A NEW APPROACH TO ABSOLUTE CONTINUITY OF ELLIPTIC MEASURE, WITH APPLICATIONS TO NON-SYMMETRIC EQUATIONS

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0. Introduction

In the late 50's and early 60's, the work of De Giorgi [De Gi] and Nash [N], and then Moser [Mo] initiated the study of regularity of solutions to divergence form elliptic equations with merely bounded measurable coefficients. Weak solutions in a domain Ω, a priori only in a Sobolev space $W^2_{1,loc}(\Omega)$, were shown to be Hölder continuous of some order depending just on ellipticity, and maximum principles and Harnack inequalities were established. The Dirichlet problem for such operators, with continuous data on the boundary, was established in [LSW]. This in turn paved the way for a more systematic and detailed study of the properties of the elliptic measures $d\omega_L$ associated to $L = \operatorname{div} A\nabla$ on a domain Ω. The classical properties of existence of non-tangential limits of solutions (Fatou type theorems) and comparison principles appeared in [CFMS], but owed a great deal to the earlier work of Carleson [Ca] and Hunt and Wheeden [H-W] on harmonic functions in Lipschitz domains.

All the results mentioned above were carried out for elliptic operators $L = \operatorname{div} A\nabla$ where the matrix $A = (a_{ij})$ has bounded measurable coefficients and is symmetric. However, it turns out that the symmetry of the matrix is not needed to get these results: Morrey [Mor] first observed this in connection with the De Giorgi-Nash-Moser theory; for the results in [CFMS], this fact has not been formally observed until now. With appropriate reformulation in terms of adjoint operators, and adjoint Green's functions, the results of [CFMS] are valid without the symmetry assumption (see §1).

The investigation into the solvability of L^p boundary value problems, in the sense of non-tangential convergence and L^p estimates on the non-tangential maximal function

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of solutions, really began with the study of harmonic functions in Lipschitz domains ([D1], [D2] and [JK]). In [D1], B. Dahlberg proved that, on any Lipschitz domain Ω, the harmonic measure, $d\omega$, and the surface measure, $d\sigma$, were mutually absolutely continuous, that $d\omega \in A_\infty(d\sigma)$ (the Muckenhoupt weight class A_∞). He showed that there exists a constant C such that for any radius r and every surface ball $\Delta(r) \subseteq \partial\Omega$,

$$\left(\int_{\Delta(r)} k^2 \frac{d\sigma}{\sigma(\Delta(r))}\right)^{\frac{1}{2}} \leq C \int_{\Delta(r)} k \frac{d\sigma}{\sigma(\Delta(r))}, \text{ where } d\omega = k d\sigma. \tag{0.1}$$

The estimate (0.1) will imply solvability of the L^2 Dirichlet problem in the domain Ω. In [JK], Jerison and Kenig realized how to obtain (0.1) by means of an elementary identity of Rellich type (see (0.2)). Since this discovery, and its further applications to a more general class of divergence form operators, the theory of boundary value problems (BVP's) for second order operators has been built on the use of L^2 Rellich type identities. This holds true even for BVP's associated with systems of elliptic equations, higher order elliptic equations and parabolic equations. (See [P] and [K] for a discussion and some references.

To be precise, consider the Laplacian in a domain above the graph of a Lipschitz function $\{t > \varphi(x)\}$ with $\|\nabla\varphi\|_\infty \leq M < \infty$. The mapping $(x,t) \mapsto (x, t - \varphi(x))$ is a biLipschitzian 'flattening' of this domain and maps the Laplacian to an elliptic divergence form operator $L = \operatorname{div} A\nabla$, where $A = (a_{ij})$ is symmetric and has merely bounded coefficients. Dahlberg's result ([D]) on the L^2 solvability of the Dirichlet problem for Laplace's equation in $\{t > \varphi(x)\}$, i.e., that (0.1) holds, translates to L^2 solvability of the Dirichlet problem for L in $\mathbb{R}^n_+$. Because this is not a property universally possessed by such operators ([CFK]) (even the A_∞ condition mentioned below may fail), one asks what special property of such matrices is responsible for this phenomenon. The answer lies in the fact the coefficients of A are independent of the t-variable. And indeed, the Rellich identity of [JK] applies to all such operators (symmetric and time-independent) to yield (0.1). Specifically, let $L = \operatorname{div} A\nabla$ be an operator of this type, and u a solution to L. Then

$$\operatorname{div}[A\nabla u.\nabla u\vec{e}] = 2\operatorname{div}[D_t u \; A\nabla u], \tag{0.2}$$

where $\vec{e} = (0, \dots, 0, 1)$. Now apply the divergence theorem to (0.2) in, say, the domain $\{t > \varphi(x)\}$ where φ is Lipschitz. Here $\vec{N}$, the unit normal, exists a.e., and $\langle \vec{e}, \vec{N}\rangle$ has a positive lower bound. Then this boundary integral identity, the estimate on $\langle \vec{e}, \vec{N}\rangle$ and the ellipticity assumption on A proves that $\|A\nabla u.\vec{N}\|_{L^2(d\sigma)} \approx \|\nabla_T u\|_{L^2(d\sigma)}$. However, the derivation of (0.2) requires symmetry of the matrix. The question is: how crucial is this assumption in order to obtain the desired consequence of (0.2), namely, L^2 solvability of the Dirichlet problem. This is the problem addressed in section 3.

Another interesting, and little understood, situation where no Rellich identity is possible is the case where the matrix A and the solution to L are complex valued. Here the issues of solvability of BVP's are closely connected with fundamental questions concerning the Cauchy integral operator and analytic perturbations of operators. In [KM], the direct connection is made—see also [K] for the reformulation of a problem of Kato on square roots of such operators in terms of a BVP.

In fact, a complex valued solution to $L = \operatorname{div} A\nabla$ where A is complex elliptic can be represented as a vector solution (by separating into real and imaginary parts) of a real, elliptic but skew-symmetric system of equations. So there is a closer connection between the complex valued situation and the non-symmetric one than merely the absence of a Rellich identity. Recently, Verchota and Vogel ([VV]) have made a systematic study of non-symmetric elliptic systems in planar domains, and found some surprising positive as well as negative results.

In this paper, motivated initially by the study of non-symmetric elliptic equations, we prove two theorems which give sufficient conditions for the elliptic measure of an elliptic divergence form operator to belong to A_∞, with respect to surface measure, on the boundary of Lipschitz domain in $\mathbb{R}^n$. By the general theory of such operators ([CFMS]), this A_∞ condition implies solvability of the L^p Dirichlet problem for some value of p which depends on the operator. In section 3 we verify this general criterion for a class of divergence form non-symmetric operators. These are the 'time independent' coefficient operators in $\mathbb{R}^2$, for which (0.1) would be proven via Rellich identities in the symmetric case. Without symmetry, we only obtain A_∞, but we also provide an example to show that this is sharp. Thus the L^2 solvability of the Dirichlet problem may fail in this context, but L^p solvability, for some value of p, holds.

We have two main criteria for A_∞, in any dimension, which are both sharp as the example will show. Theorem (2.3) says that if any solution u to $Lu = 0$ can be arbitrarily well approximated in a Lipschitz domain by smooth functions satisfying a certain technical condition, then $d\omega_L$ belongs to A_∞ with respect to surface measure on the boundary of that domain. This 'ϵ-approximability' condition arises in the work of Varopoulos ([V]) and Garnett ([G]). Indeed, the first clue that such a condition may be connected to A_∞ appears in Corollary 6.2, p.348 of [G], where a 'quantitative' Fatou theorem is proved. This is explained at the beginning of §2.

Our second main theorem (2.9) results essentially from the observation that Dahlberg's proof of ϵ-approximability of harmonic functions in Lipschitz domains applies in a more general setting. That is, his proof works for any class of operators for which one has an L^p norm equivalence between the non-tangential maximal function and the square function of solutions, again for a class of domains to be specified later. (See section 1 for the relevant definitions.)

The positive results contained here should have broad applications. Indeed, the condition can be verified for a class of operators whose coefficients satisfy a Carleson condition ([LH] and [KP]). The investigation initiated in section 3 generates some interesting questions. For example, what are the higher dimensional analogs of these two dimensional results? What condition can one assume, in addition to ellipticity, which cancels the effect of non-symmetry? Finally, the true role of the existence of Rellich type identities awaits further understanding.

1. Definitions and Background "*"

In this section we give some terminology to be used throughout and state the main properties of solutions to divergence form elliptic equations that we will need.

We will usually be defining solutions in Lipschitz domains $\Omega \subseteq \mathbb{R}^n$. Such a domain satisfies uniform interior and exterior cone conditions (and hence classical Dirichlet problems for, say, the Laplacian are solvable there). There follows a definition which pays closer attention to the constants involved in measuring the 'Lipschitz character' of these domains.

Definition. *$Z \subseteq \mathbb{R}^n$ is an M-*cylinder of diameter d *if there exists a coordinate system (x,t) such that*

$$Z = \{(x,t) : |x| \le d, -2Md \le t \le 2Md\}$$

and, for $s > 0$

$$sZ = \{(x,t) : |x| \le sd, -2sMd \le t \le 2sMd\}.$$

Definition. *$\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with character (M, N, c_o) if there exists a positive scale r and there exists at most N M-cylinders $\{Z_j\}_{j=1}^N$ of diameter d, with $\frac{r}{c_0} \le d \le c_0 r$ such that*

(i) $4Z_j \cap \partial\Omega$ is the graph of a Lipschitz function φ_j (in the coordinate system of Z_j) where $\|\varphi_j\|_\infty \le M$, and $\varphi_j(0) = 0$.

(ii) $\partial\Omega = \bigcup_j (Z_j \cap \partial\Omega)$; and $Z_j \cap \Omega \supseteq \{(x,t) : |x| \le d,\ dist((x,t), \partial\Omega) \le d/2\}$

If $Q \in \partial\Omega$ and $B_r(Q) = \{X : |X - Q| \le r\}$, then $\Delta_r(Q)$ (or sometimes just Δ_r) will denote $B_r(Q) \cap \Omega$. The Carleson region above $\Delta_r(Q)$ is $T(\Delta_r) = \Omega \cap B_r(Q)$.

For Ω a Lipschitz domain, we define non-tangential approach regions, for each $Q \in \partial\Omega$,

$$\Gamma(Q) = \Gamma_\alpha(Q) = \{X \in \Omega : |X - Q| \le (1+\alpha)\text{dist}\,(X, \partial\Omega)\}$$

where α is taken large enough (only depending on the Lipschitz character). In [D4], Dahlberg defines a collection of non-tangential approach regions $\{\Gamma(Q)\}$ which he calls a regular family of cones. Essentially these are right circular cones, with respect to a coordinate system defining the Lipschitz graph, which are contained in the domain. We shall sometimes use this terminology.

Let Ω be Lipschitz and $\{\Gamma_\alpha(Q)\}_{Q\in\partial\Omega}$ a regular family of cones (or non-tangential approach regions). Let $\Gamma_\alpha^d(Q) = \Gamma_\alpha(Q) \cap B_d(Q)$ be the d-truncated cone. If $v(X)$ is continuous in Ω, we define $N_{\alpha,d}v(Q) = \sup\{|v(X)| : X \in \Gamma_\alpha^d(Q)\}$, a *non-tangential maximal function* of v in Ω. The square function of v at Q relative to the family $\{\Gamma_\alpha^d(Q)\}$ is

$$S_{\alpha,d}v(Q) = \left\{ \int_{\Gamma_\alpha^d(Q)} |\nabla v(X)|^2 \left(\mathrm{dist}\,(X, \partial\Omega)\right)^{2-n} dX \right\}^{\frac{1}{2}}.$$

When α and d are understood we will suppress the dependence and just use the notation Nv and Sv.

Let now $A(X) = (a_{ij}(X))_{i,j=1}^n$ be a real $n\times n$ matrix, $a_{ij} \in L^\infty$, satisfying the uniform ellipticity condition:

(1.1) There exists a $\lambda > 0$ such that for all $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2.$$

The matrix A will not be assumed symmetric.

Remark: Future reference to 'the ellipticity constant' will mean a constant that depends on both λ and $\|a_{ij}\|_{L^\infty}$.

The space $W^2_{1,loc}(\Omega)$ denotes $\{f \in L^2_{loc}(\Omega) : \varphi f \in W^2_1(\Omega)\ \forall \varphi \in C_0^\infty(\Omega)\}$ where $W_1^2(\Omega)$ is the usual Sobolev space $\{f \in L^2(\Omega) : \int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 < +\infty\}$.

Definition 1.2. *A function $u \in W^2_{1,loc}(\Omega)$ is a solution in Ω to $Lu = div\, A(X)\nabla u = 0$ if*

$$\int_\Omega a_{ij}(X) D_i u\, D_j\varphi = 0 \qquad \forall\, \varphi \in C_0^\infty(\Omega). \tag{1.2}$$

The main ingredients of the De Giorgi-Nash-Moser theory for solutions to elliptic divergence form equations hold as well for the case where $A(\cdot)$ is not symmetric. This was observed by Morrey ([Mor]). The starting point for these regularity results is the following fundamental estimate. (The abbreviation $⨍_E f d\mu$ is employed for the average $(\int_E f d\mu/\mu(E))$.)

(1.3) (Caccioppoli). *If $u \geq 0$ is an L-subsolution in Ω (i.e. the integral in (1.2) is non-positive) and if $B_{2r}(X) \leq \Omega$, then*

$$\fint_{B_r(X)} |\nabla u(Z)|^2 dZ \leq \frac{C}{r^2} \fint_{B_{2r}(X)} u(Z)^2 dZ,$$

where C depends on ellipticity and dimension.

The interior regularity estimates are as follows. Here, $\operatorname{osc}_{B_r} u = \sup_{B_r} u - \inf_{B_r} u$, denotes the oscillation of u over the ball B_r.

(1.4) If u is a nonnegative subsolution in Ω and $\overline{B_{2r}} \subset \Omega$ then

$$\sup_{B_r} u \leq C \left(\fint_{B_{2r}} u^p \right)^{\frac{1}{p}}$$

for any $p > 0$ and $C = C(\lambda, n, p)$.

(1.5) (interior Hölder continuity). If u is a solution to L in Ω then

$$\operatorname{osc}_{B_r} u \leq C \left(\frac{r}{R} \right)^{\alpha} \left(\fint_{B_R} u^2 \right)^{\frac{1}{2}},$$

for some $0 < \alpha < 1$, $\alpha = \alpha(\lambda, n)$ and $0 < r < R < \operatorname{dist}(X, \partial\Omega)$.

The important fact here is that the Hölder continuity rate of the solution only depends on the ellipticity of the operator.

(1.6) (Harnack inequality). If u is a nonnegative solution to L in Ω and $B_{2r} \subset \Omega$, then

$$\sup_{B_r} u \leq C \inf_{B_r} u.$$

(1.7) If u is a solution to L in Ω and $B_{2r} \subset \Omega$ then there is a $p > 2$, $p = p(\lambda, n)$, such that

$$\left(\fint_{B_r} |\nabla u|^p \right)^{\frac{1}{p}} \leq C \left(\fint_{B_{2r}} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

(1.8) (Maximum principle). If u is a solution to L in Ω, which is continuous in a neighborhood of $\partial\Omega$, then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u.$$

For domains whose boundary has some regularity (including the class of Lipschitz domains) there are boundary analogs of the Hölder continuity and other interior estimates above. Such regularity estimates hold when solutions vanish on a portion of the boundary. Under the same hypotheses as their interior analogs, we have

(1.3.B) (Boundary Cacciopoli) If $u \equiv 0$ on Δ_{2r}, then

$$\fint_{T(\Delta_r)} |\nabla u|^2 \leq \frac{C}{r^2} \fint_{T(\Delta_{2r})} |u|^2,$$

whenever $Lu = 0$ in $T(\Delta_{2r})$.

(1.5.B) If $Lu = 0$ in $T(\Delta_{2r})$, and if $u \equiv 0$ on Δ_{2r}, then

$$\operatorname*{osc}_{T(\Delta_\rho)} u \leq C \left(\frac{\rho}{r}\right)^\alpha \left(\fint_{T(\Delta_{2r})} u^2\right)^{\frac{1}{2}},$$

where $\rho < r$ and the surface balls Δ_{2r} and Δ_ρ have the same center.

From (1.5.B) one can deduce an estimate for nonnegative solutions u of L in a region $T(\Delta_{2r}(Q))$, which vanish on $\Delta_{2r}(Q)$

$$u(X) \leq C \left(\frac{|X - Q|}{r}\right)^\alpha \sup_{T_{2r}(Q)} u \tag{1.9}$$

where $\alpha = \alpha(\lambda, r)$ and X is any point of $T(\Delta_r(Q))$.

The results of Littman, Stampacchia and Weinberger ([LSW]) are also valid in the non-symmetric setting. In particular, a Lipschitz domain Ω is *regular for the Dirichlet problem*, meaning that for every $g \in \mathrm{Lip}(\partial\Omega)$, the generalized solution to $Lu = 0$ in Ω, $u = g$ on $\partial\Omega$, given by Lax-Milgram, is in fact continuous in $\overline{\Omega}$. Thus the mapping $g \mapsto u_g(X)$ which is defined for $g \in C(\partial\Omega)$ and for which $u_g(X)$ is the solution to the Dirichlet problem with data g is a bounded positive linear functional. The Riesz representation theorem implies the existence of a family of elliptic probability measures $\{d\omega_L^X\}$ associated to L. Since, by Harnack's inequality, these are all mutually absolutely continuous, as X varies over Ω, we shall fix a point X_0 in Ω and call $d\omega_L = d\omega_L^{X_0}$ *the* elliptic measure associated to $\partial\Omega$, so that

$$u_g(X_0) = \int_{\partial\Omega} g(Q) d\omega_L(Q), \qquad \forall\, g \in C(\partial\Omega). \tag{1.10}$$

We are interested in the relationship between the elliptic measure $d\omega_L$ and the surface measure $d\sigma$ for a given domain Ω. Examples ([CFK]) show that even in the symmetric case, $d\omega_L$ and $d\sigma$ may be singular if the coefficients of the matrix are merely bounded and measurable. What further assumptions on the coefficients are required to insure, say, mutual absolute continuity, or other stronger connections between these measures (see [FKP]). To study these questions, we need to introduce the Green's function and determine its relationship to elliptic measure. In [GW], Gruter and Widman made a systematic study of the Green's function, without assuming symmetry of the matrix.

Theorem 1.11. ([GW]) *There exists a positive function $G(X,Y)$ with values in $\mathbb{R}\cup\{+\infty\}$ such that for all $Y\in\Omega$ and any $r>0$,*

(i) $G(\cdot,Y)\in W^2_1(\Omega\backslash B_r(Y))\cap \overset{\circ}{W}{}^1_1(\Omega)$

(ii) $\forall\, \varphi\in C^\infty_0(\Omega)$,

$$\int_\Omega a_{ij}(X)D_iG(X,Y)D_j\varphi(X)=\varphi(Y)$$

(iii) $G(Y,X)=G^*(X,Y)$, *where G^* satisfies (i) and (ii) for A^*, the adjoint of A*

(iv) $G(X,Y)\le C(\lambda)|X-Y|^{2-n}$ *for all $X,Y\in\Omega$*

(v) $G(X,Y)\ge c(\lambda)|X-Y|^{2-n}$ *for all $X,Y\in\Omega$ with $|X-Y|\le\frac{1}{2}\mathit{dist}(Y,\partial\Omega)$*

(vi) $G(\cdot,Y)\in \overset{\circ}{W}{}^p_1(\Omega)$ *for all $1\le p\le n/n-1$, uniformly in Y.*

(vii) $G(X,Y)\le c(\lambda)\{\mathit{dist}(Y,\partial\Omega)\}^\alpha|X-Y|^{2-n-\alpha}$, $\alpha=\alpha(\lambda,n)$.

(viii) $|G(X,Y)-G(Z,Y)|\le C_\lambda|X-Zz|^\alpha\{|X-Y|^{2-n-\alpha}+|Z-Y|^{2-n-\alpha}\}$

Note that in dimension $n=2$ the singularity in the bounds on the Green's function would be logarithmic.

If the coefficients of A and the boundary of Ω were C^∞, Green's theorem would give:

$$\begin{aligned}
u(Y) &= \int_\Omega L^*G^*(X,Y)u(X)dX\\
&= \int_\Omega \operatorname{div}[A^*\nabla G^*(X,Y)u(X)]dX\\
&\quad -\int_\Omega A^*\nabla G^*(X,Y).\nabla u(X)dX\\
&= \int_{\partial\Omega} u(Q)A^*(Q)\nabla G^*(Q,Y).\vec{N}(Q)d\sigma(Q)\\
&\quad +\int_\Omega G^*(X,Y)Lu(Y)dY.
\end{aligned}$$

where $\vec{N}(Q)$ is the unit normal to the boundary. That is, we find that $d\omega_L^Y(Q)=A^*(Q)\nabla G^*(Q,Y).\vec{N}(Q)d\sigma(Q)$, and the solution to the Dirichlet problem with data g

is given by

$$u(X) = \int_{\partial\Omega} g(Q)A^*(Q)\nabla G^*(Q,X).\vec{N}(Q)d\sigma(Q).$$

In general, to establish the relationship between the Green's function and elliptic measure is more delicate. This was carried out in [CFMS] (owing a great deal to the estimates in [HW]) for symmetric elliptic operators L. However, a careful inspection of the proofs of the results therein will show that all the estimates remain valid (with G replacing G^* where appropriate) even in the non-symmetric case. We summarize these below.

