and}$$

$$\left|\iint_{\mathbf{R}^n_+} w_i F_i \mathbf{w} \cdot \nabla \xi x_n \, dx\right| \le C \int_{Q^*} \mathcal{C}_0 S_{\phi,a}(\nabla \xi \circ \Phi^{-1}) N_{\phi,a}(\mathbf{v})^2 \, dx.$$

Thus, we find

$$|\mathrm{II}_1| \le C\mathcal{C}_0(\|N_{\phi,a}(\mathbf{v})\|^2_{L^2(Q^*)} + \|N_{\phi,a}(\mathbf{v})\|_{L^2(Q^*)} \|S^{3r/a}_{\phi,a}(\nabla \mathbf{v})\|_{L^2(Q^*)}).$$

The terms II_3 and III can also be dealt with in this way. So, combining all of these estimates and summing in i, we obtain the lemma. ■

Our second lemma is very similar to Lemma 4.1, but it controls the square function by the boundary value of the gradient $\nabla u = \mathbf{v}$.

Lemma 4.2 *Let u solve* (1) *and set $\mathbf{v} = \nabla u$ so $\mathbf{v}$ solves system* (10)*. Let ϕ be a non-negative Lipschitz function. Then for each $b > 0$, there exists a constant C, depending only on λ, Λ, $\mathcal{C}_0$, $\|\nabla \phi_+\|_{L^\infty(\mathbf{R}^{n-1})}$ and b, such that*

$$\|S_{\phi,b}(\mathbf{v})\|^2_{L^2(\partial\Omega_\phi)} \le C(\|\mathbf{v}\|^2_{L^2(\partial\Omega_\phi)} + \mathcal{C}_0 \|N_{\phi,b}(\mathbf{v})\|^2_{L^2(\partial\Omega_\phi)} + \mathcal{C}_0 \|N_{\phi,b}(\mathbf{v})\|_{L^2(\partial\Omega_\phi)} \|S_{\phi,b}(\nabla \mathbf{v})\|_{L^2(\partial\Omega_\phi)}).$$

Proof. We will repeat the proof of Lemma 4.1 and so obtain an estimate localised to a cube Q. The cut-off function ξ and the dilation factor of Q^* may need to be modified when b is large, however, otherwise the result remains the same. We then will pass to the limit $r := \operatorname{diam}(Q) \to \infty$ to obtain the lemma. First we obtain (15) and observe that the second term on the right-hand side will tend to zero as $r \to \infty$ by our a priori smoothness assumptions, which mean w has sufficient decay, and the dominated convergence theorem, because $|\nabla \xi| \le C/r$. The same is true of III. The term I is again combined with the first term on the right-hand side of (16), but this time bounded below to obtain

$$\iint_{\mathbf{R}^n_+} \nabla w_i \cdot \bar{A} \nabla w_i \xi 2x_n \, dx \ge C \|S^r_{0,b}(\nabla w_i)\|^2_{L^2(Q)}.$$

Once again, the dominated convergence theorem can be used to see that the last integral in (16) will tend to zero as $r \to \infty$. Terms II_1 and II_3 can all be dealt with as before and controlled by

$$C\mathcal{C}_0(\|N_{\phi,b}(\mathbf{v})\|^2_{L^2(Q^*)} + \|N_{\phi,b}(\mathbf{v})\|_{L^2(Q^*)} \|S_{\phi,b}(\nabla \mathbf{v})\|_{L^2(Q^*)}).$$

Summing in i and taking the limit $r \to \infty$ gives the lemma. ■

For any continuous function $\mathbf{v}\colon \Omega_\phi \to \mathbf{R}^n$ and $\mu \in \mathbf{R}$, define

$$h_{\phi,\mu,a}(\mathbf{v})(x') = \sup\{x_n \,|\, x_n \geq \phi(x') \text{ and } \sup_{\Gamma_a((x',x_n))} |\mathbf{v}| > \mu\}.$$

Lemma 4.3 *If $\mathbf{v}$ is such that $h_{\phi,\mu,a}(\mathbf{v}) < \infty$, then $h_{\phi,\mu,a}(\mathbf{v})$ is Lipschitz with constant $1/a$.*

Proof. See, for example, [15, Lem 3.5]. ■

Lemma 4.4 *Let u solve (1) and set $\mathbf{v} = \nabla u$ so $\mathbf{v}$ solves system (10) and let $\{Q_j\}_j$ be a Whitney decomposition of $\{x' \,|\, N_{\phi,a}(\mathbf{v})(x') > \mu/24\}$. Given $a > 0$ and ϕ, a Lipshitz function such that $a\|\nabla\phi\|_{L^\infty(\mathbf{R}^{n-1})} < 1$, let $E^j_{\mu,\rho}$ be the intersection of a cube Q_j with*

$$\{x' \,|\, N_{\phi,a/12}(\mathbf{v})(x') > \mu, \text{ and } \mathcal{C}_0 N_{\phi,a}(\mathbf{v})(x') + \mathscr{C}^{(Q_j)^*}_{\phi,a}(\nabla\mathbf{v})(x') \leq \rho\mu\}.$$

There exists a sufficiently small choice of ρ, independent of Q_j, so that, for each $x' \in E^j_{\mu,\rho}$, there is a cube R with $x' \in 6R$ and $R \subseteq Q^*_j$ and for which

$$|\mathbf{v}(z', h_{\phi,\mu,a/12}(v)(z'))| > \mu/2$$

for all $z' \in R$.

Proof. See [14, 3.14]. Let $x' \in E^j_{\mu,\rho}$ and so, by definition, $h_{\phi,\mu,a/12}(\mathbf{v})(x') > \phi(x)$ and so there exists an y on $\partial\Gamma_{a/12}(x', h_{\phi,\mu,a/12}(\mathbf{v})(x'))$ such that $|\mathbf{v}(y)| = \mu$ and $h_{\phi,\mu,a/12}(\mathbf{v})(y') = y_n$. Set $r_0 = y_n - \phi(x') > 0$ and $K = \Gamma_{a/12}((x', \phi(x')))\cap \{z \,|\, |z_n - y_n| \leq r_0/6\}$. Since Q_j is a Whitney cube, $r_0 \leq (120/11)\mathrm{diam}(Q_j)/a$ and so $3K \subset t^\phi_a(16Q_j)$

Now by [10, Thm 8.17] we have that

$$\mathrm{osc}_K(\mathbf{v}) \leq C(r_0^{-n/2}\|\mathbf{v} - \mathbf{c}\|_{L^2(2K)} + r_0^{1-n/q}\|(\nabla A)\mathbf{v}\|_{L^q(2K)}),$$

for any constant $\mathbf{c}$. But, by property (i), $|(\nabla A)\mathbf{v}|(z) \leq C r_0^{-1}\mathcal{C}_0 N_{\phi,a}(\mathbf{v})(x')$ for $z \in 2K$, so

$$r_0^{1-n/q}\|(\nabla A)\mathbf{v}\|_{L^q(2K)} \leq C\mathcal{C}_0 N_{\phi,a}(\mathbf{v})(x')$$

and so, using Poincaré's inequality,

$$\begin{aligned} |\mathbf{v}(z) - \mathbf{v}(y)| &\leq \mathrm{osc}_K(\mathbf{v}) \leq C(r_0^{1-n/2}\|\nabla\mathbf{v}\|_{L^2(3K)} + \mathcal{C}_0 N_{\phi,a}(\mathbf{v})(x')) \\ &\leq C(\mathscr{C}^{(Q_j)^*}_{\phi,a}(\nabla\mathbf{v})(x') + \mathcal{C}_0 N_{\phi,a}(\mathbf{v})(x')) \leq C\rho\mu, \end{aligned}$$

for any $z \in K$. Thus, we may choose ρ sufficiently small so that $|\mathbf{v}(z) - \mathbf{v}(y)| \leq \mu/2$. Then, clearly, $|\mathbf{v}(z', h_{\phi,\mu,a/12}(\mathbf{v})(z'))| \geq \mu/2$ for $|z'-y'| \leq ar_0/72$ and the lemma is proved. ■

Theorem 4.5 *Let u solve* (1) *and set $\mathbf{v} = \nabla u$ so $\mathbf{v}$ solves system* (10). *Let $\{Q_j\}_j$ be a Whitney decomposition of $\{x' \,|\, N_{\phi,a}(v)(x') > \mu/24\}$. Fix a cube R and set F_j equal to the intersection of Q_j with*

$$\{x' \,|\, N_{\phi,a/12}(\mathbf{v})(x') > \mu,\ \left[(\mathcal{C}_0 M_{R^{**}}(N_{\phi,a}(\mathbf{v})^2) + M_{R^{**}}(\mathscr{C}_{\phi,a}^{R^{**}}(\nabla\mathbf{v})^2))(x')\right]^{\frac{1}{2}} \leq \rho\mu\}$$
$$\cap\, \{x' \,|\, \left[M_{R^{**}}(N_{\phi,a}(\mathbf{v})^2)(x')\right]^{\frac{1}{2}} \left[M_{R^{**}}(S_{\phi,a}^{3r/a}(\nabla\mathbf{v})^2)(x')\right]^{\frac{1}{2}} \leq \rho\mu\},$$

where $r = diam(R^*)$ and $M_{R^{**}}$ is the Hardy-Littlewood maximal function applied to functions restricted to R^{**}, that is $M_{R^{**}}(f) = M(\chi_{R^{**}} f)$ where $\chi_{R^{**}}$ is the characteristic function of R^{**}. Given sufficiently small $a > 0$ and a Lipschitz function $\phi \geq 0$ such that $a\|\nabla\phi\|_{L^\infty(\mathbf{R}^n)} < 1$, there exist constants c and $c(\rho)$, independent of j, such that for all $\mu > 0$

$$|F_j| \leq c(\rho)|Q_j|,$$

provided $Q_j \subseteq R$. Moreover, $c(\rho)$ tends to zero as $\rho \to 0$.