Properties of the elliptic measure

(1.12)
$$\omega_L^X(\Delta_r(Q)) \geq c_0$$

for all $X \in B_{cr}(A_r(Q))$, where the point $A_r(Q) \in \Omega$ is chosen so that $dist(A_r(Q), \partial\Omega) \simeq |A_r(Q) - Q| \approx r$, and $c = c(M)$, $M =$ the Lipschitz character of Ω (see [K], pg. 8, for a more detailed discussion of the required geometric properties of domains for which these estimates hold.)

(1.13) For $X \in {}^cB_{cr}(A_r(Q)) \cap \Omega$,

(i)
$$r^{n-2}G(X, A_r(Q)) \leq C\omega_L^X(\Delta_{2r}(Q)).$$

(ii)
$$\begin{aligned}\omega_L^X(\Delta_r(Q)) &\leq Cr^{n-2}G(X, A_r(Q)) \\ &= Cr^{n-2}G^*(A_r(Q), X).\end{aligned}$$

(1.14) (Comparison principle). *If u, v are nonnegative solutions in $T(\Delta_{2r}(Q))$, continuous in $\overline{T(\Delta_{2r})}$ and vanishing on $\Delta_{2r}(Q)$, then there exists a constant $C = C(M)$, such that $\forall\ X \in T(\Delta_r)$,*

$$C^{-1}\frac{u(A_r(Q))}{v(A_r(Q))} \leq \frac{u(X)}{v(X)} \leq C\frac{u(A_r(Q))}{v(A_r(Q))}.$$

The kernel function $K(X,Q)$ is defined to be $K(X,Q) = \frac{d\omega_L^X}{d\omega_L}$, the Radon-Nikodym derivative. It satisfies the following two estimates.

(1.15) (i) If $X \in \Gamma_\alpha(P)$ with $|X - P| \approx r \approx \text{dist}\,(X, \partial\Omega)$ then $K(X,Q) \approx \frac{1}{\omega_L(\Delta_r(P))}$, for all $Q \in \Delta_r(P)$.

(ii) for all $X \in \Omega$, $|K(X,Q_1) - K(X,Q_2)| \leq C_X|Q_1 - Q_2|^\alpha$ where α depends on the Lipschitz character of Ω (and on L).

We are interested, for the purposes of solving boundary value problems, in the relationship between $d\omega_L$ and $d\sigma$, on the boundary of Ω. We need the following definitions, which involve dilation invariant conditions—those which are most natural in the context of Lipschitz domains.

Definition 1.16. [G-C, RdeF] *Let Δ denote a surface ball contained in $\partial\Omega$.*

(i) $d\mu \in A_\infty(d\nu)$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $E \subseteq \Delta$,

$$\frac{\nu(E)}{\nu(\Delta)} < \delta \Rightarrow \frac{\mu(E)}{\mu(\Delta)} < \epsilon.$$

(ii) $d\mu \in B_q(d\nu)$ if $d\mu$ is absolutely continuous with respect to $d\nu$ and $f = \frac{d\mu}{\partial\nu}$ satisfies

$$\left(\int_\Delta f^q \frac{d\nu}{\nu(\Delta)}\right)^{\frac{1}{q}} \leq C\left(\int_\Delta f \frac{d\nu}{\nu(\Delta)}\right).$$

Definition 1.17. *The Dirichlet problem $(D)_p$ with data in $L^p(d\sigma)$ is solvable in Ω for L if whenever $f \in C(\partial\Omega)$, the solution u to the classical Dirichlet problem $(u|_{\partial\Omega} = f \in C(\partial\Omega);\ u \in C(\overline{\Omega}))$ satisfies the estimate*

$$\|N(u)\|_{L^p(d\sigma)} \leq C\|f\|_{L^p(d\sigma)} \tag{1.18}$$

where C depends only on the Lipschitz character of Ω, and the ellipticity of L.

Because $N(u)(Q)$ is comparable to

$$M_{\omega_L}(f)(Q) = \sup_{\Delta \ni Q} \int_\Delta f(P) \frac{d\omega_L(P)}{\omega_L(\Delta)}$$

when $u = f$ on $\partial\Omega$, the theory of weights ([M]) tells us that $(D)_p$ is solvable for L if and only if $d\omega_L \in B'_p(d\sigma)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, since $A_\infty = \bigcup_{q>1} B_q$, it follows that $d\omega_L \in A_\infty(d\sigma)$ if and only if there exists a $p < +\infty$ for which $(D)_p$ is solvable for L.

2. Square function estimates and A_∞.

We shall prove two main theorems in this section—each valid in $\mathbb{R}^n$ for any n, and for solutions to elliptic divergence form operators which are not necessarily assumed to be symmetric.

Definition 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $L = div A\nabla$, an elliptic divergence form operator whose matrix has coefficients which are bounded and measurable. A weak solution u to $Lu = 0$ in Ω, with $\|u\|_\infty \leq 1$, is said to be*

ϵ-approximable if there exists a $\varphi \in C^\infty(\Omega)$ such that $\|u-\varphi\|_\infty < \epsilon$ in Ω and such that for all surface balls $\Delta(r,Q) = \partial\Omega \cap B(r,Q)$,

$$\int_{T(\Delta(r,Q))} |\nabla\varphi| dX \le C_\epsilon \sigma(\Delta), \tag{2.2}$$

where $T(\Delta(r,Q)) = B(r,Q)\cap\Omega$ is the Carleson region associated to $\Delta(r,Q)$, and C_ϵ depends also on the Lipschitz character of Ω.

The concept of ϵ-approximability arises quite naturally and has been studied extensively for harmonic functions. Consider $L = \Delta$ and Ω the (unbounded) domain $\mathbb{R}^n_+ = \{(x,y) \in \mathbb{R}^{n-1} \times R : y > 0\}$. If u is a bounded harmonic function, or more generally, the Poisson extension of a BMO function, then the quantity $y|\nabla u(x,y)|^2 dx\, dy$ is a Carleson measure ([G]). That is, for every cube $I \subset \mathbb{R}^{n-1}$, and if $\ell(I) =$ diameter of I, then $\int_{x\in I}\int_{y=0}^{\ell(I)} y|\nabla u(x,y)|^2 dx\ dy \le C\|u\|^2_{BMO}|I|$ which is precisely the statement (2.2) for this domain. A natural question, inspired by methods of proof of both $H^1 - BMO$ duality ([F-St]) and the Corona Theorem ([Ca2] and [G]), is whether in fact the simpler expression $|\nabla u| dxdy$ is Carleson. This is not true, but the knowledge that u may be arbitrarily well approximated by a continuous function φ whose gradient gives rise to a Carleson measure provides alternate methods of proof of both these results. For harmonic functions in the upper half space, a construction which proves this may be found in Garnett's book [G], building on earlier work of Varopoulos ([V]). Indeed, Garnett draws a corollary, [p.348, of [G]], which he calls a 'quantitative Fatou theorem', and which provides the first solid connection between ϵ-approximability and quantitative properties of harmonic measure. Later, Dahlberg, in [D5], extended Garnett's result to harmonic functions in bounded Lipschitz domains. We shall make some further remarks about Dahlberg's extension later in connection with our second main theorem, which turns out to be essentially a small observation on a proof in [D1].

Theorem 2.3. *Let $L = div A\nabla$ be elliptic, where $A = (a_{ij})$ is a (not necessarily symmetric) matrix of bounded measurable functions. Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, containing* 0*. Then there exists an ϵ, depending on ellipticity of L and the Lipschitz character of Ω such that if every solution u to $Lu = 0$, with $\|u\|_\infty \le 1$, is ϵ-approximable on Ω, then $d\omega_L$ belongs to $A_\infty(d\sigma)$, where $d\sigma =$ surface measure on $\partial\Omega$. That is, given $\eta > 0$, there exists a δ depending on ϵ, ellipticity, the Lipschitz character of Ω and approximation constants such that whenever $E \subseteq \Delta_r \subseteq \partial\Omega$, we have $\sigma(E)/\sigma(\Delta_r) < \eta$ implies $\omega_L(E)/\omega_L(\Delta_r) < \delta$.*

We will need some results on the elliptic measure $d\omega_L = d\omega$ on $\partial\Omega$ to establish (2.3). Fix Ω to be a bounded Lipschitz domain containing the unit ball of $\mathbb{R}^n$, B_1, and contained in B_M, the ball of radius M. Let M also be an upper bound for the number of coordinate patches required to cover $\partial\Omega$ by graphs of Lipschitz functions whose Lipschitz constant will also be no greater than M. This domain then possesses a *dyadic grid* (see [Ch]), a collection of subsets $\{I_{j,l}\}$ of $\partial\Omega$, where for each fixed $j \geq 0$:

(i) $\bigcup_l I_{j,l} = \partial\Omega$; $I^0_{j,l_1} \cap I^0_{j,l_2} = \emptyset$ if $l_1 \neq l_2$ and $\omega(\partial I_{j,l}) = 0$ for all j, l.

(ii) Both $\emptyset$ and $\partial\Omega$ belong to $\{I_{j,l}\}_{j,l}$

(iii) $\Delta_{j,l} \subseteq I_{j,l} \subseteq M\Delta_{j,l}$, where $\Delta_{j,l} = B(2^{-j}, Q_l) \cap \partial\Omega$. Q_l is called the center of $I_{j,l}$.

(iv) If $\overset{\circ}{I}_{j,l} \cap \overset{\circ}{I}_{j',l} \neq \emptyset$, then either $I_{j,l} \subseteq I_{j',l}$ or $I_{j',l} \subseteq I_{j,l}$. And there exist a $C(M) < 1$ such that $\omega(I_{j,l}) < C(M)\omega(I_{j',l})$ when $I_{j,l} \subseteq I_{j',l}$.

(v) Any open set $\mathscr{O} \subset \partial\Omega$ can be decomposed as $\mathscr{O} = \bigcup_{j,l} I_{j,l}$, where the $I_{j,l}$ are non-overlapping. Moreover, for each $I_{j,l}$ in this decomposition, there exists a $P_{j,l} \in \partial\Omega \backslash \mathscr{O}$ such that $\operatorname{dist}(P_{j,l}, I_{j,l}) \simeq \operatorname{diam}(I_{j,l})$.

We note that if the domain Ω contains an r-ball B_r and is contained in B_{Mr}, then there is a rescaled version of this dyadic grid in which the constants 2^{-j} are replaced by $2^{-j}r$, and the other constants do not depend on r.

Definition 2.4. *Let ϵ_0 be given and small. If $E \subseteq \partial\Omega$, a good ϵ_0-cover for E of length k is a collection of nested open sets $\{\mathscr{O}i\}_{i=1}^k$ with $E \subseteq \mathscr{O}_k \subseteq \mathscr{O}_{k-1} \subseteq \cdots \subseteq \mathscr{O}_0 = \partial\Omega$ where each $\mathscr{O}_l = \bigcup_{i=1}^{\infty} S_i^{(l)}$ and so that*

(i) each $S_i^{(l)}$ belongs to the dyadic grid for $\partial\Omega$, and

(ii) for all $1 \leq l \leq k$, $\omega(\mathscr{O}_l \cap S_i^{(l-1)}) \leq \epsilon_0 \omega(S_i^{(l-1)})$.

Note that condition (ii) of Definition (2.4) above implies that each $S_i^{(l)}$ is properly contained in some $S_j^{(l-1)}$. To see this, observe that since $\mathscr{O}_l \subseteq \mathscr{O}_{l-1}$, $S_i^{(l)}$ must intersect some $S_j^{(l-1)}$. The inclusion $S_j^{(l-1)} \subseteq S_i^{(l)}$ is not possible for $\omega(S_j^{(l-1)}) \leq \omega(S_j^{(l-1)} \cap \mathscr{O}_l)$ and (ii) gives a contradiction.

If in (2.4) above we can take $k = +\infty$ then $\{\mathscr{O}_l\}$ is called a good cover of infinite length.

Lemma 2.5. *If $\{\mathscr{O}_i\}$ is a good ϵ_0-cover of E of length k and $k \geq l > m \geq 1$, then $\omega(S_j^{(m)} \cap \mathscr{O}_l) \leq \epsilon_0^{l-m} \omega(S_j^{(m)})$.*

Proof. From the remark following the definition above, we have

$$\mathscr{O}_{m+1} \cap S_j^{(m)} = \bigcup \{S_i^{(m+1)} : S_i^{(m+1)} \subset S_j^{(m)}\}$$

and the inequality (ii) of (2.4) can be iterated $l - m$ times.

Lemma 2.6. *Given $\epsilon_0 > 0$, there exists a $\delta_0 > 0$ such that if $E \subseteq \partial\Omega$ and $\omega(E) \leq \delta_0$, then E has a good ϵ_0-cover of length k, with $k \to \infty$ as $\omega(E) \to 0$. (In fact, $k \approx \epsilon_0 \log\left(\frac{C}{\omega(E)}\right)$.)*

Proof. Let $0 < \epsilon_0' < 1$ be fixed–to be determined later. Let U be an open set containing E with $\omega(U) < 2\omega(E)$, and set $\mathscr{O}_k = \{x : M_\omega(X_U)(x) > \epsilon_0'\}$, where

$$M_\omega(g)(x) = \sup\{\int_\Delta g \frac{d\omega}{\omega(\Delta)} : \Delta \ni x, \Delta \subseteq \partial\Omega\}.$$

Since U is open, $U \subseteq \mathscr{O}_k$ and since ω is doubling, $\omega(\mathscr{O}_k) \leq \frac{C}{\epsilon_0'}\omega(U) < \frac{2C}{\epsilon_0'}\omega(E)$. If $\frac{2C}{\epsilon_0'}\omega(E)$ is less than $\frac{1}{2}$, then $\mathscr{O}_k$ has a Whitney decomposition, $\mathscr{O}_k = \bigcup_i S_i^{(k)}$, and for each $S_i^{(k)}$ there exists a point $P_i^{(k)} \in {}^c\mathscr{O}_k$ such that $\operatorname{dist}(P_i^{(k)}, S_i^{(k)}) \simeq \operatorname{diam}(S_i^{(k)})$. Since $P_i^{(k)} \in {}^c\mathscr{O}_k$, if Δ is any surface ball containing $P_i^{(k)}$, then $\frac{\omega(U\cap\Delta)}{\omega(\Delta)} \leq \epsilon_0'$. Therefore, there is a choice of ϵ_0' which depends only on the doubling constant of ω and on ϵ_0 which guarantees that $\frac{\omega(U\cap S_i^{(k)})}{\omega(S_i^{(k)})} \leq \epsilon_0$. Thus, given ϵ_0, choose ϵ_0' as above, and then choose δ_0 so that $2C\omega(E) < \epsilon_0'/2$. Let k be the largest integer such that $\left(\frac{C}{\epsilon_0'}\right)^k \omega(E) < \frac{1}{4}$. For $k - 1 \leq j \leq 1$, set $\mathscr{O}_{j-1} = \{x : M_\omega(X_{\mathscr{O}_j}) > \epsilon_0'\}$. It is straightforward to verify that $\{\mathscr{O}_i\}_{i=1}^k$ is a good ϵ_0-cover.

Remark 2.7. If Ω is an arbitrary Lipschitz domain and ω is a doubling measure on $\partial\Omega$, then we may dilate Ω to get a new domain Ω' with $B_1 \subseteq \Omega' \subseteq B_M$ and apply Lemma (2.6) to Ω'. Because the proof of (2.6) depends only on the doubling constant of ω, this rescaling will prove the lemma for arbitrary Lipschitz domains as well.

We now draw a corollary of the approximation hypothesis on bounded solutions u, which is a small modification of Corollary (6.2) of [G]. The cones $\{\Gamma(Q)\}_{Q\in\partial\Omega}$ form a regular family, i.e. non-tangential approach regions. We shall use them to define the 'oscillation function' of a solution u. Let $r < 1$ and let $\Gamma_r(Q) = \Gamma(Q) \cap B_r(Q)$ be the r-truncated cone at Q. If $X_j = (x_j, y_j) \in \Gamma_r(Q)$, let $y(X_j - Q)$ denote y_j, the second coordinate. Define, for $\theta < 1$, the oscillation function $N(\Gamma_r, \epsilon, \theta, Q)$ by

$$N(\Gamma_r, \epsilon, \theta, Q) \geq k \quad \text{if there exists } k \text{ points } X_1, \dots, X_k \in \Gamma_r(Q)$$
$$\text{such that } y(X_j - Q) < \theta y(X_{j-1} - Q), \tag{2.8}$$
$$\text{and for which } \ |u(X_j) - u(X_{j-1})| \geq \epsilon.$$

Lemma 2.9. *Suppose u is $\frac{\epsilon}{4}$-approximable in $\Omega \subset \mathbb{R}^n$. Then*

$$\int_{\partial\Omega \cap B_r(Q)} N(\Gamma_r, \epsilon, \theta, Q) d\sigma(Q) \leq C r^{n-1},$$

where C depends on ϵ, θ and the Lipschitz constant of Ω.

Proof. Let $\{\widetilde{\Gamma}(Q)\}$ be another family of regular cones with $\widetilde{\Gamma}(Q) \supset \Gamma(Q)$. Set $A_r(\varphi)(Q) = \int_{\widetilde{\Gamma}_r(Q)} |\nabla\varphi| \frac{dX}{|X-Q|^{n-1}}$, where $\widetilde{\Gamma}_r(Q) = \widetilde{\Gamma}(Q) \cap B_r(Q)$. We claim that if $N(\Gamma_r, \epsilon, \theta, Q) \geq k$ and φ approximates u in the sense of (2.1) for $\epsilon_0 = \frac{\epsilon}{4}$ then $A_r(\varphi)(Q) \geq k C_{\epsilon,\theta}$. Because

$$\int_{\Delta_r} A_r(\varphi)(Q) d\sigma(Q) \leq C \int_{T(\Delta_r)} |\nabla\varphi(x)| dX$$

which is bounded by Cr^{n-1}, $C = C(M, \epsilon)$, the claim proves the lemma. Moreover, by a dilation it suffices to prove the claim for $r = 1$.

To see the claim, we can assume that $Q = 0$ and that the cones $\Gamma(Q) \backslash \widetilde{\Gamma}(Q)$ are of the form $\{(x, y) : |x| < \alpha y\}$. Suppose that $N_\epsilon^\Gamma(0) > k$ and fix the points $X_j = (x_j, y_j)$, $|x_j| < \alpha y_j$, $0 \leq y_k \leq y_{k-1} \leq \cdots \leq y_1 \leq 1$, $y_j \leq \theta y_{j-1}$ for which $|u(X_j) - u(X_{j-1})| \geq \epsilon$. Because u is Holder continuous and $\|u\|_\infty \leq 1$, there exists a δ, depending only on ϵ and the ellipticity of L, such that $|u(X) - u(X_j)| < \epsilon/8$ whenever $X \in \{(x, y_j) : |x - x_j| < \delta y_j\} = l_j$. A similar statement holds at X_{j-1} for all $Y \in l_{j-1}$ and hence, for any $X \in l_j$ and $Y \in l_{j-1}$, $|u(X) - u(Y)| \geq 3\epsilon/4$. We may also choose δ to insure that both segments l_j, l_{j-1} belong to the cone $\widetilde{\Gamma}(0)$.

Let φ be a smooth $\frac{\epsilon}{4}$-approximant to u in the sense of (2.1). Then $|\varphi(X) - \varphi(Y)| \geq \epsilon/4$ when $X \in l_j$ and $Y \in l_{j-1}$. For $(z, y_j) \in l_j$, and $1 \leq t \leq t_j = y_{j-1}/y_j$, set

$$X_t = \left((z - x_j)t + \left(1 - \frac{t-1}{t_j - 1}\right) x_j + \frac{t-1}{t_j - 1} x_{j-1}, t y_j \right).$$

Then $X_t \in \widetilde{\Gamma}(0)$ and at $t = t_j$, $X_{t_j} \in l_{j-1}$, while $X_1 \in l_j$ by assumption. Thus, $\left| \int_1^{t_j} \frac{\partial}{\partial t} \varphi(X_t) dt \right| \geq \epsilon/4$. Also,

$$\begin{aligned} \frac{\partial}{\partial t} X_t &= \left((z - x_j) - \frac{1}{t_j - 1} x_j + \frac{1}{t_j - 1} x_{j-1}, y_j \right) \\ &= \left((z - x_j) + \frac{x_{j-1} - x_j}{t_j - 1}, y_j \right), \end{aligned}$$

and so $\left|\frac{\partial}{\partial t} X_t\right| \leq \delta y_j + \frac{2\alpha y_{j-1}}{t_j - 1} + y_j \leq C y_j$, since $y_{j-1} - y_j \geq (1 - \theta) y_{j-1}$.

Consider the change of variables $\rho : (z,t) \mapsto X_t = (x,s)$, where $|z - x_j| \le \delta y_j$, $1 \le t \le t_j$. The mapping is one to one and we have that $dzdt = \frac{(y_j)^{n-2}}{s^{n-1}} dxds$, since the Jacobian is given by the inverse of

$$\det \begin{pmatrix} t & & (*) \\ & \ddots & \\ & & t \\ (0) & & y_j \end{pmatrix} = t^{n-1} y_j = (ty_j)^{n-1} y_j^{2-n} = s^{n-1} y_j^{2-n}.$$

Therefore, if $\widetilde{\Gamma}_j = \widetilde{\Gamma} \cap \{(x,y) : y_j \le y \le y_{j-1}\}$,

$$\begin{aligned} \int_{\widetilde{\Gamma}_j} |\nabla \varphi| \frac{dxds}{s^{n-1}} &\ge C_\delta \left\{ \frac{1}{\delta y_j^{n-1}} \int_{|z-x_j| \le \delta y_j} \int_1^{t_j} \left| \frac{\partial \varphi}{\partial t}(X_t)) \right| dtdz \right\} \\ &\ge C_\delta \frac{\epsilon}{4}, \end{aligned}$$

and summing in j we conclude that the claim holds.