Proof. Fix j, set $Q := Q_j$ and $F := F_j$. Let $Q_h^* = \{(x', h_{\phi,\mu,a/12}(\mathbf{v})(x')) \,|\, x' \in Q^*\}$. By Lemma 4.4, for each $x' \in F$ and sufficiently small ρ, we have that $M_h(\mathbf{v}\chi_{Q_h^*})(x') > \mu/12$, where M_h is the Hardy-Littlewood maximal function on the graph of the function $h_{\phi,\mu,a/12}(\mathbf{v})$. By property (b) of Definition 2.2, there exists a point q' such that $\mathrm{dist}(Q, q') \leq 4\mathrm{diam}(Q)$ and $N_{\phi,a}(\mathbf{v})(q') \leq \mu/24$. Therefore we can choose $y \in Q$ such that $y_n - \phi(y') \leq 11\mathrm{diam}(Q)/a$ such that $|\mathbf{v}(y)| \leq \mu/24$. By Lemma 4.4, we have

$$M_h((\mathbf{v} - \mathbf{v}(y))\chi_{Q^h})(x') \geq M_h(\mathbf{v}\chi_{Q^h})(x') - |\mathbf{v}(y)| > \frac{\mu}{24}$$

for all $x' \in Q$, and applying the weak-type estimate for the maximal function, we obtain

$$|F| \leq \frac{C}{\mu^2} \int_{Q_h^*} (\mathbf{v} - \mathbf{v}(y))^2.$$

Now we can apply Lemma 4.1 with $x_r = y$, Q replaced with Q^*, $\phi_+ = h_{\phi,\mu,a/12}(\mathbf{v})$ and v_i replaced with $v_i - v_i(y)$, provided a is sufficiently small. We may do this as $\phi_+ - \phi \leq 12\mathrm{diam}(Q^*)/a$ on Q^{**} by (b) of Definition 2.2, as above. Observe that since we obviously have $|Q|^{1/2}|\mathbf{v}(x_r)| \leq C\|N_{\phi,a}(\mathbf{v})\|_{L^2(Q^*)}$, we get, for any $x' \in F$,

$$\begin{aligned}
\mu^2|F| &\leq C(|Q^{**}| \min_{Q^{**}} \mathscr{C}_{\phi,a}^{Q^{**}}(\nabla\mathbf{v})^2 + \mathcal{C}_0\|N_{\phi,a}(\mathbf{v})\|^2_{L^2(Q^{**})} \\
&\quad + \|N_{\phi,a}(\mathbf{v})\|_{L^2(Q^{**})}\|S_{\phi,a}^{3r/a}(\nabla\mathbf{v})\|_{L^2(Q^{**})}) \\
&\leq C|Q|\big(M_{R^{**}}(\mathscr{C}_{\phi,a}^{R^{**}}(\nabla\mathbf{v})^2)(x') + \mathcal{C}_0 M_{R^{**}}(N_{\phi,a}(\mathbf{v})^2)(x') \\
&\quad + M_{R^{**}}(N_{\phi,a}(\mathbf{v})^2)(x')^{\frac{1}{2}} M_{R^{**}}(S_{\phi,a}^{3r/a}(\nabla\mathbf{v})^2)(x')^{\frac{1}{2}}\big) \\
&\leq C\rho^2\mu^2|Q|,
\end{aligned}$$

since $Q \subset R$ and so the proof is complete. ■

Corollary 4.6 *Let u solve* (1) *and set $\mathbf{v} = \nabla u$ so $\mathbf{v}$ solves system* (10)*. Let and $\{Q_k^0\}_k$ be a Whitney decomposition of $\{x' \,|\, N_{\phi,a}(v)(x') > \mu_0/24\}$ and let $G_k^0 = \{x' \,|\, N_{\phi,a/12}(v)(x') > \mu_0\} \cap Q_k^0$. For each $q > 2$, Lipschitz function $\phi \geq 0$ and sufficiently small $a > 0$ such that $a\|\phi'\|_{L^\infty(\mathbf{R})} < 1$, there exists constants C and $c(\rho)$ such that $c(\rho) \to 0$ as $\rho \to 0$ and*

$$\begin{aligned}
&\|N_{\phi,a/12}(\mathbf{v})\|_{L^q(G_k^0)} \\
&\leq C\left(\|\mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla\mathbf{v})\|_{L^q((Q_k^0)^{**})} + \|N_{2\phi,a}(\mathbf{v})\|^{\frac{1}{2}}_{L^q((Q_k^0)^{**})}\|S_{\phi,a}^{3r/a}(\nabla\mathbf{v})\|^{\frac{1}{2}}_{L^q((Q_k^0)^{**})}\right) \\
&\quad + (c(\rho) + C\mathcal{C}_0)\|N_{\phi,a}(\mathbf{v})\|_{L^q((Q_k^0)^{**})},
\end{aligned}$$

where $r = diam((Q_k^0)^*)$.

Proof. Fix k. Let $\{Q_j\}_j$ and $\{F_j\}_j$ be as in Theorem 4.5 with $R = Q_k^0$ and let $\{F_j^0\}_j$ (and $\{Q_j^0\}_j$) be the same, but with μ replaced by μ_0 and $R = Q_k^0$.

As we remarked below Definition 2.2, if $\mu \geq \mu_0$ and $Q_k^0 \cap Q_j \neq \emptyset$, then $Q_j \subseteq Q_k^0$. This means, using Theorem 4.5, we have, for $\mu \geq \mu_0$,

$$\begin{aligned}
|G_k^0 \cap (\cup_j F_j)| &\leq |Q_k^0 \cap (\cup_j F_j)| \\
&\leq \sum_{j:Q_k^0 \cap Q_j \neq \emptyset} |F_j| \\
&\leq c(\rho) \sum_{j:Q_k^0 \cap Q_j \neq \emptyset} |Q_j| \\
&= c(\rho)|Q_k^0 \cap (\cup_j Q_j)|
\end{aligned} \tag{17}$$

Now if $\mu < \mu_0$, then

$$G_k^0 \cap (\cup_j F_j) \subseteq F_k^0$$

and $Q_k^0 \subseteq \cup_j Q_j$. As a result of this, by Theorem 4.5, when $\mu < \mu_0$,

$$|G_k^0 \cap (\cup_j F_j)| \leq |F_k^0| \leq c(\rho)|Q_k^0| = c(\rho)|Q_k^0 \cap (\cup_j Q_j)|. \tag{18}$$

Now we can proceed by standard arguments. We have

$$\begin{aligned}
&\|N_{\phi,a/12}(\mathbf{v})\|_{L^q(G_k^0)} \\
&= \int_0^\infty 4^{-q} q\mu^{q-1} |\{x \in G_k^0 \mid N_{\phi,a/12}(\mathbf{v})(x) > \mu/4\}| \, d\mu \\
&\leq \int_0^\infty 4^{-q} q\mu^{q-1} |G_k^0 \cap (\cup_j F_j)| \, d\mu + C\mathcal{C}_0 \|(M_{(Q_k^0)^{**}}(N_{\phi,a}(\mathbf{v})^2))^{\frac{1}{2}}\|_{L^q(\mathbf{R})} \\
&\quad + C\|(M_{(Q_k^0)^{**}}(\mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla \mathbf{v})^2))^{\frac{1}{2}}\|_{L^q(\mathbf{R})} \\
&\quad + C\|(M_{(Q_k^0)^{**}}(N_{\phi,a}(\mathbf{v})^2))^{\frac{1}{2}} (M_{(Q_k^0)^{**}}(S_{\phi,a}^{3r/a}(\mathbf{v})^2))^{\frac{1}{2}}\|_{L^q(\mathbf{R})} \\
&\leq c(\rho) \int_0^\infty 4^{-q} q\mu^{q-1} |Q_k^0 \cap (\cup_j Q_j)| \, d\mu + C\mathcal{C}_0 \|N_{\phi,a}(\mathbf{v})\|_{L^p((Q_k^0))^{**}} \\
&\quad + C\|\mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla \mathbf{v})\|_{L^p((Q_k^0)^{**})} + C\|N_{\phi,a}(\mathbf{v})\|_{L^q((Q_k^0)^{**})}^{\frac{1}{2}} \|S_{\phi,a}^{3r/a}(\mathbf{v})\|_{L^q((Q_k^0)^{*})}^{\frac{1}{2}} \\
&\leq c(\rho)\|N_{\phi,a/12}(\mathbf{v})\|_{L^q(Q_k^0)} + C\mathcal{C}_0 \|N_{\phi,a}(\mathbf{v})\|_{L^p((Q_k^0)^{**})} \\
&\quad + \|\mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla \mathbf{v})\|_{L^p((Q_k^0)^{**})} + \|N_{\phi,a}(\mathbf{v})\|_{L^q((Q_k^0)^{**})}^{\frac{1}{2}} \|S_{\phi,a}^{3r/a}(\mathbf{v})\|_{L^q((Q_k^0)^{**})}^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof. ■