Proof of (2.3). Let $E \subseteq \partial\Omega$ be given, with $\frac{\omega(E)}{\omega(\Delta_r)} \le \delta$. Let Ω' be a Lipschitz domain containing $T(\Delta_r)$, with Lipschitz constant bounded by that of Ω and for which $\operatorname{diam}(\Omega') \le Mr$ and $\partial\Omega' \cap \partial\Omega \subseteq \Delta_{2r}$. Let A_r be a point of Ω' whose distance to $\partial\Omega'$ is comparable to r, with constants depending only on the Lipschitz constant of Ω. Let $d\omega_{L,\Omega'}^{A_r}$ be the elliptic measure for L in the domain Ω' with respect to the point A_r. Let's abbreviate this measure ω'. Then, by the comparison principle, since $\omega(E)/\omega(\Delta_r) \le \delta$, $\omega'(E) \le C\delta$. By Lemma 2.6, construct a good ϵ_0-cover of $E \subseteq \partial\Omega'$ of length k, where ϵ_0 will be determined. That is, we have a collection of nested open sets $\{\mathscr{O}_i\}_{i=0}^k$ with $\mathscr{O}_i = \bigcup_j S_j^{(i)}$, each $S_j^{(i)}$ is contained in some $S_{j'}^{(i-1)}$ and for each $k \ge l > m \ge 1$, we have $\omega'(S_j^{(m)} \cap \mathscr{O}_l) \le \epsilon_0^{l-m} \omega'(S_j^{(m)})$. Set $f = \sum_{m=0}^{k} (-1)^m X_{\mathscr{O}_m}$, and $u(X) = \int_{\partial\Omega'} K(X,Q) f(Q) d\omega'(Q)$, the solution to $Lu = 0$ in Ω' with data f. Note that $0 \le f \le 1$.

If both ϵ and ϵ_0 have been chosen appropriately then we will show that there is a $\theta < 1$ such that $N(\Gamma_r, \epsilon, \theta, Q) \ge ck$ for all $Q \in E$. By Lemma (2.9),

$$\begin{aligned} ck\sigma(E) &\le \int_E N(\Gamma_r, \epsilon, \theta, Q) d\sigma(Q) \\ &\le \int_{\Delta_r} N(\Gamma_r, \epsilon, \theta, Q) d\sigma(Q) \\ &\le C(\epsilon, \theta) r^{n-1}. \end{aligned}$$

Thus $\sigma(E) \le \frac{C}{k} r^{n-1}$.

To prove the estimate on the oscillation function, let m be an even integer, $0 < m \le k$ and Q be any point of E. Then $Q \in \mathscr{O}_m$ and so there is an element $S^{(m)}_{j_0} \subseteq \mathscr{O}_m$ of the dyadic grid which contains Q. Let Q_{j_0} denote the center of $S^{(m)}_{j_0}$ and pick a point $X^{(m)}_{j_0}$ in Ω' with $\operatorname{dist}(X^{(m)}_{j_0}, \partial\Omega') \approx |X^{(m)}_{j_0} - Q_{j_0}| \approx \operatorname{diam}(S^{(m)}_{j_0})$. Any such $X^{(m)}_{j_0}$ is in $\Gamma_r(Q)$. Moreover,

$$\begin{aligned} u(X^{(m)}_{j_0}) &\ge \int_{S^{(m)}_{j_0}} K(X^{(m)}_{j_0}, P) f(P) d\omega'(P) \\ &\ge \frac{C}{\omega'(S^{(m)}_{j_0})} \int_{S^{(m)}_{j_0}} f(P) d\omega'(P) \end{aligned}$$

by estimate (1.15) on $K(X^{(m)}_{j_0}, Q)$ for $Q \in S^{(m)}_{j_0}$, and the doubling properties of ω'. Also,

$$\begin{aligned} \frac{1}{\omega'(S^{(m)}_{j_0})} \int_{S^{(m)}_{j_0}} f(P) d\omega'(P) &= \frac{1}{\omega(S^{(m)}_{j_0})} \int_{S^{(m)}_{j_0}} \sum_{l=0}^{m} (-1)^l X_{\mathscr{O}_l} d\omega' \\ &\quad + \frac{1}{\omega'(S^{(m)}_{j_0})} \int_{S^{(m)}_{j_0}} \sum_{l=m+1}^{k} (-1)^l X_{\mathscr{O}_l} d\omega' \\ &= I + II. \end{aligned}$$

since $S^{(m)}_{j_0} \subseteq \mathscr{O}_l$, for $l = 0, \dots, m$ and m is even, $\sum_{l=0}^{m} (-1)^l = 1$, thus term $I = 1$. Term II is, in absolute value, bounded above by

$$\begin{aligned} \frac{1}{\omega'(S^{(m)}_{j_1)}} \sum_{l=m+1}^{k} \omega'(\mathscr{O}_l \cap S^{(m)}_{j_0}) &\le \frac{1}{\omega'(S^{(m)}_{j_0})} \sum_{l=m+1}^{k} \epsilon_0^{l-m} \omega'(S^{(m)}_{j_0}) \\ &\le 2\epsilon_0, \end{aligned}$$

provided that $\epsilon_0 < \frac{1}{2}$.

Therefore $u(X^{(m)}_{j_0}) \ge 1 - 2\epsilon_0$. Our objective now is to find points Y_j, for $j \le k$ and j odd, such that $u(Y_j) \le c_0$ where $1 - 2\epsilon_0 - c_0 \ge \epsilon > 0$ determines ϵ and this gives the lower bound on $N(\Gamma_r, \epsilon, \theta, Q)$.

Let m be odd, $0 < m \le k$ and let $Q \in E$ so that there is an $S^{(m)}_{j_0}$ such that $Q \in S^{(m)}_{j_0}$. If Q_{j_0} denotes the center of $S^{(m)}_{j_0}$, choose $X^{(m)}_{j_0,\eta} \in \Omega$ such that

$$\operatorname{dist}(X^{(m)}_{j_0,\eta}, \partial\Omega) \approx |X^{(m)}_{j_0,\eta} - Q_{j_0}| \approx \eta \operatorname{diam}(S^{(m)}_{j_0}),$$

for $\eta < 1$ to be determined. The Hölder continuity estimate (1.9) guarantees that

$$\begin{aligned}\int_{^cS_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, P) f(P) d\omega'(P) &\leq \int_{^cB_{2^{-m}}(Q_{j_0})} K(X_{j_0,\eta}^{(m)}, P) f(P) d\omega'(P) \\ &= 1 - \int_{B_{2^{-m}}(Q_{j_0})} K(X_{j_0,\eta}^{(m)}, P) f(P) d\omega'(P),\end{aligned}$$

so

$$\int_{^cS_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, P) f(P) d\omega'(P) \leq C\eta^\alpha.$$

Therefore

$$\begin{aligned}u(X_{j_0,\eta}^{(m)}) &\leq \int_{S_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, P) f(P) d\omega'(P) + C\eta^\alpha \\ &= \int_{S_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, P) \left(\sum_{l=0}^{m} (-1)^l X_{\mathscr{O}_l} \right) d\omega' \\ &\quad + \int_{S_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, P) \left(\sum_{l=m+1}^{k} (-1)^l X_{\mathscr{O}_l} \right) d\omega' + C\eta^\alpha.\end{aligned}$$

On $S_{j_0}^{(m)}$, $\sum\limits_{l=0}^{m} (-1)^l X_{\mathscr{O}_l} = 0$ since m is odd and so

$$\begin{aligned}u(X_{j_0,\eta}^{(m)}) &\leq C\eta^\alpha + \int_{S_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, P) \left(\sum_{l=m+1}^{k} (-1)^l X_{\mathscr{O}_l} \right) d\omega' \\ &\leq C\eta^\alpha + \sum_{l=m+1}^{k} \int_{S_{j_0}^{(m)}} K(X_{j_0,\eta}^{(m)}, Q) X_{\mathscr{O}_l} d\omega'.\end{aligned}$$

By Harnack's inequality for positive solutions, and the doubling property of the elliptic measure we have $K(X_{j_0,\eta}^{(m)}, Q) \leq \frac{C_\eta}{\omega'(S_{j_0}^{(m)})}$ for $Q \in S_{j_0}^{(m)}$, and this yields

$$\begin{aligned}u(X_{j_0,\eta}^{(m)}) &\leq C\eta^\alpha + C_\eta \sum_{l=m+1}^{k} \epsilon_0^{l-m} \\ &\leq C\eta^\alpha + C_\eta \epsilon_0.\end{aligned}$$

η and ϵ_0 will be chosen later, at this point we assume they satisfy $C\eta^\alpha \leq 1/8$, and $C_\eta \epsilon_0 \leq 1/8$. Choose $Y_m \in \Gamma_r(Q)$ such that

$$\operatorname{dist}(Y_{(m)}, \partial\Omega) \approx |Y_{(m)} - Q| \approx \eta \operatorname{diam}(S_{j_0}^{(m)}),$$

then $|Y_m - X^{(m)}_{j_0,\eta}| \le C(\eta+1)\mathrm{diam}(S^{(m)}_{j_0})$. Note that $1-u$ is a non negative harmonic function in Ω'. Harnack's inequality guarantees that

$$1 - u(Y_m) \ge C\eta(1 - u(X^{(m)}_{j_0,\eta})) \ge C\eta(1 - C\eta^\alpha - C_\eta\epsilon_0) \ge C'\eta.$$

Hence $u(Y_m) \le 1 - C'\eta$. From now we also assume that $4e_0 \le C'\eta$.

For $Q \in E$, consider the sequence $\{X_m\}_{m=0}^k$, where $X_m = X^{(m)}_{j_0}$ for m even and $X_m = Y_m$ for m odd, $m = 0, 1, \ldots, k$. The estimates above show that provided $C_\eta\epsilon_0 \le 1/8$ and $4\epsilon_0 \le C'\eta$, $|u(X_m) - u(X_{m'})| \ge \frac{C'\eta}{2}$ whenever m is odd and m' is even. Moreover note that

$$\begin{aligned}|y(X_{2\ell+1} - Q)| \le |X_{2\ell+1} - Q| &\le C\eta\mathrm{diam}(S^{(2\ell+1)}_{j_0})\\ &\le C\eta\mathrm{diam}(S^{(2\ell)}_{j_0}) \le C\eta\mathrm{dist}\,(X_{2\ell}, \partial\Omega)\\ &\le C\eta|X_{2\ell} - Q| \le C''\eta|y(X_{2\ell} - Q)|.\end{aligned}$$

Here $C'' > 0$ depends of the aperture of the cone. We now choose $\eta \in (0,1)$ satisfying $C\eta^\alpha \le 1/8$ and $C''\eta \le \sqrt{\eta}$. ϵ_0 is chosen accordingly, satisfying the conditions specified above. Under these assumptions $|y(X_{2\ell+1} - Q)| \le \sqrt{\eta}|y(X_{2\ell} - Q)|$. To insure that heights $y(X_m - Q)$ decrease as well, we need to choose a new sequence $\{\overline{X}_m\}$. In order to do that, note that for $p \ge 1$,

$$\begin{aligned}|y(X_{2p+2\ell} - Q)| \le |X_{2p+2\ell} - Q| &\le C\mathrm{diam}(S^{(2p+2\ell)}_{j_0})\\ &\le C2^{-p}\mathrm{diam}(S^{(2\ell+1)}_{j_0}) \le C'\frac{2^{-p}}{\eta}|X_{2\ell+1} - Q|\\ &\le C''\frac{2^{-p}}{\eta}|y(X_{2\ell+1} - Q)|.\end{aligned}$$

Choose $p \ge 1$ such that $C''\frac{2^{-p}}{\eta} \le \sqrt{\eta}$. This guarantees that $|y(X_{2p+2\ell} - Q)| \le \sqrt{\eta}|y(X_{2\ell+1} - Q)|$. Let $\overline{X}_0 = X_0$, $\overline{X}_1 = X_1$ and $\overline{X}_2 = X_{(2p)}$, $\overline{X}_3 = X_{(2p+1)}$ and in general, $X_{2\ell} = X^{(2\ell p)}$. By skipping this fixed number of points in the sequence, we obtain a new sequence, $\{\overline{X}_m\} \subset \Gamma_r(Q)$, of length a fixed fraction of k. Moreover $|y(\overline{X}_{m+1} - Q)| \le \sqrt{\eta}|y(\overline{X}_m - Q)|$, and $|u(\overline{X}_m) - u(\overline{X}_{m+1})| \ge C\eta/2$. Thus $N(\Gamma_r, C\eta/2, \sqrt{\eta}, Q) \ge ck$. Here $\eta \in (0,1)$ only depends on the aperture of the cone, and on the Lipschitz character of the domain Ω.

Our second main theorem provides a criterion for testing when ϵ-approximability holds. The condition is useful—it can be verified in nontrivial instances. The next section is devoted to one such instance: two dimensional non-symmetric elliptic divergence form equations with non-smooth coefficients, independent of one of the variables. A particular example computed there shows that Theorem (2.3) (as well as Theorem 2.9) is sharp in the sense that no stronger conclusion than A_∞ can be drawn.

Theorem 2.9. *Suppose that for all bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^n$ and any solution u to $Lu = div A\nabla u = 0$, with u vanishing at some fixed point in Ω, where L is elliptic, A is bounded and measurable, one can prove the estimates*

$$\int_{\partial\Omega} N^2(u)d\sigma \leq C_1 \int_{\Omega} \delta(X)|\nabla u(X)|^2 dX \leq C_2 \int_{\partial\Omega} N^2(u)d\sigma,$$

for $\delta(X) = dist(X, \partial\Omega)$, with constants depending only the Lipschitz character of Ω. Then, on any such domain Ω, $d\omega_L^\Omega$ belongs to $A_\infty(d\sigma)$.

Remark (2.10). In [D5], B. Dahlberg proved that harmonic functions in Lipschitz domains are ϵ-approximable for any $\epsilon > 0$. His proof used the square function estimates (2.9) for harmonic functions that he had recently shown in [D4]. The other properties of harmonic functions used in the proof, like the mean value property and the pointwise estimates of gradients in terms of the function itself, may all be replaced by interior estimates, Harnack's inequality, maximum principles, L^2 averages of gradients and Caccioполи inequalities. In other words, Dahlberg's proof is valid for any class of solutions which possess the properties which follow from the De Giorgi-Nash-Moser theory and, in addition, satisfy (2.9). As a final comment, we note that it suffices, by purely real variable arguments, to prove square function estimates in any L^p, $0 < p < \infty$, from which (2.9)—the $p = 2$ case—may be derived.

It may also be important to note that the same conclusion of the theorem may be drawn from slightly weaker hypothesis. Suppose one wishes to verify that $d\omega$ belongs to $A_\infty(d\sigma)$ on a domain $\Omega \subset \mathbb{R}^n$. Then, it suffices to prove that (2.9) holds on any Lipschitz domain which is a subdomain of Ω. This is apparent from the construction in Dahlberg's paper.

An application of Theorem 2.9 which yields a new result follows in the next section. Theorem 2.9 may also be applied to Laplace's equation in Lipschitz domains to prove that harmonic measure is an A_∞ weight relative to surface measure. This conclusion is not, of course, the sharp result, but the argument is fairly elementary. First, the results of [DKPV] show that $\int_{\partial\Omega} S^2(u)d\sigma \leq c \int_{\partial\Omega} N^2(u)d\sigma$ for $\Delta u = 0$ in Ω, and also that

$$\int_{\partial\Omega} u^2 d\sigma \leq c \int_{\partial\Omega} S^2(u)d\sigma + c \left(\int_{\partial\Omega} S^2(u)d\sigma \right)^{\frac{1}{2}} \cdot \left(\int_{\partial\Omega} N^2(u)d\sigma \right)^{\frac{1}{2}}$$

for (normalized) solutions $\Delta u = 0$ in Ω. Then, the stopping time argument for 3.15 of the next section shows how to get $\int_{\partial\Omega} N^2(u)d\sigma \leq c \int_{\partial\Omega} S^2(u)d\sigma$ from this latter inequality.

3. Non-symmetric Elliptic Equations in $\mathbb{R}^2$.

Our aim in this section is to prove the following theorem, by showing that the square function estimates of Theorem 2. hold.

Theorem 3.1. *Let $L = div A\nabla$ be an elliptic operator in $\mathbb{R}^2$ with bounded measurable coefficients. Suppose that there exists a fixed unit vector $\vec{e}$ such that $A(x,t) = A((x,t)\cdot\vec{e})$. Then, the elliptic measure dw_L belongs to $A_\infty(\partial\Omega, d\sigma)$ on any bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$.*

The theorem has an interesting corollary, pointed out to us by L. Escauriaza. In dimension 2, if $L = \Sigma a_{ij}(x) D_i D_j$ is a non-divergence form operator, symmetric and elliptic, then $Lu = 0$ is equivalent to $\widetilde{L}u = 0$, where $\widetilde{L}$ is a (non-symmetric) elliptic operator in divergence form. Thus, in two dimensions, the Dirichlet problem for such symmetric non-divergence elliptic operators (L^∞ coefficients but independent of the variable) is solvable with data in $L^p(\partial\Omega)$ for some p.

The theorem 3.1 is sharp in the sense that A_∞ is the best possible conclusion. The example which shows this is as follows

(3.2) Example for Poor regularity of the harmonic measure.

Let H be the upper half plane in $\mathbb{R}^2$ given by $t > 0$, where $z = (x,t)$ is a point of $\mathbb{R}^2$. Consider the problem:

$$u_{tt} + u_{xx} + D_t m D_x u - D_x m D_t u = 0$$

with Dirichlet boundary data and $m(x) \in L^\infty$. Thus, u is a weak solution if for all $\varphi \in \overset{\circ}{W}{}^{1,2}$, $\int_H (u_t + m u_x)\varphi_t + (u_x - m u_t)\varphi_t = 0$. Let $L = \operatorname{div} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \nabla$ denote the operator from $\overset{\circ}{W}{}^{1,2}$ to $W^{-1,2}$ and $\bar{L}$ its adjoint.

Let $G(z,\widetilde{z})$ denote the Green's function for L, i.e.,

$$L \int_H G(z,\widetilde{z}) f(\widetilde{z}) d\widetilde{z} = f(z)$$

and

$$\bar{L} \int_H G(z,\widetilde{z}) g(z) dz = g(\widetilde{z}),$$

so that harmonic measure for L at z in H is given by $D_t G(z,(x,t))|_{t=0}$. Let

$$m(x) = \begin{cases} -k, & \text{for } x < 0 \\ k, & \text{for } x > 0 \end{cases}$$

where k is some constant and denote this operator L_k, and its adjoint by L_{-k}.

Theorem (3.2.1). *The harmonic measure $d\omega^z$ is given by $h(x)dx$ where there exists a $c > 0$ such that $c^{-1} \le h(x)|x|^\beta \le c$ for $\beta = \frac{2\arctan k}{\pi}$ and for $|x| < 1$.*

Remark. As a corollary of the theorem, we see that A_∞ is the strongest conclusion one can draw since $\beta \to 1$ as $k \to \infty$.

Proof of Theorem (3.2.1). The theorem follows from the comparison principle and the computation of an explicit solution to L_{-k} in H, which is zero at $t = 0$. We claim that if $\alpha = 1 - \beta$, where $\beta = \beta(k)$ is defined in (3.2.1), then

$$u(x,t) = \operatorname{Im} \begin{cases} (x+it)^\alpha, & \text{for } x > 0 \\ (-x+it)^\alpha, & \text{for } x < 0 \end{cases}$$

satisfies $L_{-k}u = 0$ in H. The computation is simplified by the following observations:

(1) Any solution to $L_{-k}(L_k)$ is harmonic in the quarter planes $\{x > 0, t > 0\}$ and $\{x < 0, t > 0\}$.
(2) Any solution which is 0 at $t = 0$ is smooth in these quarter planes up to the boundary if one omits $(0,0)$. This can be seen by writing the problems as a system of elliptic equations for which the regularity is standard.

Thus, u is a solution to the adjoint problem if and only if u is harmonic in the quarter planes, smooth up to the boundary (omitting $(0,0)$), continuous at $t = 0$ and satisfies the transmission condition:

$$[u_x^- - u_x^+] - 2ku_t = 0, \text{ on } \{x = 0\}.$$

This latter condition follows from

$$\begin{aligned} 0 = &\int_{H^-} (u_t + ku_x)\varphi_t + (u_x - ku_t)\varphi_x \\ &+ \int_{H^+} (u_t - ku_x)\varphi_t + (u_x + ku_t)\varphi_x \\ = &\int_{\mathbb{R}^+} [(u_x^- - u_x^+) - 2ku_t]\varphi dt \end{aligned}$$

Then, to complete the proof of (3.2.1), we compute the derivatives of u at $x = 0$:

$$\begin{aligned} u_x^- &= -\alpha \operatorname{Im}(i^{\alpha-1}t^{\alpha-1}) \\ u_x^+ &= \alpha \operatorname{Im}(i^{\alpha-1}t^{\alpha-1}) \\ u_t &= \alpha \operatorname{Im}(i^\alpha t^{\alpha-1}) \end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}(i^{\alpha-1}) &= \sin((\alpha-1)\pi/2)\\ &= -\sin(\beta\pi/2),\\ \operatorname{Im}(i^{\alpha}) &= \cos((\alpha-1)\pi/2)\\ &= \cos(\beta\pi/2).\end{aligned}$$

Hence, u is a solution if and only if $k = \tan(\beta\pi/2)$.

Our strategy for proving (3.1) is to establish the L^2 norm equivalence of the non-tangential maximal function (N) and the square function (S) of solutions to L on any bounded Lipschitz domain. The proof is complicated so we outline the main steps below. The precise statements can be found in the lemmas which follow the outline. Step 1 requires the most work, and much of what follows is devoted to its proof. Without loss of generality, assume $A(x,t) = A(x)$ from now on.

Step 1. We prove a localized version of the L^2 equivalence in the special case where:

(i) Ω is the domain above a graph.
(ii) the graph which gives the boundary of Ω is Lipschitz with respect to some coordinate system (i.e., in any direction).
(iii) the matrix A is upper triangular.
(iv) the Lipschitz constant of the graph is small.

By 'localized version' we shall mean an integral over a portion of the boundary, and there will be error terms of lower (estimable) order. Thus there are three assumptions to be removed: The fact that A is triangular, that the boundary is a graph of a single function, and that the Lipschitz constant is small.

Step 2: The L^2 norm equivalence between (N) and (S) is established for solutions in any bounded Lipschitz domain (with *small* Lipschitz constant) to $L = \operatorname{div} A\nabla$, when A is upper triangular.

That is, we remove the restriction that $\partial\Omega$ is a graph.

Step 3: On any bounded Lipschitz domain, with arbitrary Lipschitz constant, the L^p (for any $0 < p < \infty$) equivalence between (N) and (S) is established for solution to $L = \operatorname{div} A\nabla$ with A upper triangular. The 'build-up scheme' of G. David [Da] is used here to remove the restriction on the smallness of the Lipschitz constant.

Step 4: Establish the L^p estimates of step 3 for A upper triangular for solutions above a graph (in any coordinate system), with arbitrary Lipschitz constant.

We remark that Step 4 differs from Step 1 in that the Lipschitz constant need not be small. Because the proof here uses good-λ inequalities, we needed to first establish the L^p estimates on all bounded domains (Step 3).

Step 5: Establish the results of step 4 for any A as in theorem 3.1, but only for graphs with small Lipschitz constant. That is, the restriction that A be upper triangular is removed, but only with this extra assumption. This change-of-variable argument uses two dimensions in a crucial way. It may therefore be possible to prove higher dimensional analogs of Theorem 3.1 for matrices of a special form, obviating the need for this special change of variable.

Step 6: Establish the results of Step 1 for general matrices A. This is a localized version of the results of Step 5.

Step 7: The arguments of Step 2 may be repeated to show the result of Step 3, but for general matrices A, completing the proof.

We are assuming that for $(x,t) \in \mathbb{R}^2$ and that $A(x,t) = A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ is real and elliptic: $\exists \lambda$ s.t. $A(x)\binom{\xi}{\eta} \cdot \binom{\xi}{\eta} \geq \lambda^{-1}(|\xi|^2_|\eta|^2)$ and $\|A\|_\infty \leq \lambda$. Then u is a solution of $L = \operatorname{div} A\nabla$ in $\Omega \subseteq \mathbb{R}^2$ if $\int_\Omega A\nabla u \cdot \nabla\varphi = 0$ $\forall \varphi \in Lip_0(\Omega)$. We shall make use of various changes of variables in what follows and so we record here how such changes of variables transform solutions. Suppose $\operatorname{div} A\nabla u = 0$ in Ω and $\Phi : \widetilde{\Omega} \to \Omega$ is the change of variables $\Phi(z,s) = (\Phi_1(z,s), \Phi_2(z,s))$ for $(z,s) \in \widetilde{\Omega}$, $\Phi(z,s) = (x,t) \in \Omega$. Define $v(z,s) = u \circ \Phi$ in $\widetilde{\Omega}$, and denote $D\Phi(z,s) = \begin{pmatrix} \Phi_{1,z} & \Phi_{2,z} \\ \Phi_{1,s} & \Phi_{2,s} \end{pmatrix}$, $J\Phi(z,s) = |\det D\Phi|$. Then $dx\,dt = J\Phi(z,s)dz\,ds$, $\nabla u \circ \Phi = (D\Phi)^{-1}\nabla v$ and changing variables in (3.2) one obtains:

$$0 = \int_{\widetilde{\Omega}} A \circ \Phi \cdot (D\Phi)^{-1}\nabla v (D\Phi)^{-1}\nabla(\varphi \circ \Phi)|J\Phi| dz\,ds.$$

That is, $\operatorname{div} B\nabla v = 0$ in $\widetilde{\Omega}$ where $B = |J\Phi|(D\Phi^{-1})^t A \circ \Phi (D\Phi)^{-1}$.

Definition 3.3. *Let $\vec{e}$ be a unit vector and $\vec{e}_\perp$ be a unit vector orthogonal to $\vec{e}$. A* Lipschitz graph domain in the direction $\vec{e}$ *is a domain Ω of the form*

$$\{(x,t) : \vec{e}_\perp \cdot (x,t) > \varphi((x,t).\vec{e})\} = \Omega_{\vec{e},\varphi},$$

where φ is Lipschitz ($\|\nabla\varphi\|_\infty \leq M$).

We shall generally assume, where convenient and without loss of generality that $\varphi(0) = 0$. We shall first argue that it is possible to consider three special choices of $\vec{e}$ above and consider only those special domains $\Omega_{\vec{e},\varphi}$ corresponding to these choices. To use this reduction in each of the steps above, we shall need to prove that there is no harm in simultaneously assuming that the Lipschitz constant of φ is small.

Lemma 3.4. *Given a graph φ and associated direction $(e_1, e_2) = \vec{e}$, $e_1, e_2 \geq 0$, then for $\epsilon > 0$ and $\epsilon' > 0$, there exists a $\delta > 0$ $\delta = \delta(\epsilon)$ such that*

(i) *If $\|\varphi'\|_\infty \leq \epsilon/4$ and $e_2 \leq \delta e_1$ then $\Omega_{\vec{e},\varphi} = \Omega_{(1,0),\psi}$ where $\|\psi'\|_\infty \leq \epsilon$.*

(ii) *If* $\|\varphi'\|_\infty \le \epsilon/4$ *and* $e_1 \le \delta e_2$*, then* $\Omega_{\vec{e},\varphi} = \Omega_{(0,1),\psi}$ *where* $\|\psi'\|_\infty \le \epsilon$.
(iii) *If* $e_1 \ge \delta e_2$, $e_2 \ge \delta e_1$ *and* $\|\varphi'\|_\infty \le \epsilon'\delta^3/3$*, then*

$$\Omega_{\vec{e},\varphi} = \{e_1 t > e_2 x + \psi(x)\}$$

where $\|\psi'\|_\infty \le \epsilon'$.

We first note that the restriction $e_1, e_2 \ge 0$ of the Lemma is eliminable. For suppose $\Omega_{\vec{e},\varphi}$ is given with, say, $e_1 < 0$ and $e_2 \ge 0$. Let $\Phi(z,s) = (-z,s) = (x,t) \in \Omega$ be a map $\Phi : \widetilde{\Omega} \mapsto \Omega$; that is, $\widetilde{\Omega} = \{(z,s) : e_1 \cdot (-z,s) \ge \varphi((-z,s)\cdot\vec{e})$. Then if $\vec{\alpha} = (-e_1, e_2)$, we have $\widetilde{\Omega} = \{(z,s) : \alpha_\perp \cdot (z,s) \ge \varphi((z,s)\cdot\vec{\alpha})\}$. Observe that the Lipschitz constant remains unchanged and that the structure of the matrix A in $\operatorname{div} A\nabla$ (as well as the size of its coefficients) is not changed by such a transformation.

Proof of Lemma 3.4 Let $\epsilon > 0$ be given and $\|\varphi'\|_\infty < \epsilon/2$. For $\delta > 0$ to be determined, assume first that $\vec{e} = (e_1, e_2)$ satisfies $e_2 \le \delta e_1$. Then $1 = e_1^2 + e_2^2 \le (1+\delta^2)e_1^2$. We search for $\psi = \psi(x)$ with $\|\psi'\|_\infty < \epsilon$ such that $\Omega_{\vec{e},\varphi} = \Omega_{(1,0)\psi}$, i.e.

$$e_\perp \cdot (x,t) = \varphi((x,t)\cdot\vec{e}) \text{ if and only if } t = \psi(x). \tag{3.5}$$

To solve for $\psi(x)$, let $h(x) = e_1 x + e_2\psi(x)$. Then (3.5) is the condition $-e_2 x + e_1\psi(x) = \varphi(h(x))$. If h is $1-1$, then $-e_2 h^{-1}(x) + e_1 \psi\circ h^{-1}(x) = \varphi(x)$. Also, from the definition of h, $x = e_1 h^{-1} + e_2 \psi\circ h^{-1}$ and therefore, $-e_2 h^{-1} + \frac{e_1}{e_2}(x - e_1 h^{-1}) = \varphi$, or $h^{-1}(x) = e_1 x - e_2\varphi(x)$. Because $\|\varphi'\|_\infty < \epsilon/2$, $\|(h^{-1})'\|_\infty \ge (1-\delta\epsilon/2)e_1 > 0$ when $\delta < 1$ and so h^{-1} is increasing. This determines h and hence ψ since $e_1\psi(x) = e_2 x + \varphi\circ h(x)$. And $\|\psi'\|_\infty \le \frac{e_2}{e_1} + \|\varphi'\|_\infty \cdot \|h'\|_\infty \le \epsilon$ as long as $\delta < \epsilon/2$, and $(1-\delta\epsilon/2)e_1 \ge \frac{(1-\epsilon^2/4)}{\sqrt{1+\delta^2}} \ge \frac{1}{2}$.

The case $e_1 \le \delta e_2$ with $\delta < \epsilon/2$ results in $\Omega_{\vec{e},\varphi} = \Omega_{(0,1),\psi}$ for a ψ satisfying $\|\psi'\|_\infty < \epsilon$. So we consider now the case where $e_2 > \delta e_1$ and $e_1 > \delta e_2$. Then $1 = e_1^2 + e_2^2 \le e_i^2(1+\delta^{-2})$ implies that both e_1 and e_2 are larger than $\frac{1}{(1+\delta^{-2})^{\frac{1}{2}}}$. In this case, we claim that there exists a ψ s.t.

$$e_\perp \cdot (x,t) = \varphi((x,t)\cdot\vec{e}) \text{ if and only if } e_1 t = e_2 x + \psi(x). \tag{3.6}$$

Condition (3.6) says that ψ must be defined by

$$\begin{aligned}\psi(x) &= \varphi(e_1 x + \frac{e_2^2}{e_1}x + \frac{e_2}{e_1}\psi(x)) \\ &= \varphi(\frac{1}{e_1}x + \frac{e_2}{e_1}\psi(x)).\end{aligned}$$

Let $h(x) = \frac{1}{e_1}x + \frac{e_2}{e_1}\psi(x)$. Then $\psi = \varphi \circ h$, or $\psi \circ h^{-1} = \varphi$. Since $x = h \circ h^{-1}(x) = \frac{1}{e_1}h^{-1} + \frac{e_2}{e_1}\psi \circ h^{-1}$, solving for h^{-1}, we find that $h^{-1}(x) = e_1 x - e_2\varphi(x)$ and $(h^{-1})'(x) = e_1 - e_2\varphi'(x)$. Then

$$\begin{aligned}(h^{-1})' &\geq \frac{1}{(1+\delta^{-2})^{1/2}} - \|\varphi'\|_\infty \\ &= \frac{\delta - \|\varphi'\|_\infty}{(1+\delta^2)^{1/2}} \geq \frac{\delta - \epsilon'\delta^3/3}{(1+\delta^2)^{1/2}} \\ &\geq 0.\end{aligned}$$

and so

$$\begin{aligned}\|\psi'\|_\infty &\leq \|\varphi'\|_\infty \|h'\|_\infty \\ &\leq \frac{\epsilon'\delta^3}{3}\frac{\sqrt{2}}{\delta}\frac{3}{3-\epsilon'\delta^2} = \frac{\sqrt{2}\delta^2\epsilon'}{3-\epsilon'\delta^2},\end{aligned}$$

which is less than ϵ' when also $\sqrt{2}\delta^2/3 - \delta^2\epsilon' \leq 1$, i.e. $\delta < 1$.

Remark on Approximation arguments In carrying out the steps of the argument to come, in particular in Step 1, we may assume that the solutions are a priori smooth and that the coefficients of the matrix are smooth. For if A is elliptic (but not necessarily symmetric) and $\{A_j\}$ is a smooth approximating sequence to A, i.e., $A_j \to A$ and A_j has C^∞ coefficients, then $dw_j^X \to dw^X$ weakly as measures, and uniformly for X in compact subsets. Thus if dw_j is shown to belong to $A_\infty(d\sigma)$, uniformly in j, then dw will also. The convergence of the approximating measures dw_j to dw was proven in Section 7 of [KP1], under the assumption that A was symmetric. This assumption can be eliminated, and all the lemmas there will hold in our non-symmetric case once the following is established.

Approximation Lemma. *Let $A_j \to A$ a.e. and in L^2, and suppose u_j, $u \in \overset{\circ}{W}{}^2_1(\Omega)$ are such that $L_j u_j \equiv div\, A_j \nabla u_j = div\, A_j \nabla f$ and $Lu \equiv div\, A\nabla u = div\, A\nabla f$, for $f \in Lip(\bar{\Omega})$, then $\int A\nabla u_j \cdot \nabla u_j \to \int A\nabla u \cdot \nabla u$.*

Proof. Consider

$$\int A\nabla u_j \cdot \nabla u_j = \int A_j \nabla u_j \cdot \nabla u_j + \int (A - A_j)\nabla u_j \cdot \nabla u_j.$$

To bound the second integral above, we use the fact that there exists a $p_0 > 2$ such that $u_j \in \overset{\circ}{W}{}^{p_0}_1$ uniformly in j (see Lemma 7.1 of [KP1]), obtaining, by Hölder's inequality,

$$|\int (A - A_j)\nabla u_j, \nabla u_j| \leq (\int |A - A_j|^{p_0'} |\nabla u_j|^{p_0'})^{1/p_0'} \cdot (\int |\nabla u_j|^{p_0})^{1/p_0}$$

and a further use of Hölder's inequality on the integral with $|A-A_j|^{p_0'}$ shows that this tends to zero as $j \to \infty$. Then

$$\int A_j \nabla u_j \cdot \nabla u_j = \int A_j \nabla f \cdot \nabla u_j$$
$$= \int (A_j - A)\nabla f \cdot \nabla u_j + \int A \nabla f \cdot \nabla u_j.$$

Again, the first integral tends to zero as $j \to \infty$ and $\int A\nabla f \cdot \nabla u_j \to \int A\nabla f \cdot \nabla u$ because u_j tends weakly to u in $\overset{\circ}{W}{}^2_1$, and indeed each derivative $D_{x_k} u_j$ tends weakly in L^2 to the corresponding derivative $D_{x_k} u$.

To see this, note that for any $\varphi \in \overset{\circ}{W}{}^2_1(\Omega)$ $\int A\nabla u_j \cdot \nabla \varphi \to \int A \nabla u \cdot \nabla \varphi$ and by Lax-Milgram this convergence suffices to conclude that $u_j \rightharpoonup u$ weakly in $\overset{\circ}{W}{}^2_1(\Omega)$, i.e., $\int \nabla u_j \cdot \nabla \psi \to \int \nabla u \cdot \nabla \psi$ as $j \to \infty$. The component-wise convergence of ∇u_j follows from the fact that, by passing to a subsequence, the uniform boundedness in L^2 of $|D_{x_k} u_j|$ insures weak convergence and the weak limit must then be $D_{x_k} u$.

It also suffices, for the simple convergence of the measures dw_j to dw, to argue that a subsequence of solutions u_j converges in $C^\alpha(\bar{\Omega})$ norm to u, and hence uniformly on compact sets. This follows, in dimension $n = 2$, simply from the compactness of the embedding of $\overset{\circ}{W}{}^{p_0}_1$ in $C^\alpha(\bar{\Omega})$ for some $\alpha > 0$.

We now begin **Step 1** in the proof of Theorem (3.1). We assume that the matrix A has coefficients independent of the t-variable and is upper triangular and elliptic; that is, $A = \begin{pmatrix} 1 & b(x) \\ 0 & \gamma(x) \end{pmatrix}$. There are two inequalities to prove for the equivalence in norm of the expressions $N(\cdot)$ and $S(\cdot)$ on three different types of graphs. The first result is a *localized* version of the domination of N by S in L^2 for solutions above graphs $t = \varphi(x)$, where φ is Lipschitz and satisfies $\|\varphi'\|_\infty < \epsilon$. The expressions $N_{(a,d)}$ and $S_{(a,d)}$ denote, as usual, N and S defined with respect to cones Γ^d_a of aperture a and truncated at height d. And Δ_r denotes a surface ball on the graph of $t = \varphi(x)$ of radius r centered at the origin $(0, \varphi(0))$.

Theorem 3.7. *Let $\mathscr{O} = \{(x,t) : |x| < 2,\ \varphi(x) < t < \varphi(x) + 2\}$ and suppose that $Lu = div A\nabla u = 0$ in $\mathscr{O}$. There exists $\epsilon > 0$ so that there are constants $C_1, a = a(\epsilon)$ and $C_2 = C_2(a)$ such that*

$$\int_{\Delta_{1/4}} N^2_{(a,1/2)}(u) d\sigma \le C_2 \int_{\Delta_{7/8}} S^2_{(4a,3/2)}(u) d\sigma + C_2 \iint_K u^2 dX, \tag{3.7.1}$$

where K is a compact subset of $\mathscr{O}$ at distance $\frac{1}{4}$ from the graph of φ.

The theorem will follow from a stopping time argument, a localization, and good-λ inequalities via the next lemma.

Lemma 3.8. *Let u be a solution to L in $\mathscr{O}$, as in Theorem 3.7. Then there exists an $a_0 = a_0(\epsilon)$ such that for all $a > a_0$ and any $\alpha > 0$, there is a compact set $K \subset\subset \mathscr{O}$ so that*

$$\begin{aligned}(3.8.1)\quad \int_{\Delta_{1/2}} u^2 d\sigma \le C_\alpha \int_{\Delta_{3/4}} S^2_{(4a,1)}(u) d\sigma + C_\alpha \iint_K u^2 dX \\ + (\alpha + C\|\varphi'\|_\infty^{1/2}) \int_{\Delta_{3/4}} N^2_{(a,\alpha)}(u) d\sigma \\ + C_\alpha \left(\int_{\Delta_{3/4}} S^2_{(4a,1)}(u) d\sigma \right)^{\frac{1}{2}} \cdot \left(\int_{\Delta_{3/4}} N^2_{(a,\alpha)}(u) d\sigma \right)^{\frac{1}{2}}.\end{aligned}$$

Proof of Lemma 3.8: We remark that the apertures a, $4a$ of $N(\cdot)$ and $S(\cdot)$ depend on the Lipschitz constant. The truncation α of $N(\cdot)$ can be chosen arbitrarily small—and will be chosen to depend on a in order to prove Theorem 3.7. It may also be assumed a priori that the coefficients of A are smooth.

Let $\theta(x)$ be a C^∞ bump function supported in $\{|x| < \frac{3}{4}\}$ with $\theta \equiv 1$ when $|x| < \frac{1}{2}$ and let $\mu(t)$ be C^1 with support in $|t| < 2\alpha$, $\mu \equiv 1$ when $|t| < \alpha$. Let $\rho : \mathbb{R}^2_+ \to \{t > \varphi(x)\}$ be defined by $\rho(z,s) = (z, F(z,s))$ where $F(z,s) = s + \eta_s * \eta_s * \varphi$. This is a variant on the Dahlberg-Kenig-Stein adapted distance function [D3]. The C^∞ function η is an approximate identity, supported in the interval $\{|x| < \frac{1}{2}\}$—so that $(z,0) \mapsto (z, \varphi(z))$. Set $v = u \circ \rho$. Set $G(z,s) = \eta_s * \eta_s * \varphi$. Then v verifies a divergence form equation, namely $\operatorname{div} B \nabla v = 0$ where

$$B = \begin{pmatrix} 1 + G_s & -G_z + b \\ -G_z & \frac{G_z^2 - G_z b + \gamma}{1 + G_s} \end{pmatrix}$$

and b and γ depend only on z.