We now need to turn the local L^q inequality for $q > 2$ into a global (on $\partial\mathbf{R}_+^n$) L^2 estimate. To do this we will need a more local version of the notion of a global point of density from [2]. An $x' \in \mathbf{R}^{n-1}$ is a point of *dyadic regional γ-density* with respect to F over R if, for each dyadic cube Q such that $x' \in 16Q \subseteq 160R$, $|F \cap 16Q|/|16Q| \geq \gamma$. The set of such x' will be denoted $\mathscr{D}_\gamma^R(F)$. If $R_0 \subseteq R$, then a point of dyadic regional γ-density over R is a point of dyadic regional γ-density over R_0.

Given a set $S \subseteq \mathbf{R}^{n-1}$, we define a *sawtooth domain* over S to be

$$\Gamma_b(S) := \cup_{x' \in S} \Gamma_b((x', 0)).$$

Lemma 4.7 *Let $\alpha < 2a < 2\alpha < b$, with $a < 1/2$, and $Q \subset R$. Suppose that the graph of ϕ gives the boundary of $\Gamma_\alpha(\mathscr{D}_\gamma^R(F))$ and $32Q \cap \mathscr{D}_\gamma^R(F) \neq \emptyset$, then there exists a constant $C(\gamma, a, \alpha)$ such that*

$$\iint_{t_a^\phi(16Q)} |\psi(y)| y_n \, dy \leq C(\gamma, a, \alpha) \int_{F \cap Q_0} \left(\iint_{\Gamma_b((x',0))} |\psi(y)|/y_n^{n-2} \, dy \right) dx',$$

where $Q_0 = 160Q$.

Proof. We follow [2, Lem 2]. After appyling Fubini's theorem, it clearly suffices to prove that if $y \in t_a^\phi(16Q) \subseteq \Gamma_\alpha(\mathscr{D}_\gamma^R(F))$, then

$$\int_{F\cap Q_0} \chi((x'-y')/(by_n))\,dx' \geq C(\gamma,a,\alpha)y_n^{n-1}, \tag{19}$$

where χ is the characteristic function of the unit ball.

Since $y \in t_a^\phi(16Q) \subseteq \Gamma_\alpha(\mathscr{D}_\gamma^R(F))$, there exists $\bar{x}' \in \mathscr{D}_\gamma^R(F)$ such that $|y'-\bar{x}'| \leq \alpha y_n$ and since $32Q \cap \mathscr{D}_\gamma^R(F) \neq \emptyset$, $y_n \leq (\frac{2}{\alpha}+\frac{1}{a})16\mathrm{diam}(Q)$ and $\bar{x}' \in 64Q$. Observe that since $2\alpha < b$, $B(\bar{x}',\alpha y_n) \cap B(y',by_n)^c = \emptyset$, therefore

$$\begin{aligned}
&|F\cap Q_0 \cap B(\bar{x}',\alpha y_n)|\\
&\leq |F\cap Q_0\cap B(\bar{x}',\alpha y_n)\cap B(y',by_n)| + |F\cap Q_0\cap B(\bar{x}',\alpha y_n)\cap B(y',by_n)^c|\\
&\leq |F\cap Q_0\cap B(y',by_n)| + |B(\bar{x}',\alpha y_n)\cap B(y',by_n)^c|\\
&= |F\cap Q_0\cap B(y',by_n)|.
\end{aligned}$$

Now, because $y_n \leq (\frac{2}{\alpha}+\frac{1}{a})16\mathrm{diam}(Q)$ and $\bar{x}' \in 64Q$, there exists a dyadic cube Q' with diameter comparable to y_n such that $\bar{x}' \in 16Q' \subseteq B(\bar{x}',\alpha y_n) \cap Q_0 \subseteq Q_0 \subseteq 160R$, so

$$|F\cap Q_0\cap B(y',b_0y_n)| \geq |F\cap Q_0\cap B(\bar{x}',\alpha y_n)| \geq |F\cap 16Q'| \geq \gamma|16Q'|.$$

This proves (19) and with it the lemma. ■

Theorem 4.8 *Let u solve* (1) *and set $\mathbf{v} = \nabla u$ so $\mathbf{v}$ solves system* (10)*. For a sufficiently small constant a, there exist constants C and b such that*

$$\|N_{0,a/12}(\mathbf{v})\|_{L^2(\mathbf{R}^{n-1})} \leq C(\|S_{0,b}(\nabla\mathbf{v})\|_{L^2(\mathbf{R}^{n-1})} + \mathcal{C}_0\|N_{0,b}(\mathbf{v})\|_{L^2(\mathbf{R}^{n-1})})$$