Observe that $G_z^2 - G_z b + \gamma = \begin{pmatrix} 1 & b \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} -G_z \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -G_z \\ 1 \end{pmatrix} \ge \lambda(1 + G_z^2)$ and so we can bound $\int_{\Delta_{1/2}} u^2 d\sigma$ from above by

$$\begin{aligned}
&\int [G_z^2 - G_z b + \gamma]\theta(z)\mu(0)v^2 dz \\
&= -\int_{s=0}^{\infty}\int_{\mathbb{R}} D_s([G_z^2 - G_z b + \gamma]\theta(z)\mu(s)v^2(z,s))dz\, ds \\
&= -\int_0^{\infty}\int_{\mathbb{R}} [G_z^2 - G_z b + \gamma]\theta(z)\mu'(s)v^2 dz\, ds \\
&\quad -\int_0^{\infty}\int_{\mathbb{R}} [G_z^2 - G_z b + \gamma]\theta(z)\mu(s)2v\, D_s v\, dz\, ds \\
&\quad -\int_0^{\infty}\int_{\mathbb{R}} D_s[G_z^2 - G_z b + \gamma]\theta(z)\mu(s)v^2(z,s)dz\, ds \\
&= ① + ② + ③.
\end{aligned}$$

The term ③ is the delicate one—we leave this argument for last. Term ① is bounded by $C \iint_{K_0} v^2 dz\, ds$ where $C = C(\lambda)$ and $K_0 = \operatorname{supp}(\mu')$. Any such expression, in turn, can be bounded by

$$\iint_K v^2 dz\, ds + \int_{|x|<3/4} S^2(v)dx,$$

where K is at a fixed distance from the boundary $\{s = 0\}$, say distance $\frac{1}{4}$ by appropriately choosing the aperture of $S(\cdot)$. This (fairly standard) uses a variant of Poincaré's inequality to introduce derivatives of v—see [St, p. 213], for the argument for harmonic functions, and substitute interior estimates for the mean value property.

In term ②, we introduce the expression $1 = D_s(s)$ in order to integrate by parts:

$$\begin{aligned}
② &= -\int_0^{\infty}\int_{\mathbb{R}} 2[G_z^2 - G_z b + \gamma]\theta\mu v\, D_s(v)D_s(s)dzds \\
&= \int_0^{\infty}\int_{\mathbb{R}} 2vD_s v\theta\mu D_s[G_z^2 - G_z b + \gamma]sdzds \\
&\quad + \int_0^{\infty}\int_{\mathbb{R}} 2vDsv\theta\mu'[G_z^2 - G_z b + \gamma]sdzds \\
&\quad + \int_0^{\infty}\int_{\mathbb{R}} 2vD_{ss}v\theta_u[G_z^2 - G_z b + \gamma]sdzds \\
&\quad + \int_0^{\infty}\int_{\mathbb{R}} 2(D_s v)^2\theta\mu[G_z^2 - G_z b + \gamma]sdzds \\
&= ②_a + ②_b + ②_c + ②_d.
\end{aligned}$$

The integral $②_d$ is dominated by a square function expression:

$$
\begin{aligned}
②_d &\le C \iint \theta\mu s |D_s v|^2 dz ds \\
&\le C \iint_{X\in\mathscr{O}} \delta(X)|\nabla u(X)|^2 dX \le C\int_{\Delta_{3/4}} S^2(u) d\sigma.
\end{aligned}
$$

We shall often suppress the dependence of the constant C on the ellipticity of the matrix and of the apertures of $S(\cdot)$ and of $N(\cdot)$ on the constants $a = a(\epsilon)$ and the truncation. It will be important, in the stopping time argument which follows, to keep track of them however.

The Cauchy-Schwarz inequality guarantees that the term $②_b$, is bounded by $(\iint \theta\mu s|D_s v|^2 dz\, ds)^{\frac{1}{2}} \cdot (\iint_K v^2 dz\, ds)^{\frac{1}{2}}$, where K is a compact subset of $\rho^{-1}(\mathscr{O})$. For $②_a$ we use the fact that $s|G_{zs}|^2$ is a Carleson measure to bound this integral by

$$
\begin{aligned}
&C\left(\iint s|D_s v|^2 \theta\mu dz\, ds\right)^{\frac{1}{2}} \cdot \left(\iint v^2\theta\mu s|G_{zs}|^2 dz\, ds\right) \\
&\le C\left(\iint_{X\in G} \delta(x)|\nabla u(x)|^2 dX\right)^{\frac{1}{2}} \cdot \left(\int_{\Delta_{3/4}} N^2(u)d\sigma\right)^{\frac{1}{2}} \\
&\le C\left(\int_{\Delta_{3/4}} S^2(u)d\sigma\right)^{\frac{1}{2}} \cdot \left(\int_{\Delta_{3/4}} N^2(u)d\sigma\right)^{\frac{1}{2}}.
\end{aligned}
$$

For term $②_c$, we use the equation that v satisfies. Since

$$
\begin{aligned}
&D_s\left(\frac{[G_z^2 - G_z b + \gamma]}{G_s+1} D_s v\right) \\
&= \frac{1}{G_s+1} D_s([G_z^2 - G_z b+\gamma]D_s v) - \frac{G_{ss}}{(G_s+1)^2}[G_z^2 - G_z b + \gamma]D_s v,
\end{aligned}
$$

we have

$$
\begin{aligned}
D_{ss}v[G_z^2 - G_z b + \gamma] &= (G_s+1)D_s\left(\frac{[G_z^2 - G_z b+\gamma]}{G_s+1}D_s v\right) \\
&\quad - D_s[G_z^2 - G_z b+\gamma]D_s v + \frac{G_{ss}}{G_s+1}[G_z^2 - G_z b+\gamma]D_s v.
\end{aligned}
$$

The last two summands contain terms which are handled exactly as in $②_a$ above, and so we consider how the equation transforms the first summand above.

$$
\begin{aligned}
(3.9)\qquad &(G_s+1)D_s\left(\frac{[G_z^2 - G_z b+\gamma]}{G_s+1}D_s v\right) \\
&= -(G_s+1)D_z([G_s+1]D_z v) - (G_s+1)D_z((-G_z+b)D_s v) \\
&+ (G_s+1)D_s(G_z D_z v).
\end{aligned}
$$

Inserting this expression into the integral in ②$_c$ yields three expressions in which we integrate by parts. The first is

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}} 2sv(z,s)(G_s+1)D_z([G_s+1]D_z v)\theta\mu \, dz\, ds \\
=& \int_0^\infty \int_{\mathbb{R}} 2[G_s+1]D_z v \; D_z\{\theta\mu(G_s+1)v\} s\, dz\, ds.
\end{aligned}
$$

This gives rise to terms bounded by $\int S^2(u)d\sigma$ (when D_z falls on v), and to a product $(\int N^2(u)d\sigma)^{\frac12} \cdot (\int S^2(u)d\sigma)^{\frac12}$ (when D_z falls on G and one invokes the Carleson measure property of $|\nabla\nabla G|^2 s\, dz\, ds$). There is also a term of the form

$$
\begin{aligned}
(3.10) \qquad & \int_0^\infty \int_{\mathbb{R}} 2sv\theta'(z)\mu(s)(G_s+1)^2 D_z v\, dz\, ds \\
& \le C\left(\iint s|\theta'|\mu v^2 dz\, ds\right)^{\frac12} \cdot \left(\iint s|D_z v|^2\theta'\mu dz\, ds\right) \\
& \le C\sqrt{\alpha}\left(\int_{|z|\le 3/4} N^2(v)dz\right)^{\frac12} \cdot \left(\int_{|z|\le 3/4} S^2(v)dz\right)^{\frac12}
\end{aligned}
$$

since $s \le 2\alpha$. (We remark that the non-tangential maximal functions $N(v)$, $N(u)$ above may be assumed to be truncated at height α, as the difference is absorbed in the error terms $\iint_K v^2$, together with the integrals involving square functions.

The second term from the right hand side of (3.9) is

$$
\begin{aligned}
& -\iint 2sv\theta\mu(G_s+1)D_z((-G_z+b)D_s v)dz\, ds \\
= \quad & \iint 2s(D_s v\cdot D_z v)^2(G_s+1)(-G_z+b)\theta\mu dz\, ds \\
& +\iint 2sv\theta\mu G_{sz}(-G_z+b)D_s v\, dz\, ds \\
& +\iint 2sv\theta'\mu(G_s+1)(-G_z+b)D_s(v)dz\, ds.
\end{aligned}
$$

The first integral above is bounded by $\int S^2(v)dx$; the second integral is bounded by $C(\iint s|G_{sz}|^2 v^2\theta\mu dz\, ds)^{\frac12} \cdot (\iint s|D_s v|^2\theta\mu dz\, ds)$ which is dominated by a product of $\|N(v)\|_{L^2} \cdot \|S(v)\|_{L^2}$ via the Carleson measure property of $s|\nabla\nabla G|^2 dz\, ds$ as usual. The third integral above is handed just as (3.10).

The third term from (3.9) yields an integral:

$$\begin{aligned}&\iint 2sv\theta\mu(G_s+1)D_s(G_zD_zv)dz\,ds\\ =&-\iint 2sv\theta\mu(G_s+1)G_{zs}D_zv dz\,ds\\ &-\iint 2sv\theta\mu(G_s+1)G_zD_{zs}v\,dz\,ds.\end{aligned}$$

For the first integral above we invoke the Carleson property of G_{zs} as usual and for the second integral above we integrate by parts one more time but in the z variable. All the expressions which arise are similar to those we have handled before.

Finally, term ③ is equal to

$$\iint v^2\theta\mu[2G_zG_{sz}-bG_{sz}]dz\,ds. \tag{3.10.1}$$

Recall that $G_z=\eta_s*\eta_s*\varphi'$ and so $G_{zs}=2D_s\eta_s*\eta_s*\varphi'=2D_z(\psi_s*\eta_s*\varphi')$ where $\widehat{\psi}(\xi)=\widehat{\eta}'(\xi)$, and η is chosen to be even. Because, whenever $f\in L^\infty$, the expression $|\psi_s*f|^2\frac{dzds}{s}$ is a Carleson measure, we have

$$\begin{aligned}\iint v^2\theta\mu G_zG_{zs}&=-2\iint D_z(v^2\theta\mu G_z)\psi_s*\eta_s*\varphi'dz\,ds\\ &=-4\iint D_zv\,v\,\theta\mu G_z\psi_s*\eta_s*\varphi'dz\,ds\\ &\quad-2\iint v^2\theta'\mu G_z\psi_s*\eta_s*\varphi'dz\,ds\\ &\quad-2\iint v^2\theta\mu G_{zz}\psi_s*\eta_s*\varphi'dz\,ds.\end{aligned}$$

The first integral above is bounded by a constant times:

$$\begin{aligned}&\left(\iint|D_zv|^2s\theta\mu dz\,ds\right)^{\frac12}\cdot\left(\iint v^2\theta\mu\frac{|\psi_s*\eta_s*\varphi'|^2}{s}dzds\right)^{\frac12}\\ \le&\left(\int_{|x|<3/4}S^2(v)dx\right)^{\frac12}\cdot\left(\int_{|x|<3/4}N^2(v)dx\right)^{\frac12}.\end{aligned}$$

A bound of $\alpha\int_{\Delta_{3/4}}N^2_{a,\alpha)}(u)d\sigma$ comes from the second integral—handled like (3.10)—since $|\psi_s*\eta_s*\varphi'|\le\|\varphi'\|_\infty$, and the third integral is bounded by:

$$\begin{aligned}&\left(\iint v^2\theta\mu|G_{zz}|^2sdzds\right)^{\frac12}\cdot\left(\iint v^2\theta\mu\frac{|\psi_s*\eta_s*\varphi'|}{s}dzds\right)\\ \le&\, C\|\varphi'\|_\infty\int N^2(v)dx.\end{aligned}$$

We must use here the fact that the Carleson measure norm (of either quantity) is small (since $\|\varphi'\|_\infty < \epsilon$) because this term must ultimately be regarded as an error term in the main inequality. It remains to handle the integral $\iint v^2\theta\mu b G_{zs} dzds$ which is more delicate because a straightforward integration by parts in z is impossible as $b(z)$ is not differentiable.

The following calculation shows how the special double convolution form of the change of variable is used.

$$\begin{aligned}\iint v^2\theta\mu b G_{zs} dzds &= 2\iint v^2\theta\mu b\eta_s * D_z(\psi_s * \varphi')dzds \\ &= 2\iint \widetilde{\theta}(z)\eta_s * (v^2\theta b)\mu D_z(\psi_s * \varphi')dzds,\end{aligned} \tag{3.11}$$

where $\widetilde{\theta}$ is C^∞ and $\widetilde{\theta} \equiv 1$ in $|z| \le 3/4$ and supported in $|z| < 7/8$. We split the left hand side of (3.11) into two terms $T_1 + T_2$, where

$$T_1 = \iint v^2\mu(s)\eta_s * (\theta b)\widetilde{\theta}(z)D_z(\psi_s * \varphi')dzds$$

and

$$T_2 = \iint D_z(\psi_s * \varphi')\widetilde{\theta}(z)\mu(s)[\eta_s * (v^2\theta b) - v^2\eta_s * (\theta b)]dzds.$$

In term T_1, we integrate by parts in z obtaining

$$\begin{aligned}T_1 = &-\iint \psi_s * \varphi' 2v D_z v\mu\eta_s * (\theta b)\widetilde{\theta}(z)dzds \\ &-\iint \psi_s * \varphi' v^2\mu\widetilde{\theta}'(z)\eta_s * (\theta b)dzds + \\ &-\iint (\psi_s * \varphi')v^2\mu\widetilde{\theta}D_z(\eta_s * (\theta b))dzds.\end{aligned}$$

Since $\frac{|\psi_s * \varphi'|^2}{s}dzds$ is a Carleson measure, the first two integrals can be estimated by the usual arguments. The third integral is equal to

$$\begin{aligned}&-\iint (\psi_s * \varphi')v^2\mu\widetilde{\theta}(\eta')_s * (\theta b)\frac{1}{s}dzds \\ &\le \left(\iint \frac{|\psi_s * \varphi'|^2}{s}v^2\mu\widetilde{\theta}dzds\right)^{\frac{1}{2}} \cdot \left(\iint \frac{|(\eta')_s * (\theta b)|^2}{s}v^2\mu\widetilde{\theta}dzds\right)^{\frac{1}{2}} \\ &\le C\|\phi'\|_\infty^{\frac{1}{2}} \int_{|z|<7/8} N^2(v)dz,\end{aligned}$$

where, again, we have used the fact that the Carleson norms are small.

Setting $\widetilde{\psi} = \psi'(z)$, we have

$$|T_2| \leq \iint_{(z,s)} \frac{|\widetilde{\psi}_s * \varphi'|}{s} |\eta_s * (v^2\theta b) - v^2\eta_s * \theta b| \cdot \widetilde{\theta}\mu dz ds.$$

Since,

$$\begin{aligned}
|\eta_s * (v^2\theta b) - v^2\eta_s * (\theta b)(z)| &= \left| \int \eta_s(z'-z)\theta b(z') \cdot [v^2(z',s) - v^2(z,s)]dz' \right| \\
&\leq \left| \int \eta_s(z'-z)|\theta b|[v(z',s) - v(z,s)]^2 dz' \right| \\
&\quad + 2|v(z,s)| \left| \int \eta_s(z'-z)\theta b(z')[v(z',s) - v(z,s)]dz' \right|
\end{aligned}$$

we have $|T_2| \leq T_2^1 + T_2^2$, where

$$\begin{aligned}
T_2^2 \leq &\left(\iint_{(z,s)} \widetilde{\theta}\mu \frac{|\widetilde{\psi}_s * \varphi'|^2}{s} |v(z,s)|^2 dz ds \right)^{\frac{1}{2}} \cdot \|b\|_\infty \\
&\cdot \left(\iint_{(z,s)} \widetilde{\theta}\mu \frac{1}{s} \left| \int \eta_s(z'-z)[v(z',s) - v(t,s)]dz' \right|^2 dz ds \right)^{\frac{1}{2}}
\end{aligned}$$

by an application of the Cauchy inequality. The first integral in the above product is bounded by $\epsilon \left(\int_{|x|<3/4} N^2(v)dx \right)^{\frac{1}{2}}$ and the square of the second is bounded by

$$\begin{aligned}
&\left(\iint_{(z,s)} \widetilde{\theta}\mu \frac{1}{s} \int \eta_s(z'-z)|v(z',s) - v(z,s)|^2 dz' dz ds \right) \\
&\leq C \iint_{(z,s)} \widetilde{\theta}\mu \left(\int_{|z-z'|<2s} |\nabla v(z',s)|^2 s^2 dz' \right) dz ds \\
&\leq C \int_{(z',s)} \widetilde{\theta}\mu s |\nabla v(z',s)|^2 dz' ds \leq C \int_{|z|<3/4} S^2(v)dx.
\end{aligned}$$

This also shows how to estimate T_2^1, since

$$\begin{aligned}
T_2^1 &= \left| \iint_{(z,s)} \frac{|\widetilde{\psi}_s * \varphi'|}{s} \widetilde{\theta}\mu \left| \int \eta_s(z'-z)\theta b(z')[v(z',s) - v(z,s)]^2 dz' \right| dz ds \right| \\
&\leq \|\varphi'\|_\infty \iint_{(z,s)} \frac{\widetilde{\theta}\mu}{s} \left| \int \eta_s(z'-z)[v(z',s) - v(z,s)]^2 dz' \right| dz ds
\end{aligned}$$

and the argument proceeds as for T_2^2 After changing variables to recover the solution u, this proves the inequality of Lemma 3.8.

The proof of theorem 3.7 requires two lemmas, both of which will be used repeatedly. The first lemma is a stopping time argument which is used to prove bounds on $\|N(u)\|$, from an inequality only involving $\|u\|$.

Let us fix a graph $t = \psi(x)$—we shall make use of the fact that the inequality (3.8.1) is available for any Lipschitz graph whose Lipschitz constant is sufficiently small.

Let $Lu = 0$ in the region $\mathscr{O}$ of Theorem 3.7 and let $v(x,t) = u(x,t)\theta(x)\mu_\alpha(t-\psi(x))$, where $\mu_\alpha = \begin{cases} 1, & 0 < t < \alpha/2 \\ 0, & t > \alpha \end{cases}$, and $\theta(x) = 1$ for $|x| \le \frac{3}{4}$, $\operatorname{supp}\theta \subseteq \{|x| < \frac{7}{8}\}$, $0 \le \theta \le 1$. The constant α is to be determined, which is related to the size of the aperture of the cones used to define N and S.

Fix an integer $j > 0$, and fix an aperture $a > 0$. The choice of a will depend on $\|\psi'\|_\infty$ and will be determined later. Set $E_j = \{(x,\psi(x) : N_{(a,\alpha/2)}u(x,\psi(x)) > 2^j, S_{(4a,\beta)}(u)(x,\psi(x)) \le \rho 2^j\} \cap \{|x| < \frac{1}{4}\}$. The truncation β is chosen so that $\Gamma_{4a}^{\beta}(x,\psi(x)) \subseteq \{|x| \le \frac{3}{4}, t < 1\}$ when $|x| < \frac{1}{4}$; that is, $\beta \approx \frac{c}{a}$. Now fix $\alpha = \frac{2}{3}\beta$. The constant $\rho = \rho(a)$ will be determined later. Define

$$h_j(x) = \sup\{t \ge \psi(x) : \sup_{(z,s)\in\Gamma_a(x,t)} |v(z,s)| > 2^j\}$$

where $\Gamma_a(x,t) = \{(z,s) : |z-x| \le a(s-t), s > t\}$.

Lemma 3.13. *The function $h_j(x)$ is Lipschitz with constant $\frac{1}{a}$.*

Proof: Let x_1, x_2 be given with $x_2 > x_1$ and suppose $h_j(x_1) = t_1$. Let $\widetilde{t} = t_1 + \frac{1}{a}(x_2 - x_1)$. Since $\Gamma_a(x_2,\widetilde{t}) \subset \Gamma_a(x_1,t_1) \subseteq \{t > \psi(x)\}$, we see that $\widetilde{t} > \psi(x_2)$. Then $h_j(x_2) \le \widetilde{t}$; for if not, there would exist a cone $\Gamma_a(x_2,t_2)$, properly contained in $\Gamma_a(x_1,t_1)$, for which $|v(z,s)| > 2^j$ for some $(z,s) \in \Gamma_a(x_2,t_2)$. To see this, note that $\Gamma_a(x_2,\widetilde{t}) \subseteq \Gamma_a(x_1,t_1)$ so one may choose $t_2 \in (\widetilde{t}, h_j(x_2))$. But this would imply $h_j(x_1) > t_1$, a contradiction. Moreover, by a similar argument, if $\widetilde{\widetilde{t}} = t_1 - \frac{1}{a}(x_2 - x_1)$, then it can be seen that $h_j(x_2) > \widetilde{\widetilde{t}}$. Altogether, $|h_j(x_2) - h_j(x_1)| \le \frac{1}{a}|x_2 - x_1|$.

We now claim:

(3.14). *There exists a $\rho = \rho(a)$ such that for $x \in E_j$, there is an interval J, with $4J \subseteq \{|x| < \frac{3}{4}\}$, such that $|u(z,h_j(z))| > 2^{j-1}$ when $z \in J$ and such that $x \in 4J$.*

Let $x \in E_j$ (so that $|x| < \frac{1}{4}$ and also $N_{(a,\alpha/2)}(u)(x,\psi(x)) > 2^j$). For such x, $h_j(x) > \psi(x)$, since there exists $(z,s) \in \Gamma_a(x,\psi(x))$ such that $|v(z,s)| > 2^j$ (and we may assume that v is continuous up to the boundary). For any $\psi(x) < t < h_j(x)$, there exists a point $(z, s) \in \Gamma_a(x)$ where $|v(z, s)| > 2^j$, but for any $\delta > 0$, $|v| \leq 2^j$ in $\Gamma_a(x, h_j(x)+\delta)$. Thus $|v| \leq 2^j$ in $\mathrm{int}(\Gamma(x, h_j(x)))$ and there exists an (x_0, t_0) on $\partial\Gamma(x, h_j(x))$ such that $|v(x_0, t_0)| = 2^j$. Also, by definition of h_j, $h_j(x_0) = t_0$.

Set now $A = \{(z, s) : |z - x_0| \leq a(t_0 - \psi(x)),\ |s - t_0| \leq \frac{1}{2}(t_0 - \psi(x))$. Then $A \subseteq \Gamma^{\beta}_{4a}(x, \psi(x))$: for if $(z, s) \in A$, then $|z-x| \leq |z-x_0|+|x-x_0| \leq a(t_0 -\psi(x))+a(t_0 - h_j(x))$, since $|x-x_0| = a(t_0-h_j(x))$. Thus $|z-x| \leq 2a(t_0-\psi(x)) \leq 4a(s-\psi(x))$. And if $|s - t_0| \leq \frac{1}{2}(t_0 - \psi(x))$, then $s - \psi(x) \leq \frac{3}{2}(t_0 - \psi(x)) \leq \frac{3}{2}\alpha \leq \beta$ since $t_0 - \psi(x) \leq \alpha$ be the choice of the cut-off function μ_α.

Let $B = \{(z, s) : |z - x_0| < (a - 1)(t_0 - \psi(x)),\ |s - t_0| < \frac{1}{4}(t_0 - \psi(x))\}$. Then we shall see that $|u(z, s) - u(x_0, t_0)| \leq C\sqrt{a}(\int_A |\nabla u|^2)^{\frac{1}{2}}$. To show this, write B as a union of boxes I_j of side lengths $(t_0 - \psi(x)) \times \frac{1}{4}(t_0 - \psi(x))$ whose doubles $\widetilde{I}_j$ are contained in A and such that no more than two of the $\widetilde{I}_j$ overlap. Because u is a solution in each I_j, we have the estimate:

$$\operatorname*{osc}_{I_j}(u) \leq C(t_0 - \psi(x)) \int_{\widetilde{I}_j} |\nabla u|.$$

Adding these estimates, we obtain

$$|u(z, s) - u(x_0, t_0)| \leq \frac{C}{t_0 - \psi(x)} \int_A |\nabla u| \leq \frac{C}{(t_0 - \psi(x))} (\int_A |\nabla u|^2)^{\frac{1}{2}} \cdot \sqrt{a}(t_0 - \psi(x)).$$

Because $A \subseteq \Gamma^{\beta}_{4a}(x)$, $(\int_A |\nabla u|^2)^{\frac{1}{2}} \leq \rho 2^j$ and so if $(z, s) \in B$, then $|u(z, s)-u(x_0, t_0)| \leq C\rho\sqrt{a}2^j$. Now choose ρ so that $C\rho\sqrt{a} \leq \frac{1}{2}$. Set $J = \{z : |z - x_0| \leq \frac{1}{4}a(t_0 - \psi(x))\}$. If $z \in J$, $(z, h_j(z)) \in B$ and hence $u(z, h_j(z)) > 2^{j-1}$. ($|u(x_0, t_0)| > |v(x_0, t_0)| = 2^j$.) Also, $|x - x_0| = a(t_0 - h_j(x)) \leq a(t_0 - \psi(x))$, so that $x \in 4J \subseteq \{|x|) < \frac{3}{4}\}$, which proves claim 3.14.

Lemma 3.15. *Let Λ be the graph of a Lipschitz function $\varphi(x)$, with $\|\varphi'\|_\infty < \frac{\epsilon}{4}$, where ϵ is as in Theorem 3.7. Let $M(\cdot)$ denote the Λ-maximal function, for f defined on Λ: $Mf(x, \varphi(x)) = \sup_{\substack{x \notin I \\ I \subseteq \mathbb{R}}} \frac{1}{|I|} \int_I |f(y, \varphi(y)|dy$. If $d\sigma =$ surface measure on Λ, and with the notation of Lemma 3.8, where a denotes the aperture of the cones, we have the following good-λ inequality:*

Given $\gamma < 1$ there exists $C(\gamma)$, $C(\gamma) \to 0$ as $\gamma \to 0$ such that

$$\sigma\{N_a(u) > 2\lambda, M(S_{4a}(u)) \leq \gamma\lambda, (M(S^2_{4a}(u)))^{\frac{1}{4}} \cdot (M(N^2_a(u)))^{\frac{1}{4}} \leq \gamma\lambda,$$
$$(M((\alpha + C\|\varphi'\|_\infty)^{\frac{1}{2}} N^2_a(u)))^{\frac{1}{2}} \leq \gamma\lambda\} \leq C(\gamma)\sigma\{N_a(u) > \lambda/32\}.$$

We first state a rescaled version of Lemma 3.8. (We also need rescaled versions of 3.13 and 3.14).

(3.16). *Let $\mathcal{O}_r = \{(x,t) : |x| \le 2r,\ \psi(x) < t < \psi(x) + 2r\}$ with $\|\psi'\|_\infty \le \epsilon$, and suppose $Lu = 0$ in $\mathcal{O}_r$. Then, if A_r is any point of $\mathcal{O}_r$ with distance to $\partial\mathcal{O}_r$ approximately r, there exists an $a_0 = a_0(\epsilon)$ such that for all $a > a_0$ and for all $0 < \alpha < 1$,*

$$\int_{\Delta_{r/2}} u^2 d\sigma \le C_\alpha \int_{\Delta_r} S^2_{(4a,r)}(u) d\sigma + C_\alpha u^2(A_r) r + (\alpha + C\|\varphi'\|_\infty^{\frac{1}{2}}) \int_{\Delta_r} N^2_{(a,\alpha r)}(u) d\sigma + C_\alpha \Big(\int_{\Delta_r} S^2_{4a,r}(u) d\sigma\Big)^{\frac{1}{2}} \cdot \Big(\int_{\Delta_r} N^2_{(a,\alpha r)}(u) d\sigma\Big)^{\frac{1}{2}}. \tag{3.16}$$

This follows easily from Lemma 3.8 by rescaling. The term $\iint_{K_r} u^2 dX$, where K is a compact subset of $\mathcal{O}_r$ has been replaced by the quantity $u^2(A_r) \cdot r + \int_{\Delta_r} S^2_{(4a,r)}(\mu) d\sigma$.

We combine (3.16) with the stopping time functions $h_j(x)$ as follows; it suffices to prove (3.15) for $\lambda = 2^j$. Let $\{\Delta_i\}$ be a Whitney decomposition of $\{N_a(u) > \frac{\lambda}{32}\}$. (Notice that the cones here are infinite cones—there is no truncation parameter.) We assume $F_j \subseteq \Delta_i \ne \emptyset$, and $\gamma < \rho$ in 3.14, where F_j is the set appearing on the right hand side of the inequality in 3.15, with $\lambda = 2^j$. That is, $\Delta_i \subset \Lambda \cap B_i$, where B_i is a ball of radius r_i and there exists a point $P_i \in 2B_i$ with dist $(P_i, \Lambda) \ge r_i$ such that $|u(P_i)| \le \frac{1}{32} 2^j$. If γ is sufficiently small, then the truncated maximal function $N_{(a,\alpha/2)}(u)(x, \psi(x))$ is still larger than $2^j/2$ for $x \in \Delta_i \cap F_j$. (See [D4] or [DJK]). Let $h_j(x)$ be defined as before, relative to the domain Λ, so that if $M_j(\cdot)$ denotes the maximal function with respect to the graph of h_j, then $M_j(u\chi_{4\Delta_i})(x, h_j(x)) > 2^j/16$ when $(x, \psi(x)) \in \Lambda \cap F_j \cap \Delta_i$. In fact, if $\widetilde{u}(x) = u(x) - u(P_i)$, then $M_j(\widetilde{u}\chi_{4\Delta_i})(x, h_j(x)) > 2^j/32$. Thus, by the maximal function theorem,

$$\sigma(F_j \cap \Delta_i) \le C \left(\frac{32}{2^j}\right)^2 \int_{4\Delta_i} \widetilde{u}^2(x, h_j(x)) d\sigma_j(x).$$

We now apply (3.16) with $\Delta_{r/2} = 4\Delta_i$ (i.e. $r = 8r_i$) and observe that $\widetilde{u}(A_r) = 0$ if A_r is chosen to be P_i. Also note that $\|h_j'\|_\infty < \epsilon$. There are several terms which result as an upper bound for $2^{-2j} \int_{4\Delta_i} \widetilde{u}^2(x, h_j(x)) d\sigma_j(x)$, and the first is: $C2^{-2j} \int_{5\Delta_i} S^2_{(4a,r_i),h_j}(u) d\sigma$. Here the subscript in the $S(\cdot)$ function means that we are using the square function relative to the domain above the graph of h_j. But the cones $\Gamma_{(4a,r_i),h_j}$ used to define these square functions at a point $(x, h_j(x))$ are contained in the cones $\Gamma_{(4a,r_i)}$ at $(x, \varphi(x))$. Therefore this first term is bounded by

$$2^{-2j}\int_{4\Delta_i} S^2_{(4a)}(u)d\sigma \cdot |4\Delta_i| \leq$$
$$\leq 2^{-2j}|4\Delta_i| \cdot M(S^2_{(4a)}(u))(x_0, \varphi(x_0))$$

for any $(x_0, \varphi(x_0)) \in \Delta_i \cap F_j$, which is then less than $C|\Delta_i| \cdot \gamma^2$, by the definition of F_j and the fact that $F_j \cap \Delta_i \neq \emptyset$. The other terms which arise from (3.16) are handled in the same way, introducing maximal functions, and using the bounds on these maximal functions from the definition of F_j.

The good-λ inequality of Lemma 3.15 can be used to prove that, for any Λ, with small Lipschitz constant, depending on p, and for appropriate choice of $\gamma = \gamma(p)$, the L^p-inequality, valid for $p > 2$:

$$\|N_a(u)\|_{L^p(d\sigma)} \leq C_p\|S_{4a}(u)\|_{L^p(d\sigma)}. \tag{3.17}$$

(Here we have used the fact that $\alpha + C\|\varphi'\|_\infty^{\frac{1}{2}}$ is small.)

In order to recover the localized L^2 version of this L^p inequality we state and prove a localization theorem, a version of which will also be needed in Step 6 of the proof.

Theorem 3.18. *Let $L = div A\nabla$ and let $\Omega_{\vec{e},\varphi}$ be the domain above the graph of φ, $\|\varphi'\|_\infty < \epsilon$. Assume that estimate (3.17), as well as its converse—the domination of $\|S(u)\|_p$ by $\|N(u)\|_p$—holds on all Lipschitz graphs with Lipschitz constant bounded by 2ϵ contained in $\Omega_{\vec{e},\varphi}$, for some $p > p_0$, with p_0 depending only on ellipticity, and for any solution u to L in $\Omega_{\vec{e},\varphi}$. Let $B_1 = \{(x,t) : |(x,t)\cdot\vec{e}^\perp| < 1,\ \varphi((x,t)\cdot\vec{e}^\perp) < (x,t)\cdot\vec{e}_1 < \varphi((x,t)\cdot\vec{e}^\perp) + 1\}$, and suppose that $Lu = 0$ in $\widetilde{\Omega} = \Omega_{\vec{e},\varphi} \cap B_1$.*

Then, we have

$$\int_{\partial\widetilde{\Omega}\cap|(x,t)\cdot\vec{e}^\perp|<\frac{1}{2}} N^p(u)d\sigma \leq C\int_{\partial\widetilde{\Omega}\cap\{|(x,t)\cdot\vec{e}^\perp|<\frac{3}{4}\}} S^p(u)d\sigma + C_K \max|u|^p \tag{3.19}$$

where K is a compact subset of $\widetilde{\Omega}$.

Proof: We will assume that $\vec{e} = (1, 0)$. The proof will show that there is no loss of generality in doing this. We will also work with non-tangential maximal functions and square functions defined with respect to truncated cones which remain in $\widetilde{\Omega}$. (The truncation will be then of the order of ϵ and the aperture will depend on ϵ as well.) Consider now ψ Lipschitz, with $\varphi \equiv \psi$ on $|x| > 1$, $|x| < \frac{5}{8}$, $\|\psi'\|_\infty \leq \frac{3}{2}\epsilon$, and such that for $\frac{5}{8}+\frac{1}{16}\le |x|\le \frac{3}{4}$, we have $\varphi(x)<\psi(x)$. We now consider the domain $\tilde{\tilde{\Omega}}\subset\tilde{\Omega}$, given by

$$\tilde{\tilde{\Omega}}=\{(x,t):|x|<\frac{3}{4},\psi(x)<t<\psi(x)+\frac{3}{4}\}.$$

Let now $\tilde{\tilde{K}}=\partial\tilde{\tilde{\Omega}}\backslash\{(x,\psi(x)):|x|<\frac{5}{8}+\frac{1}{16}\}$, and note that $\tilde{\tilde{K}}\subset\subset\tilde{\Omega}$. Thus, by interior regularity we can find $\tilde{K}\subset\subset\Omega$ and $\alpha=\alpha(\lambda)>0$, so that

$$\sup_{X\in\tilde{\tilde{K}}}|u(X)|+\sup_{X,X\in\tilde{\tilde{K}}}\frac{|u(X)-u(X')|}{|X-X'|^\alpha}\le C_{\tilde{K}}\max_{\tilde{K}}|u|.$$

Fix now $\theta_1(x)$, $0\le\theta_1\le 1$, $\theta_1\in C^\infty$, with $\theta_1\equiv 1$ on $|x|\le\frac{5}{8}+\frac{1}{16}$, $\operatorname{supp}\theta_1\subset\{|x|<\frac{5}{8}+\frac{3}{32}\}$. We now split $u=u_1+u_2$, in $\tilde{\tilde{\Omega}}$, where $Lu_i=0$ in $\tilde{\tilde{\Omega}}$, and

$$u_2|_{\partial\tilde{\tilde{\Omega}}}=\begin{cases}0 & \text{on top part and lateral parts of } \partial\tilde{\tilde{\Omega}}\\ u(x,\psi(x))\cdot\theta_1(x) & \text{on } \{t=\psi(x)\}\bigcup\partial\tilde{\tilde{\Omega}}\end{cases}$$

Note that $u_1|_{\partial\tilde{\tilde{\Omega}}}\in C^\alpha(\partial\tilde{\tilde{\Omega}})$, with norm controlled by $C_{\tilde{K}}\max_{\tilde{K}}|u|$. The same follows then for u_1 in $\tilde{\tilde{\Omega}}$, by boundary regularity. From this, it easily follows that $\int_{\partial\tilde{\tilde{\Omega}}}S^p(u_1)+N^p(u_1)\le C_{\tilde{K}}\max_{\tilde{K}}|u|^p$. We now turn to estimating u_2. Let $\theta\in C_0^\infty$, $0\le\theta\le 1$ be identically 1 on $|x|<\frac{5}{8}+\frac{3}{32}+\frac{1}{256}$, with $\operatorname{supp}\theta\subseteq\{|x|<\frac{5}{8}+\frac{3}{32}+\frac{1}{128}\}$, and let $\mu\in C_0^\infty$ be supported in $|t|<\frac{1}{2}$, $\mu\equiv 1$ for $|t|<\frac{1}{4}$. Finally, let $v(x,t)=\theta(x)\mu(t-\psi(x))\cdot u_2(x,t)$, be defined in Ω_ψ. Decompose $v=v_1+v_2$, in Ω_ψ, where

$$\begin{cases}Lv_1=0 & \text{in } \Omega_\psi\\ v_1|_{\partial\Omega_\psi}=v|_{\partial\Omega_\psi}\end{cases}$$

and $Lv_2=Lv$ in Ω_ψ, with $v_2|_{\partial\Omega_\psi}\equiv 0$.

Claim (3.20). *(Here N_ψ and S_ψ denote the non-truncated versions.)*

$$\int_{\partial\Omega\psi}(N_\psi(v_2))^p+S_\psi^p(v_2)d\sigma\le C_{\tilde{F}}\sup_{\tilde{F}}|u_2|^p,$$

where $\tilde{F}\subset\bar{\tilde{\tilde{\Omega}}}$, and $\tilde{F}\subset\subset\tilde{\Omega}$.

From (3.17) it follows that $\int_{\partial\Omega_\psi}N_\psi(v_1)^pd\sigma\le C\int_{\partial\Omega_\psi}S_\psi^p(v_1)$, but

$$\begin{aligned}\int_{\partial\Omega_\psi}S_\psi(v_1)^p&\le C_p\int_{\partial\Omega_\psi}S_\psi(v_1+v_2)^p+S_\psi(v_2)^p\\ &\le C_p\int_{\partial\Omega_\psi}S_\psi(v)^p+C_p\int_{\partial\Omega_\psi}S_\psi(v_2)^p\end{aligned}$$

Now,

$$\nabla v = \theta(x)\mu(t-\psi(x))\nabla u_2 \\ + (u_2(x,t)\cdot \frac{\partial \theta}{\partial x}\mu(t-\psi(x) - \frac{\partial \psi}{\partial x}\mu'(t-\psi(x))\theta(x), u_2(x,t)\theta(x)(\mu'(t-\psi(x))).$$

Note that $\operatorname{supp} \frac{\partial\theta}{\partial x} \subset \{\frac{5}{8}+\frac{3}{32}+\frac{1}{256} < |x| < \frac{5}{8}+\frac{3}{32}+\frac{1}{128}\}$ and $\operatorname{supp}\mu' \subset \{\frac{1}{4} < |t| < \frac{1}{2}\}$. Thus, there exists a set $F \subset \overline{\overline{\widetilde{\widetilde{\Omega}}}}$, with $F \subset\subset \widetilde{\Omega}$, such that the second term in the sum is bounded by $C\sup_F |u_2|$. We then obtain

$$\int_{\partial\Omega\psi} S_\psi(v)^p \le \int_{\partial\widetilde{\widetilde{\Omega}}\cap\{(x,\psi(x)):|x|<\frac{5}{8}+\frac{3}{32}+\frac{1}{64}\}} S^p(u_2) + C\max_{\widetilde{F}} |u_2|^p,$$

where $\widetilde{F} \subset \overline{\overline{\widetilde{\widetilde{\Omega}}}}$, and $\widetilde{F} \subset\subset \widetilde{\Omega}$. A similar, but simpler, argument shows that

$$\int_{\partial\widetilde{\Omega}\cap\{(x,\varphi(x):|x|<\frac{1}{2}\}} N(u_2)^p d\sigma \le \int_{\partial\Omega_\psi} N_\psi(v)^p + C\sup_{\widetilde{F}} |u_2|,$$

where $\widetilde{F}$ is as above. Gathering our estimates, we obtain

$$\begin{aligned} \int_{\partial\widetilde{\Omega}\cap\{(x,\varphi(x)):|x|<\frac{1}{2}\}} N(u_2)^p d\sigma &\le C\sup_{\widetilde{F}} |u_2|^p + \int_{\partial\Omega_\psi} N_\psi(v_2)^p + C_p\int_{\partial\Omega_\psi} S_\psi(v_1)^p \\ &\le C_p\sup_{\widetilde{F}} |u_2|^p + C_p\int_{\partial\Omega_\psi} N_\psi(v_2)^p + C_p \int_{\partial\widetilde{\widetilde{\Omega}}\cap\{(x,\psi(x)):|x|<\frac{5}{8}+\frac{3}{32}+\frac{1}{64}\}} S(u_2)^p \\ &\quad + C_p\int_{\partial\Omega_\psi} S(v_2)^p \\ &\le C_p\sup_{\widetilde{F}} |u_2|^p + C_p \int_{\partial\widetilde{\widetilde{\Omega}}\cap\{|x|<\frac{5}{8}+\frac{3}{32}+\frac{1}{64}\}} S(u_2)^p, \end{aligned}$$

by (3.20). But then,

$$\begin{aligned}
\int_{\partial\widetilde{\Omega}\cap\{|x|<\frac{1}{2}\}} N(u)^p d\sigma &\le \int_{\partial\widetilde{\Omega}\cap\{|x|<\frac{1}{2}\}} N(u_2)^p d\sigma + \int_{\partial\widetilde{\Omega}\cap\{|x|<\frac{1}{2}\}} N(u_1)^p d\sigma \\
&\le C\sup_{\widetilde{K}}|u|^p + C\sup_{\widetilde{F}}|u_2|^p + C\int_{\partial\widetilde{\widetilde{\Omega}}\cap\{|x|<\frac{5}{8}+\frac{3}{32}+\frac{1}{64}\}} S(u_2)^p \\
&\le C\sup_{\widetilde{K}}|u|^p + C\sup_{\widetilde{F}}|u|^p + C\sup_{\widetilde{F}}|u_1|^p \\
&\quad + C\int_{\partial\widetilde{\Omega}\cap\{|x|<\frac{3}{4}\}} S(u)^p + C\int_{\partial\widetilde{\widetilde{\Omega}}\cap\{|x|<\frac{5}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\}} S(u_1)^p \\
&\le C\sup_{K}|u|^p + C\int_{\partial\widetilde{\Omega}\cap\{|x|<\frac{3}{4}\}} S(u)^p \quad ,
\end{aligned}$$

as desired, where $K \subset\subset \widetilde{\Omega}$.