Proof. Choose a sufficiently small so that we may apply Corollary 4.6 with $\|\nabla\phi\|_{L^\infty(\mathbf{R}^{n-1})} \leq 1/a$ and suppose $b > 2a$. Set

$$E_{\mu_0}^{l,\rho} = \{x' \mid N_{0,a/12}(\mathbf{v})(x') > \mu_0, S_{0,b}(\nabla\mathbf{v})(x') \leq \rho\mu_0, N_{0,b}(\mathbf{v})(x') \leq l\mu_0\}.$$

and

$$F_{\mu_0}^{l,\rho} = \{x' \mid S_{0,b}(\nabla\mathbf{v})(x') \leq \rho\mu_0, N_{0,b}(\mathbf{v})(x') \leq l\mu_0\}.$$

By standard arguments we have

$$\|N_{0,a/12}(v)\|^2_{L^2(\mathbf{R}^{n-1})} = \int_0^\infty 2\mu_0 |\{N_{0,a/12}(\mathbf{v})(x') > \mu_0\}|\, d\mu_0$$
$$\leq \int_0^\infty 2\mu_0 (|E^{l,\rho}_{\mu_0}| + |\{S_{0,b}(\nabla \mathbf{v})(x') > \rho\mu_0\}|$$
$$+ |\{N_{0,c}(\mathbf{v})(x') > l\mu_0\}|)\, d\mu_0$$
$$\leq \int_0^\infty 2\mu_0 |E^{l,\rho}_{\mu_0}|\, d\mu_0$$
$$+ \frac{1}{\rho^2}(\|S_{0,b}(\nabla \mathbf{v})\|^2_{L^2(\mathbf{R})} + \frac{1}{l^2}\|N_{0,b}(\mathbf{v})\|^2_{L^2(\mathbf{R}^{n-1})}.$$

Using [19, p62], we may pick l sufficiently large so that the last term may be hidden on the left-hand side.

Now let $\{Q^0_k\}_k$ be the Whitney decomposition of $\{x' \mid N_{\phi,a}(v)(x') > \mu_0/24\}$. We consider two cases. The first is k for which $\mathscr{D}^{(Q^0_k)^{**}}_\gamma (Q^0_k \cap F^{l,\rho}_{\mu_0})$ is empty. Then, for each $x' \in 160(Q^0_k)^{**}$, there exists a dyadic cube $R_{x'}$, such that $x' \in 16R_{x'} \subseteq 160(Q^0_k)^{**}$ and $|F^{l,\rho}_{\mu_0} \cap 16R_{x'}| \leq \gamma |16R_{x'}|$. Obviously,

$$\cup_{x' \in 160(Q^0_k)^*} 16R_{x'} = 160(Q^0_k)^{**}.$$

By the properties of dyadic cubes, we can pick out the maximal subcollection of $\{R_{x'}\}_{x'}$, which we label $\{R_k\}_k$. Now the $\{R_{x'}\}_{x'}$ will be disjoint and have the property

$$\sum_k |R_k| \leq |160(Q^0_k)^{**}| \leq C|Q^0_k|,$$

and so

$$|Q^0_k \cap F^{l,\rho}_{\mu_0}| \leq |\cup_k (F^{l,\rho}_{\mu_0} \cap 16R_k)| \leq \sum_k |16R_k| \frac{|(F^{l,\rho}_{\mu_0} \cap 16R_k)|}{|16R_k|}$$
$$\leq C\gamma 16^{n-1} \sum_k |R_k| \leq C\gamma |Q^0_k|.$$

Thus, if we fix γ sufficiently small, we have by [19, p62] that

$$
\begin{aligned}
&\sum_k \int_0^\infty 2\mu_0 |Q_k^0 \cap E_{\mu_0}^{l,\rho}|\, d\mu_0 \\
&\sum_k \int_0^\infty 2\mu_0 |Q_k^0 \cap F_{\mu_0}^{l,\rho}|\, d\mu_0 \\
&\leq C\gamma \sum_k \int_0^\infty 2\mu_0 |Q_k^0|\, d\mu_0 \\
&\leq C\gamma \|N_{0,a}(v)\|^2_{L^2(\mathbf{R})} \\
&\leq (1/4)\|N_{0,a/12}(v)\|^2_{L^2(\mathbf{R})},
\end{aligned}
\tag{20}
$$

where the sum in k is only over those k for which $\mathscr{D}_\gamma^{(Q_k^0)^{**}}(Q_k^0 \cap F_{\mu_0}^{l,\rho})$ is empty.