We next turn to the proof of (3.20). We first compute

$$\begin{aligned}
Lv_2 &= \operatorname{div} A(\theta(x)\mu(x-\psi(t)))\nabla u_2 + \operatorname{div} Au_2\nabla(\theta(x)\mu(t-\psi(x))) \\
&= A\nabla(\theta(x)\mu(x-\psi(t)))\cdot\nabla u_2 + \operatorname{div} Au_2\nabla(\theta(x)\mu(t-\psi)).
\end{aligned}$$

Note that, as in the estimate for $S(v)$ above, $\nabla(\theta(x)\mu(t-\psi(x)$ is supported in $\widetilde{F} \subset \widetilde{\widetilde{\Omega}}$, with $\widetilde{F} \subset\subset \widetilde{\Omega}$. In fact, note that

$$\begin{aligned}
&\operatorname{supp}\nabla(\theta(x)\mu(t-\psi(x))) \subset \\
&\qquad \{(x,t)\in\widetilde{\widetilde{\Omega}} : \frac{5}{8}+\frac{3}{32}+\frac{1}{256} < |x| < \frac{5}{8}+\frac{3}{32}+\frac{1}{128}, 0 < t-\psi(x) < \frac{1}{2}\} \\
&\qquad \cup\{(x,t)\in\widetilde{\widetilde{\Omega}} : |x| < \frac{5}{8}+\frac{3}{32}+\frac{1}{128}, \frac{1}{4} < t-\psi(x) < 1\} = E_1\cup E_2.
\end{aligned}$$

Note that the E_2 is compactly contained in $\widetilde{\Omega}$, while for the first set, note that $\operatorname{supp} u_2(x,\psi(x)) \subset \{(x,\psi(x)) : |x| < \frac{5}{8}+\frac{3}{32}\}$. This implies by the Cacciopoli inequality (which is valid up to the boundary for functions vanishing on the boundary) that $\iint_{E_1}|\nabla u_2|^2 \le C\iint_{\widetilde{\widetilde{E}}_1} u_2^2$, and in fact, by the N. Meyers estimate (see [Gi] for example) $\iint_{E_1}|\nabla u_2|^q \le C\iint_{\widetilde{\widetilde{E}}_1}|u_2|^q$, for $q > 2$, where q depends only on ellipticity. Here $\widetilde{\widetilde{E}}_1 = \{(x,t)\subset\widetilde{\widetilde{\Omega}} : \frac{5}{8}+\frac{3}{32}+\frac{1}{512} < |x| < \frac{5}{8}+\frac{3}{32}+\frac{1}{64}, 0 < t-\psi(x) < \frac{5}{8}\}$. Note that $\widetilde{F} = \widetilde{\widetilde{E}}_1\cup\widetilde{E}_2 \subset \widetilde{\widetilde{\Omega}}$, and $\widetilde{F} \subset\subset \widetilde{\Omega}$. These arguments show that $Lv_2 = f_2 + \operatorname{div} f_1$, where, for some $q > 2$, we have $\|f_1\|_{L^\infty(\Omega_\psi)} + \|f_2\|_{L^q(\Omega_\psi)} \leq C \max_{\widetilde{F}} |u_2|$. The vanishing of v_2 on $\partial\Omega_\psi$, the Cacciopoli estimate up to the boundary, and the N. Meyers improvement now give

$$\|\nabla v_2\|_{L^q(|x|\leq 10, 0 \leq t-\psi(x) \leq 12)} \leq C\|f_1\|_{L^\infty(\Omega_\psi)} + \|f_2\|_{L^q(\Omega_\psi)}, \tag{3.21}$$

for $q > 2$ and thus, by the Sobolev embedding theorem, we obtain

$$\|v_2\|_{C^\alpha(|x|\leq 8, 0\leq t-\psi(x)\leq 10)} \leq C \max_{\widetilde{F}} |u_2|,$$

where $\alpha = \alpha > 0$.

We now will show that v_2 decays at infinity, that is, $|v_2(x)| \leq (C \max_{\widetilde{F}} |u_2|) \cdot |x|^{-\beta}$, for $|x| > 4$, and $\beta > 0$, depending only on ellipticity. The decay estimate argument which follows is fairly elementary and general—in higher dimensions one can apply a similar argument to the ratio of the solution with the fundamental solution to also get the sharp rate of decay.

(3.22). *(Decay at ∞). Assume that $Lw = 0$ in $\Omega \cap \{|x| > 2\}, 0 \leq w \leq 1$, and that $w \equiv 0$ on $\partial\Omega \cap \{|x| > 2\}$. Then there exists $\beta > 0$ such that $w(x) \leq C|x|^{-\beta}$ for $|x|$.*

To prove (3.22), we first show that there exists a constant μ, depending only on ellipticity and on the Lipschitz constant, such that $w(x) \leq \mu$ in $\Omega \cap \{|x| \leq 4\}$. By Harnack at the boundary, since w vanishes on $\partial\Omega$, there exists an $\eta_0 > 0$ and a $\mu < 1$ such that $w(z) \leq \mu$ for all $z \in \Omega$ with $|z - 4| < \eta_0$ or $|z + 4| < \eta_0$. Consider the solution $h = 1 - w$. By interior Harnack, since h is nonnegative, there is a positive lower bound, call it ν, for h on $\Omega \cap \{|x| = 4\}$. Thus $-\nu + 1$ is an upper bound for v and by the maximum principle, we have $v \leq \mu \equiv 1 - \nu$, for all $x \in \Omega$ with $|x| \leq 4$. An iteration of this argument using μ as a pointwise bound for w leads ultimately to a decay of $|x|^{-\beta}$ for some β depending on ellipticity.

By (3.22) and (3.21), as well as the maximum principle, and the fact that $Lv_2 = 0$ in $\{|x| > 2\} \cap \Omega_\psi$, we obtain the classical bound for v_2. Then, if p_0 is chosen so that $p_0\beta > 1$, we have

$$\int_{\partial\Omega_\psi} N_\psi(v_2)^p d\sigma \leq C_{\widetilde{F}} \sup_{\widetilde{F}} |u_2|^p.$$

It remains to estimate $\int_{\partial\Omega_\psi} S_\psi(v_2)^p$ to establish the claim. We use the converse inequality to (3.17) as follows: Construct a Lipschitz graph Λ, with Lipschitz constant less than ϵ such that $\Lambda \subseteq \Omega_\psi \backslash \{|x| \leq 2\}$ and $\Lambda \cap \partial\Omega_\psi = \partial\Omega_\psi \cap \{|x| \geq M\}$, for $M = M(\|\psi'\|_\infty)$. In the domain above the graph Λ, v_2 is a solution to L and hence,

$$\int_\Lambda S^p(v_2) d\sigma \leq C \int_\Lambda N^p(v_2) d\sigma.$$

This controls $\int_{\partial\Omega_\psi\setminus\{|x|\geq M\}} S^p(v_2)d\sigma$. Let $Q\in\{|x|\leq M\}\cap\partial\Omega_\psi$—and we need only consider a truncated cone $\Gamma_M(Q)$, as the infinite part of the cone is controlled just as in the previous argument. But, the estimate (3.21) (extended to the range $\{|x|<M, 0<t-\psi(Q)<2M\}$) implies that

$$\left(\int_{\Gamma_M(Q)}|\nabla v_2|^2\right)^{\frac{1}{2}}\leq C\max_{\widetilde{F}}|u_2|,\text{ and}$$

the proof is concluded. ■

Remark. Under the conditions of (3.18), we also have the inequality

$$\int_{\partial\widetilde{\Omega}\cap|(x,t)\cdot\vec{e}|<\frac{1}{2}} S^p(u)d\sigma\leq C\int_{\partial\widetilde{\Omega}\cap|(x,t)\cdot\vec{e}|<\frac{3}{4}} N^p(u)d\sigma+C_K\max_K|u|^p,$$

by a similar argument as that given for (3.18).

From the localized L^p estimate (3.19) it is standard, by means of good-λ inequalities to obtain (localized) L^r estimates for any r, ([F-St], for example). Some care must be taken to keep the Lipschitz constant small. To use our localization theorem 3.18, we need, since the argument required it, the converse inequality to (3.17). In fact, the arguments are similar, but there will be no need to build up from the case of small Lipschitz constant for x-graphs. And, there is no need to use a stopping time argument.

Theorem 3.23. *Let $\mathscr{O}=\{(x,t):|x|<2,\ \varphi(x)<t<\varphi(x)+2\}$ and suppose $Lu=div A\nabla u=0$ in $\mathscr{O}$, where $\|\varphi'\|_\infty\leq M<+\infty$, for some M. Then there exists an aperture $a=a(M)$ and constant $C=C(M)$, such that*

$$\int_{\Delta_{\frac{1}{2}}} S^2_{(a,1)}(u)d\sigma\leq C\left(\int_{\Delta_1} S^2_{a,1}(u)d\sigma\right)^{\frac{1}{2}}\cdot\left(\int_{\Delta_1} N^2_{(a,1)}(u)d\sigma\right)^{\frac{1}{2}}+C\int_{\Delta_1} N^2_{(a,1)}(u)d\sigma.$$

Proof. We invoke the change of variable $\rho(z,s)=(kz,s+G(z,s)))$ where k is a constant, depending on M, chosen so that this transformation is one to one. As before, set $v=u\circ\rho$ and the equation v verifies (locally) in $\mathbb{R}^2_+$ is $\operatorname{div} B\nabla v=0$, where

$$B=\begin{pmatrix} G_s+k & -G_z+b\\ -G_z & \frac{G_z^2-G_zb+\gamma}{G_s+k}\end{pmatrix}.$$

If $\theta\in C^\infty$ has support in $|z|\leq 1$, $\theta\equiv 1$ in $|z|<\frac{3}{4}$ and $\mu(s)$ has support in $0<s<1$, $\mu\equiv 1$ in $\{s<\frac{1}{2}\}$, then it suffices to estimate

(3.24)

$$\iint_{(z,s)} s|\nabla v(z,s)|^2\theta(z)\mu(s)dz\,ds\leq C_0\iint_{(z,s)} s\theta(z)\mu(s)M\nabla v(z,s)\cdot\nabla v(z,s)dz\,ds$$

where

$$M = \begin{pmatrix} (G_s+k)^2 & (G_s+k)(-2G_z+b)/2 \\ (G_s+k)(-2G_z+b)/2 & G_z^2 - G_z b + \gamma \end{pmatrix}.$$

Expanding $M\nabla v \cdot \nabla v$ yields three integrals to evaluate, the first of which is

$$\begin{aligned}
&\iint (G_s+k)^2 |D_z v|^2 s\theta(z)\mu(s)dz\,ds \\
&\quad = -\iint v\, D_z[s(G_s+k)^2 D_z v\theta(z)\mu(s)]dz\,ds \\
&\quad = -\iint vG_{sz}(G_s+k)D_z v\theta(z)\mu(s)sdz\,ds+ \\
&\qquad -\iint_v D_z((G_s+k)D_z v)(G_s+k)\theta(z)\mu(s)sdz\,ds \\
&\qquad -\iint v(G_s+k)^2\theta'(z)D_z v\mu(s)sdz\,ds \\
&\quad = a_1 + a_2 + a_3
\end{aligned}$$

By familiar arguments (see 3.10 for example), terms a_1 and a_3 are bounded by

$$\left(\int_{\{|z|<1\}} N^2(v)\right)^{\frac{1}{2}} \cdot \left(\int_{\{|z|<1\}} S^2(v)\right)^{\frac{1}{2}}.$$

The term a_2 contains part of the equation that v satisfies and will be combined later with others to yield $\operatorname{div} B\nabla v$.

The second integral arising from (3.24) is:

$$\begin{aligned}
&\iint [G_z^2 - G_z b + \gamma]|D_s v|^2 s\theta(z)\mu(s)dz\,ds \\
&\quad = -\iint vD_s v[G_z^2 - G_z b + \gamma]\theta(z)\mu(s)dz\,ds \\
&\qquad -\iint vD_s v[G_z^2 - G_z b + \gamma]\theta(z)\mu'(s)sdz\,ds \\
&\qquad -\iint vD_s\left[\frac{(G_z^2 - G_z b + \gamma)}{(G_s+k)}D_s v\right](G_s+k)\theta(z)\mu(s)sdz\,ds \\
&\qquad +\iint vD_s v[G_z^2 - G_z b + \gamma]s\theta(z)\mu(s)\frac{G_{ss}}{(G_s+k)^2}dz\,ds \\
&\quad = b_1 + b_2 + b_3 + b_4.
\end{aligned}$$

Term b_3 will be combined with term a_2. Term b_1 is a boundary integral, $\int v^2[G_z^2 - G_z b + \gamma]\theta(z)\mu(v)dz$, plus the solid integral $\iint v^2 D_s[G_z^2 - G_z b + \gamma]\theta(z)\mu(s)dz\,ds$, which is estimated as in (3.10.1) by a sum of $\int_{|z|<1} N^2(v) + \left(\int_{|z|<1} N^2(v)\right)^{\frac{1}{2}} \cdot \left(\int_{|z|<1} S^2(v)\right)^{\frac{1}{2}}$. Terms b_2 and b_4 are also bounded by the product in the sum above.

The last integral arising from 3.16 is:

$$\begin{aligned}
&\iint s(G_s+k)(-2G_z+b)D_svD_zv\theta(z)\mu(s)dz\,ds \\
&\quad = \iint s(G_s+k)(-G_z+b)D_svD_zv\theta(z)\mu(s)dz\,ds \\
&\qquad + \iint s(G_s+k)(-G_z)D_svD_zv\theta(z)\mu(s)dz\,ds \\
&\quad = I + II.
\end{aligned}$$

In I, we integrate by parts in z obtaining

$$\begin{aligned}
&- \iint sv(G_s+k)D_z[(-G_z+b)D_sv]\theta(z)\mu(s)dz\,ds \\
&- \iint sv(-G_z+b)D_svG_{sz}\theta(z)\mu(s)dz\,ds \\
&- \iint sv(-G_z+b)D_sv(G_s+k)\theta'(z)\mu(s)dz\,ds \\
= {} & C_1 + C_2 + C_3.
\end{aligned}$$

Term C_1 contains part of the equation $\operatorname{div} B\nabla v$, Terms C_2 and C_3 are bounded by $\left(\int_{|z|<1} N^2 v\right)^{\frac{1}{2}} \cdot \left(\int_{|z|<1} S_v^2\right)^{\frac{1}{2}}$.

For II, we integrate by parts in s obtaining

$$\begin{aligned}
&- \iint (G_s+k)(-G_z)D_z(v^2/2)\theta(z)\mu(s)dz\,ds \\
&- \iint D_s[(-G_z)D_zv]v(G_s+k)s\theta(z)\mu(s)dz\,ds \\
&- \iint vD_zv(-G_z)G_{ss}s\theta(z)\mu(s)dz\,ds \\
&- \iint vD_zv(-G_z)(G_s+k)s\theta(z)\mu'(s)dz\,ds \\
= {} & d_1 + d_2 + d_3 + d_4.
\end{aligned}$$

Note that $d_2 + a_2 + c_1 + b_3 = 0$, since $\operatorname{div} B\nabla v = 0$. The only term that requires some additional manipulation is d_1. Note that, after integrating by parts in z, we need to bound the integral

$$\iint v^2 G_{zz}(G_s + k)\theta(z)\mu(s)dz\,ds + \iint v^2 G_{zs}(-G_z)\theta\mu dz\,ds + \iint v^2(G_s + k)(-G_z)\theta'\mu d_z\,ds. \tag{3.25}$$

To handle the first term introduce $1 = D_s(s)$ and integrate by parts in s again. To estimate the expression $\iint sv^2 G_{zzs}(G_s + k)\theta(z)\mu(s)dz\,ds$, integrate by parts again in z. All these manipulations have the final effect of introducing G_{zs} in place of G_{zz} in the expression (3.25) and, as before, $G_{zs} = D_z(\psi_s * \eta_s * \varphi')$ and we use the additional fact that $|\psi_s * \eta_s * \varphi'|^2 dz\frac{ds}{s}$ is a Carleson measure. Using the analogous fact for G_{ss}, we can also control $\iint v^2 G_{zz} G_{ss}\theta(z)\mu(s)dz\,ds$. The second term in (3.25) already contains G_{zs}, and thus is handled by the same argument as above—the last term in (3.25) is the simplest.

We complete the proof of Theorem 3.7 as follows. From (3.23), one may now also obtain a version of (3.23) on graphs, for any $p > 2$. This follows from a good-λ inequality of the same type as in Lemma 3.15. By (3.17) and (3.23) for graphs, we have now satisfied the hypotheses of the localization Theorem 3.18, and we obtain localized L^p estimates (3.19) for p sufficiently large. A standard argument, again using good-λ inequalities, allows us to recover the desired localized estimates in L^2—taking care not to increase too much the Lipschitz constants. Indeed, we'll get such estimates for any L^q, $q > 0$, obtaining Theorem 3.7 in particular.

We now assume we are in the situation of (ii) of Lemma 3.4, i.e. the graph has the form $x = \varphi(t)$, $\|\varphi'\|_\infty < \epsilon$, and ϵ small. Because the matrix has coefficients which are independent of the variable t, not x, the proof in this case is not merely a repetition of the earlier one.

Theorem 3.26. *If $Lu = div\, A\nabla u = 0$ in $\mathscr{O} = \{(x,t) : |t| < 2, \varphi(t) < x < \varphi(t) + 2\}$, and $\|\varphi'\|_\infty$ is sufficiently small, then the inequality (3.7.1) of Theorem 3.7 holds for u.*

Proof. The result follows from a stopping time argument, just as in the x-graph case and so it suffices to establish (3.8.1) as well as the converse inequality, which is easier to derive. For the localization, we introduce as before $\theta(s)$ and $\mu(z)$, θ supported in $|s| < \frac{3}{4}$, μ supported in $|z| \leq 2\alpha$ and the change of variable $\rho(z,s) = (F(z,s), s)$ where

$F(z,s) = z + \eta_z * \varphi(s)$ mapping $\{z > 0\} \mapsto \{(x,t) : x > \varphi(t)\}$. Then, if $v = u \circ \rho$,

$$\begin{aligned}\int_{|t|<\frac{1}{2}} v^2(0,s)ds &\leq \int v^2(0,s)\theta(s)\mu(0)ds \\ &= -\iint D_z(v^2(z,s)\theta(s)\mu(z))ds\,dz \\ &= -2\iint vD_zv\theta\mu dz\,ds - 2\iint v^2\theta\mu' dz\,ds.\end{aligned}$$

The second integral above is an error term of the form $\iint_K v^2 dz\,ds$. At this point $K = K(\alpha)$, but the dependence on α is removed by the same argument as in lemma 3.5. In the first integral, introduce $1 = D_z(z)$ and integrate by parts to obtain

$$2\iint vD_{zz}v\theta\mu z\,dz\,ds + 2\iint (D_zv)^2\theta\mu z\,dz\,ds + 2\iint zvD_zv\theta\mu'(z)dz\,ds,$$

of which only the first expression requires new arguments.

Note that $D_{zz}v(D_{xx}u\circ\rho)F_z^2 + (D_xu\circ\rho)F_{zz}$ where, by D_xu, we mean to differentiate u with respect to its first variable. And, $\iint (u\circ\rho)(D_xu\circ\rho)\theta(s)\mu(z)F_{zz}zdz\,ds$ is bounded by the product $\|N(u)\|_{L^2(\Delta_{\frac{3}{4}})}\cdot\|S(u)\|_{L^2(\Delta_{\frac{3}{4}})}$ since $z|F_{zz}|^2dz\,ds$ is a Carleson measure. Consider now

$$\begin{aligned}&\iint zF_z^2\theta\mu v(D_{xx}u\circ\rho)dz\,ds = \\ &= -\iint zF_z^2\theta\mu v(D_xbD_tu\circ\rho)dz\,ds + \\ &\quad -\iint zF_z^2\theta\mu v(\gamma D_{tt}u\circ\rho)dz\,ds \\ &= I + II.\end{aligned}$$

$$\begin{aligned}I &= -\iint zF_z\theta\mu vD_z(bD_tu\circ\rho)dz\,ds = \\ &= \iint b(D_tu\circ\rho)D_zv\theta\mu zF_zdz\,ds + \\ &\quad + \iint b(D_tu\circ\rho)v\theta\mu' zF_zdz\,ds + \\ &\quad + \iint (bD_tu\circ\rho)v\theta\mu F_zdz\,ds + \\ &\quad + \iint (bD_tu\circ\rho)v\theta\mu zF_{zz}dz\,ds.\end{aligned}$$

Claim 3.27. $\left|\iint b(D_t u \circ \rho) v\theta\mu F_z dz\, ds\right| \le (\alpha + \|b\|_\infty \epsilon) \int N^2(u) d\sigma.$

To see this, change variables again, by means of ρ^{-1} to re-express this as an integral in Ω:

$$\begin{aligned}
&\iint b(x) D_t u(x,t) u(x,t) (\theta\mu \circ \rho^{-1}) dx\, dt \\
&\quad = \iint D_t \left[b \frac{u^2}{2} \theta\mu \circ \rho^{-1} \right] dx\, dt + \\
&\qquad - \iint b(x) \frac{u^2}{2} D_t (\theta\mu \circ \rho^{-1}) dx\, dt \\
&\quad \le \left| \int_{|t|<\frac{3}{4}} b(\varphi(t)) \frac{u^2}{2} (\varphi(t), t) \varphi'(t) (\theta\mu \circ \rho^{-1})(\varphi(t), t) dt \right| \\
&\qquad + \alpha \int_{|t|<\frac{3}{4}} N^2(\mu) d\sigma,
\end{aligned}$$

by the support properties of $\theta\mu \cdot \rho^{-1}$ and the formula $\int_{-\infty}^{\infty} \int_{x>\varphi(t)} D_t w(x,t) dx\, dt = \int_{-\infty}^{\infty} w(\varphi(t), t) \varphi'(t) dt$—and the ϵ in the right hand side of (3.27) comes from the fact that $\|\varphi'\|_\infty \le \epsilon$. The other terms comprising I are handled much as before, and it remains to consider II. Here we use the identity $D_s(h \circ \rho) = D_z(h \circ \rho) \frac{F_s}{F_z} + D_t h \circ \rho$, valid for any h, to write

$$\begin{aligned}
II &= - \iint z F_z^2 \theta\mu v D_s(\gamma D_t u \circ \rho) dz\, ds + \iint \theta\mu z F_z F_s D_z(\gamma D_t u \circ \rho) v\, dz\, ds \\
&= \iint z (\gamma D_t u \circ \rho) D_s(v F_z^2 \theta\mu) dz\, ds - \iint (\gamma D_t u \circ \rho) D_z v z F_s F_z \theta\mu dz\, ds + \\
&\quad - \iint (\gamma D_t u \circ \rho) v F_s F_z \theta\mu dz\, ds - \iint (\gamma D_t u \circ \rho) v z (F_{sz} + F_{zz}) \theta\mu\, dz\, ds,
\end{aligned}$$

of which integrals only the third needs further examination. Writing $\psi = \rho^{-1}$, we express this as an integral back over the region above the graph by

$$\iint \gamma(x) D_t u(x,t) (\theta\mu \circ \psi) u(x,t) (F_s \circ \psi) dx\, dt. \tag{3.28}$$

Since $\gamma(x)$ is bounded from above and below, we can write $H'(x) = \gamma(x)$, i.e., $H(x)$ is a primitive of γ, where H is increasing. Set $\widetilde{\varphi}(t) = H \circ \varphi(t)$, a new Lipschitz graph, and make the change of variables $y = H(x)$. Then, if we also use the notation $H(x,t)$ for $(H(x), t)$, we have

$$(3.28) = \iint D_t v(y,t) v(y,t) \alpha(y,t) \beta(y,t) dy\, dt$$

where $v = u \circ H^{-1}$, $\alpha = F_t \circ \psi \circ H^{-1}$ and $\beta = \theta\mu \circ \psi \circ H^{-1}$ and the integration (for α small) is in $\widetilde{\mathscr{O}} = \{(y,t) : \widetilde{\varphi}(t) < y < \widetilde{\varphi}(t) + 1, |t| < 1\}$.