We now consider those k for which $\mathscr{D}_\gamma^{(Q_k^0)^{**}}(Q_k^0 \cap F_{\mu_0}^{l,\rho})$ is non-empty. In this case, we may form the sawtooth domain $\Gamma_\alpha(\mathscr{D}_\gamma^{(Q_k^0)^{**}}(Q_k^0 \cap F_{\mu_0}^{l,\rho}))$ for $a < \alpha < 2a$. Let ϕ be the Lipschitz function which gives the boundary of $\Gamma_\alpha(\mathscr{D}_\gamma^{(Q_k^0)^{**}}(Q_k^0 \cap F_{\mu_0}^{l,\rho}))$. Now, for $q > 2$, by Corollary 4.6 and Cauchy's inequality,

$$
\begin{aligned}
&\mu_0^q |Q_k^0 \cap E_{\mu_0}^{l,\rho}| \leq C \int_{G_k^0} N_{\phi,a/12}(\mathbf{v})^q \\
&\leq C(\varepsilon)\left(\int_{(Q_k^0)^{**}} \mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla \mathbf{v})^q\right) \\
&\qquad + C(c(\rho) + \mathcal{C}_0 + \varepsilon)\left(\int_{(Q_k^0)^{**}} (N_{\phi,b}(\mathbf{v})^q\right) \\
&\leq C(\varepsilon)\left(\int_{(Q_k^0)^{**}} \mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla \mathbf{v})^q\right) + C(c(\rho) + \mathcal{C}_0 + \varepsilon) l^q \mu_0^q |Q_k^0|,
\end{aligned}
\tag{21}
$$

since, for $y' \in (Q_k^0)^{**}$, $N_{\phi,b}(\mathbf{v})(y') \leq N_{0,b}(\mathbf{v})(x')$ for some $x' \in Q_k^0 \cap F_{\mu_0}^{l,\rho}$.

We now aim to show that, for $y' \in (Q_k^0)^{**}$,

$$
\mathscr{C}_{\phi,a}^{(Q_k^0)^{**}}(\nabla \mathbf{v})(y') \leq C\rho\mu_0. \tag{22}
$$

To see this take a dyadic cube $Q \subset (Q_k^0)^{**}$. If

$$
32Q \cap \mathscr{D}_\gamma^{(Q_k^0)^{**}}(Q_k^0 \cap F_{\mu_0}^{l,\rho}) \neq \emptyset
$$

then, by Lemma 4.7,

$$\begin{aligned}
&\iint_{t_a^\phi(16Q)} (x_n - \phi(x'))|\nabla \mathbf{v}|^2\, dx \\
&\le C(\gamma, a, \alpha) \int_{(Q_k^0 \cap F_{\mu_0}^{l,\rho}) \cap 160Q} \left(\iint_{\Gamma_b((x',0))} |\nabla \mathbf{v}|/y_n^{n-2}\, dy \right) dx' \\
&\le C \int_{(Q_k^0 \cap F_{\mu_0}^{l,\rho}) \cap 160Q} S_{0,b}(\nabla \mathbf{v})(x')^2 dx' \\
&\le C\rho^2 \mu_0^2 |Q|.
\end{aligned}$$

On the other hand, if

$$32Q \cap \mathscr{D}_\gamma^{(Q_k^0)^{**}} (Q_k^0 \cap F_{\mu_0}^{l,\rho}) = \emptyset$$

then the distance from $t_a^\phi(16Q)$ to $\partial \mathbf{R}_+^n$ is at least a fixed multiple of $\operatorname{diam}(Q) =: r$. Thus, for $x \in t_a^\phi(16Q)$, $x_n \le Cr$ and so, for b sufficiently large,

$$\begin{aligned}
\iint_{t_a^\phi(16Q)} (x_n - \phi(x'))|\nabla \mathbf{v}|^2\, dx &\le r \iint_{t_a^\phi(16Q)} |\nabla \mathbf{v}|^2\, dx \\
&\le Cr^{n-1} \iint_{t_a^\phi(16Q)} |\nabla \mathbf{v}|^2 \frac{dx}{x_n^{n-2}} \\
&\le C|Q| S_{0,b}(\nabla \mathbf{v})(y') \le C|Q|\rho^2\mu_0^2,
\end{aligned}$$

for some $y' \in F_{\mu_0}^{l,\rho}$ and fixed b sufficiently large. Putting these two facts together proves (22). Substituting this in (21) we find

$$|Q_k^0 \cap E_{\mu_0}^{l,\rho}| \le (C(\varepsilon)\rho^q + C(c(\rho) + \mathcal{C}_0 + \varepsilon))|Q_k^0|.$$

And so, choosing ε then ρ sufficiently small, we can argue as in (20) to complete the proof of the theorem. ■

We can now prove (11). By Theorem 4.8 and Lemma 4.2 we have

$$\begin{aligned}
\|N_{0,a/12}(\mathbf{v})\|_{L^2(\partial \mathbf{R}_+^n)}^2 \le C(&\|\mathbf{v}\|_{L^2(\partial \mathbf{R}_+^n)}^2 + \mathcal{C}_0 \|N_{\phi,b}(\mathbf{v})\|_{L^2(\partial \mathbf{R}_+^n)}^2 \\
&+ \mathcal{C}_0 \|N_{\phi,b}(\mathbf{v})\|_{L^2(\partial \mathbf{R}_+^n)} \|S_{\phi,b}(\nabla \mathbf{v})\|_{L^2(\partial \mathbf{R}_+^n)}).
\end{aligned}$$

Therefore, (11) follows by [19, p62], Cauchy's inequality and taking $\mathcal{C}_0$ sufficiently small so that we may hide $\|N_{\phi,b}(\mathbf{v})\|_{L^2(\partial \mathbf{R}_+^n)}^2$ on the left-hand side.

Now we move on to prove (12). Let us assume A is symmetric, so that our equation becomes $\operatorname{div}(A\nabla u) + B \cdot \nabla u = 0$ and B satisfies conditions

(i) and (ii) with B replacing ∂A. We start in the spirit of [17, (3.1)] with $e_n = (0, 0, \dots, 0, 1)$, $x = (x', t)$ and $\chi = \chi(t)$ a smooth function on $[0, \infty)$:

$$
\begin{aligned}
&\operatorname{div}((\chi A\nabla u \cdot \nabla u)e_n t)\\
&= \partial_t(\chi A\nabla u \cdot \nabla u)t + (\chi A\nabla u \cdot \nabla u)\\
&= 2(\chi A\nabla u \cdot \nabla \partial_t u)t + (\partial_t \chi A\nabla u \cdot \nabla u)t + (\chi \partial_t A\nabla u \cdot \nabla u)t\\
&\quad + (\chi A\nabla u \cdot \nabla u)\\
&= 2\operatorname{div}((\chi A\nabla u)(\partial_t u)t) - 2\chi \operatorname{div}(A\nabla u)(\partial_t ut) - 2(\partial_t \chi)(A\nabla u)(\partial_t ut)\\
&\quad - 2(\chi A\nabla u \cdot e_n)\partial_t u + (\partial_t \chi A\nabla u \cdot \nabla u)t + (\chi \partial_t A\nabla u \cdot \nabla u)t\\
&\quad + (\chi A\nabla u \cdot \nabla u)\\
&= 2\operatorname{div}((\chi A\nabla u)(\partial_t u)t) + 2\chi(B \cdot \nabla u)(\partial_t ut) - 2(\partial_t \chi)(A\nabla u)(\partial_t ut)\\
&\quad - 2(\chi A\nabla u \cdot e_n)\partial_t u + (\chi \partial_t A\nabla u \cdot \nabla u)t + (\chi + t\partial_t \chi)(A\nabla u \cdot \nabla u).
\end{aligned}
$$

Now, integrating in $\mathbf{R}^n_+$ and using the divergence theorem, we obtain

$$
\begin{aligned}
&\iint_{\mathbf{R}^n_+} 2(\chi A\nabla u \cdot e_n)\partial_t u\\
&= \iint_{\mathbf{R}^n_+} 2\chi(B \cdot \nabla u)(\partial_t ut) - 2(\partial_t \chi)(A\nabla u)(\partial_t ut) + (\chi \partial_t A\nabla u \cdot \nabla u)t\\
&+ \iint_{\mathbf{R}^n_+} (\chi + t\partial_t \chi)(A\nabla u \cdot \nabla u).
\end{aligned}
$$

Now, let us take χ to be a smooth function of t which is equal to n on $[0, 1/n)$ and vanishing on $[2/n, \infty)$. Taking the limit as $n \to \infty$ and using the fact that $\int_0^\infty (\chi + t\partial_t \chi)dt = 0$, we obtain

$$
\int_{\partial \mathbf{R}^n_+} 2(A\nabla u \cdot e_n)\partial_t u \leq \int_{\partial \mathbf{R}^n_+} 2\mathcal{C}_0 |\nabla u||\partial_t u| + \mathcal{C}_0 |\nabla u|^2.
$$

Now by ellipticity, Cauchy's inequality and knowing $\mathcal{C}_0$ can be taken to be sufficiently small, we find

$$
\int_{\partial \mathbf{R}^n_+} |\partial_t u|^2 \leq C \int_{\partial \mathbf{R}^n_+} |\nabla_T u|^2 + \frac{1}{2} \int_{\partial \mathbf{R}^n_+} |\partial_t u|^2.
$$

This proves (12) and with it Theorem 3.1.

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