We change variables again, by means of a transformation $\widetilde{\rho}(w,t) = (\widetilde{F}(w,t), t)$, defined just as ρ was defined for the region $\mathscr{O}$, which maps $\{w > 0\}$ into $\widetilde{\mathscr{O}}$. Thus,

$$
\begin{aligned}
(3.28) &= \iint (D_t v \circ \widetilde{\rho})\; v \circ \widetilde{\rho}\; \alpha \circ \widetilde{\rho}\; \beta \circ \widetilde{\rho}\; \widetilde{F}_w \, dw\, dt \\
&= \iint v \circ \widetilde{\rho}\; \alpha \circ \widetilde{\rho}\; \beta \circ \widetilde{\rho}\; \widetilde{F}_w \, D_t(v \circ \widetilde{\rho}) dw\, dt \\
&\quad - \iint v \circ \widetilde{\rho}\; \alpha \circ \widetilde{\rho}\; \beta \circ \widetilde{\rho}\; \widetilde{F}_w \left(\frac{\partial v}{\partial y} \circ \widetilde{\rho} \right) \widetilde{F}_t \, dw\, dt \\
&= A + B
\end{aligned}
$$

We have, setting $\widetilde{\psi} = \widetilde{\rho}^{-1}$,

$$
\begin{aligned}
B &= -\iint \alpha(y,t)\beta(y,t) \frac{\partial}{\partial y}(v^2) \widetilde{F}_t \circ \widetilde{\psi} dy\, dt \\
&= \iint v^2 \frac{\partial}{\partial y} [(\widetilde{F}_t \circ \widetilde{\psi})\alpha\beta] dy\, dt + \\
&\quad - \int v^2(\widetilde{\varphi}(t), t)\alpha(\widetilde{\varphi}(t), t)\beta(\widetilde{\varphi}(t), t)(\widetilde{F}_t \circ \widetilde{\psi})(\widetilde{\varphi}(t), t) dt \\
&= B_1 + B_2
\end{aligned}
$$

Since $v(\widetilde{\varphi}(t), t) = u(\varphi(t), t), \alpha(\widetilde{\varphi}(t), t) = (F_t \circ \psi)(\varphi(t), t), \beta(\widetilde{\varphi}(t), t) = (\theta\mu \circ \psi)(\varphi(t), t)$, we have $B_2 \leq \left| \int_{|t|<1} u^2(\varphi(t), t)\varphi'(t)\widetilde{\varphi}'(t)\, dt \right| \leq \epsilon \int_{\Delta_1} N^2(u) d\sigma$, for ϵ small.

We write $B_1 = B_{11} + B_{12} + B_{13}$, where

$$
\begin{aligned}
B_{11} &= \iint v^2 \frac{\partial \alpha}{\partial y} \widetilde{F}_t \circ \widetilde{\psi} \beta \, dy\, dt, \\
B_{12} &= \iint v^2 \alpha \frac{\partial}{\partial y} (\widetilde{F}_t \circ \widetilde{\psi}) \beta \, dy\, dt, \\
B_{13} &= \iint v^2 \alpha \widetilde{F}_t \circ \widetilde{\psi} \frac{\partial \beta}{\partial y} \, dy\, dt.
\end{aligned}
$$

B_{13} is handed by familiar arguments, and we now turn to B_{11}.

Recall that $\alpha(y,t) = F_t \circ \psi \circ H^{-1}$, and that $H^{-1}(y,t) = (H^{-1}(y), t)$. Hence, $\frac{\partial}{\partial y}(F_t \circ \psi) \circ H^{-1} = \frac{\partial}{\partial x}(F_t \circ \psi) \circ H^{-1} \frac{\partial}{\partial y} H^{-1}$, and since $\frac{\partial}{\partial y} H^{-1}\, dy\, dt = dx\, dt$, a change of variables gives

$$\begin{aligned} B_{11} &= \iint u^2(x,t)\frac{\partial}{\partial x}(F_t\circ\psi)\cdot \widetilde{F}_t\circ\widetilde{\psi}\circ H(\theta\mu\circ\psi)dx\,dt \\ &= \iint (u^2\circ\rho)(z,t)\frac{\partial}{\partial x}(F_t\circ\psi)\circ\rho\cdot\widetilde{F}_t\circ\widetilde{\psi}\circ H\circ\widetilde{\rho}\cdot\theta u F_z\,dz\,dt \\ &= \iint u^2\circ\rho(z,t)\frac{\partial}{\partial z}F_t(z,t)\cdot\widetilde{F}_t\circ\widetilde{\psi}\circ H\circ\widetilde{\rho}\theta\mu\,dz\,dt. \end{aligned}$$

Recall now that $\frac{\partial}{\partial z}F_t = \frac{\partial}{\partial t}Q_z * \varphi'$, and integrate by parts in t in the last integral. We then get the desired bound using the Carleson measure property of $Q_z * \varphi'$, which gives a small error term, since $\|\varphi'\|_\infty \le \epsilon$.

In order to estimate B_{12}, we change variables by $\widetilde{\rho}$, to obtain

$$\begin{aligned} B_{12} &= \iint v^2\circ\widetilde{\rho}(w,t)\alpha\circ\widetilde{\rho}\frac{\partial}{\partial y}(\widetilde{F}_t\circ\widetilde{\psi})\circ\widetilde{\rho}\,\beta\circ\widetilde{\rho}\widetilde{F}_w dw\,dt \\ &= \iint v^2\circ\widetilde{\rho}(w,t)\alpha\circ\widetilde{\rho}\frac{\partial}{\partial w}\widetilde{F}_t\,\beta\circ\widetilde{\rho}\,dw\,dt. \end{aligned}$$

Again, $\frac{\partial}{\partial w}\widetilde{F}_t = \frac{\partial}{\partial t}Q_w * \widetilde{\varphi}'$, and the term can be handled, upon integration by parts, in a similar manner as B_{11}.

Term A requires a different sort of argument, via a method first used by Dahlberg in [D3].

Let T denote the Hilbert transform and $\Lambda^{\frac{1}{2}}$ denote the operator of $\frac{1}{2}$-order derivative in the t variable. Then,

$$\begin{aligned} A &= \int_{-\infty}^{\infty}\int_{w>0} v\circ\widetilde{\rho}D_t(v\circ\widetilde{\rho})\alpha\circ\widetilde{\rho}\,\beta\circ\widetilde{\rho}\,\widetilde{F}_w\,dw\,dt \\ &= \iint \Lambda^{\frac{1}{2}}(\alpha\circ\widetilde{\rho}\,\widetilde{F}_w\beta\circ\widetilde{\rho}v\circ\widetilde{\rho})T\Lambda^{\frac{1}{2}}(v\circ\widetilde{\beta}\circ\widetilde{\rho})\,dt\,dw \end{aligned}$$

where $\widetilde{\beta}$ has similar support properties to β, and $\widetilde{\beta}\cdot\beta = \beta$. Thus

$$A \le \left(\iint |\Lambda^{\frac{1}{2}}(\alpha\circ\widetilde{\rho}\,\widetilde{F}_w\,\beta\circ\widetilde{\rho}v\circ\widetilde{\rho})|^2 dt\,dw\right)^{\frac{1}{2}}\cdot\left(\iint |\Lambda^{\frac{1}{2}}(v\circ\widetilde{\beta}\circ\widetilde{\rho})|^2 dt\,dw\right).$$

For each fixed w, the Sobolev trace theorem yields

$$\begin{aligned} \iint |\Lambda^{\frac{1}{2}}(v\circ\widetilde{\beta}\circ\widetilde{\rho})|^2\,dw\,dt &\le C\int_{w>0}\left(\int_t\int_{z>0}|\nabla_{z,t}(v\circ\widetilde{\beta}\circ\widetilde{\rho})(w+z,t)|^2 dz\,dt\right)dw \\ &\le C\iint |\nabla_{z,t}(v\circ\widetilde{\beta}\circ\widetilde{\rho})(z,t)|^2 z\,dz\,dt. \end{aligned}$$

Similarly the square of the other factor is bounded from above by a constant times

$$\iint |\nabla_{z,t}(\beta \circ \widetilde{\rho}\, \alpha \circ \widetilde{\rho}\, \widetilde{F}_w \; v \circ \widetilde{\rho})|^2 z\, dz\, dt$$

in which we will use the Carleson measure properties of $|\nabla \widetilde{F}_w|$, $|\nabla(\alpha \circ \widetilde{\rho})|$, and $|\nabla(\beta \circ \widetilde{\rho})|$.

This together with arguments repeated from the x-graph situation and the stopping time lemma, which goes over without modification, completes the proof of (3.26).

We have, in addition, the analog of theorem 3.26, the domination of the square function by the non-tangential maximal function. (Recall, this is also needed in the proof of localization for 3.26.)

Theorem 3.28. *If $\mathscr{O} = \{(x,t) : |t| < 2,\ \varphi(t) < x < \varphi(t) + 2\}$, and $Lu = div\, A\nabla u = 0$ in $\mathscr{O}$, then the conclusion of Theorem (3.23) holds, with $S(u)$, $N(u)$ defined with respect to the graph $x = \varphi(t)$.*

The proof follows from the argument given to establish 3.26. In fact, an examination of that proof shows that (with the notation of 3.26):

$$\begin{aligned}\int v^2(0,s)\theta(s)\mu(0)ds = 2\iint |D_z v|^2 \theta\mu\, z\, dz\, ds &+ 2\iint bD_t u \circ \rho\, D_s v\, F_z^2 \theta\mu\, dz\, ds\\ &+ 2\iint z\gamma D_t u \circ \rho\, D_s v\, F_z^2 \theta\mu\, dz\, ds\\ &- 2\iint z\gamma\, D_t u \circ \rho\, D_z v\, z\, F_s F_z \theta\mu\, dz\, ds + E,\end{aligned}$$

where $|E|$ is bounded from above by the left hand side of (3.23). Finally, since $D_s v = D_z v \frac{F_s}{F_z} D_t u \circ \rho$, and $D_z v = D_x u \circ \rho\, F_z$, the right hand side of the above equality becomes

$$\begin{aligned}2\iint z|D_x u \circ \rho|^2 \theta\mu F_z^2\, dz\, ds&\\ + 2\iint b\, D_t u \circ \rho\, D_x u \circ \rho\, \theta\mu F_z^2\, dz\, ds&\\ + 2\iint z\gamma |D_t u \circ \rho|^2 \theta\mu\, F_z^2\, dz\, ds + E,&\end{aligned}$$

and our claim follows from ellipticity.

Let us now suppose that we are in the situation of (iii) of Lemma 3.4. That is, $\Omega = \{e_1 t > e_2 x + \varphi(x)\}$, $\|\varphi'\|_\infty < \epsilon'$ and $e_1 \geq \delta e_2$ and $e_2 \geq \delta e_1$.

Theorem 3.30. *Let* $\mathscr{O} = \Omega \cap \{(x,t) : |t+x| < 2,\ e_2 x + \varphi(x) < e_1 t < 2 + e_2 x + \varphi(x)\}$ *and suppose that* $Lu = div A\nabla u = 0$ *in* $\mathscr{O}$. *Then given* δ, *there exists* $\epsilon' > 0$, *depending on* δ, *so that there exist constants* $C_1, a = a(\epsilon)$, $C_2(a)$ *such that if* $\|\varphi'\|_\infty < \epsilon'$

$$\int_{\Delta_{\frac{1}{4}}} N(a,d)(u)^2 d\sigma \leq C_1 \int_{\Delta_{\frac{7}{8}}} S^2_{(4a,\frac{3}{2})}(u) d\sigma + C_2 \iint_K u^2 dx, \tag{3.31}$$

where K *is a compact subset of* $\mathscr{O}$ *at distance* $\frac{1}{4}$ *from the boundary of* $\mathscr{O}$ *and* $\Delta_s = B(0,s) \cap \{e_1 t = e_2 x + \varphi(x)\}$.

We shall prove the analog of (3.8.1) for a suitable choice of ϵ'. This suffices since the stopping time argument is independent of the graph. Set $e = (e_1, e_2)$ and $e_\perp = (-e_2, +e_1)$. We define a transformation $\rho(z,s) = (z, s + H(z,s))$ mapping $\{(z,s) : (z,s)\cdot e_\perp > 0\}$ onto Ω, by setting $H(z,s) = \frac{1}{e_1} G(z, (z,s)\cdot e_\perp)$ where $G(z,\alpha) = \eta_\alpha * \eta_\alpha * \varphi(z)/e_1$. Observe that if $(z,s)\cdot e_\perp = 0$, $H(z,s) = \frac{1}{e_1} G(z,0) = \frac{\varphi(z)}{e_1}$ and therefore $(z, s + H(z,s))\cdot e_\perp = (z, s + \varphi(z)/e_1)\cdot(-e_2, e_1) = (z,s)\cdot e_\perp + \varphi(z) = \varphi(z)$.

If we set $v = u \circ \rho$, then v will satisfy an equation $\operatorname{div} B\nabla u = 0$ which we write:

$$\begin{aligned} &D_s([H_z^2 - H_z b + \gamma] D_s v) \\ &\quad = \frac{H_{ss}}{H_s + 1}[H_z^2 - H_z b + \gamma] D_s v - (H_s+1) D_z[(H_s+1) D_z v] + \\ &\qquad - (H_s+1) D_z([-H_z + b] D_s v) - (H_s+1) D_s[-H_z D_z v]. \end{aligned} \tag{3.32}$$

Again, by ellipticity of A, there exists a $\lambda > 0$ such that $H_z^2 - H_z b + d > \lambda(H_z^2 + 1) > \lambda$.. Consider

$$e_1 \int_{z\in\mathbb{R}} [H_z^2 - H_z b + \gamma] v^2 (z, \frac{e_2}{e_1} z) \theta((e_1 + \frac{e_2^2}{e_1}) z) \mu(0) dz \tag{3.33}$$

where $\theta(\cdot)$ has support in $\{|z| < 1\}$ and $\mu(\cdot)$ has support in $\{0 < s < \alpha\}$, where α is, as in Lemma 3.8, dependent on the apertures of the cones used to define $N(\cdot)$ and $S(\cdot)$.

Then

$$
\begin{aligned}
(3.33)\quad &= -e_1 \iint_{(z,s)} D_s([H_z^2 - H_z b + \gamma]v^2(z,s)\theta((z,s)\cdot e)\mu((z,s)\cdot e_\perp))dz\,ds \\
&= -e_1 \iint (2H_z H_{zs} - H_{zs}b)v^2(z,s)\theta((z,s)\cdot e)\mu((z,s)\cdot e_\perp)dz\,ds \\
&\quad - 2e_1 \iint [H_z^2 - H_z b + \gamma]v(z,s)D_s v\theta((z,s)\cdot e)\mu((z,s)\cdot e_\perp)dz\,ds \\
&\quad - e_1 \iint [H_z^2 - H_z b + \gamma]v^2(z,s)\{\theta'((z,s)\cdot e)\mu((z,s)\cdot e_\perp)e_2 \\
&\quad + \theta((z,s)\cdot e)\mu'((z,s)\cdot e_\perp)e_1\}dz\,ds \\
&= I + II + III.
\end{aligned}
$$

In II, we write $e_1 = D_s((z,s)\cdot e_\perp)$ inside the integral and integrate by parts. Thus

$$
\begin{aligned}
II &= 2\iint_{(z,s)} ((z,s)\cdot e_\perp)|D_s v|^2\theta\mu[H_z^2 - H_z b + \gamma]dz\,ds + \\
&\quad + 2\iint_{(z,s)} ((z,s)\cdot e_\perp)D_s([H_z^2 - H_z b + \gamma]D_s v)v\theta\mu dz\,ds \\
&\quad + 2\iint_{(z,s)} ((z,s)\cdot e_\perp)[H_z^2 - H_z b + \gamma]vD_s vD_s(\theta\mu)dz\,ds \\
&= II_1 + II_2 + II_3.
\end{aligned}
$$

(Note that the first term II_1 is bounded by $\int_{\{s=\frac{e_2}{e_1}z\}} S^2(v)dz$ since $(z,s)\cdot e_\perp =$ distance of (z,s) to the line $\{s = \frac{e_2}{e_1}z\}$.) In term II_2, we use the equation in the form (3.32):

$$
\begin{aligned}
II_2 &= 2\iint ((z,s)\cdot e_\perp)[H_z^2 - H_z b + \gamma]\frac{H_{ss}}{1+H_s}vD_s(v)\theta\mu dz\,ds + \\
&\quad - 2\iint ((z,s)\cdot e_\perp)(H_s+1)D_z((H_s+1)D_z v)v\theta\mu dz\,ds + \\
&\quad - 2\iint ((z,s)\cdot e_\perp)(H_s+1)D_z(-H_z + b)D_s v)\theta\mu\, dz\,ds + \\
&\quad - 2\iint ((z,s)\cdot e_\perp)(H_s+1)D_s(-H_z D_z v)v\theta\mu\, dz\,ds \\
&= II_{2,1} + II_{2,2} + II_{2,3} + II_{2,4}.
\end{aligned}
$$

Then

$$
\begin{aligned}
II_{2,2} = 2 \iint ((z,s)\cdot e_\perp)(H_s+1)^2(D_z v)^2 \theta\mu dz\, ds + \\
+ 2 \iint ((z,s)\cdot e_\perp) H_{sz}(H_s+1) D_z v v \theta \mu dz\, ds + \\
+ 2 \iint ((z,s)\cdot e_\perp)(H_s+1)^2 D_z v v D_z(\theta\mu) dz\, ds + \\
+ 2 \iint (-e_2)(H_s+1)^2 D_z(v^2/2)\theta\mu\, dz\, ds,
\end{aligned}
$$

and the fourth integral above can, in turn, be expressed:

$$
\begin{aligned}
& - 2e_2 \iint (H_s+1)^2 D_z(v^2/2)\theta\mu\, dz\, ds \\
= & -e_2 \int (H_s+1)^2 v^2(\frac{e_1}{e_2}s, s)\mu(0)\theta(\frac{e_1}{e_2}^2 s + e_2 s)\, ds + \\
& + 2e_2 \iint H_{sz}(H_s+1)v^2\theta\mu\, dz\, ds + e_2 \iint (H_s+1)^2 v^2 D_z(\theta\mu) dz\, ds.
\end{aligned}
$$

Similarly, we find

$$
\begin{aligned}
II_{2,3} = 2 \iint (z,s)\cdot e_\perp (H_s+1)(-H_z+b) D_s v D_z v \theta\mu\, dz\, ds + \\
+ 2 \iint (z,s)\cdot e_\perp H_{sz}(-H_z+b) v\, D_s v \theta\mu\, dz\, ds + \\
+ 2 \iint (z,s)\cdot e_\perp (H_s+1)(-H_z+b) v\, D_s\, v\, D_z(\theta\mu) dz\, ds + \\
+ e_2 \iint H_{ss}(-H_z+b) v^2\, \theta\mu\, dz\, ds + \\
+ e_2 \iint (H_s+1)(-H_{zs}) v^2 \theta\mu\, dz\, ds + \\
+ e_2 \iint (H_s+1)(-H_z+b) v^2\, D_s(\theta\mu) dz\, ds + \\
+ e_2 \int_{-\infty}^{\infty} (H_s+1)(-H_z+b) v^2(z, \frac{e_2}{e_1} z)\mu(0)\theta(e_1 z + \frac{e_2^2}{e_1} z) dz,
\end{aligned}
$$

and

$$
\begin{aligned}
II_{2,4} = 2 \iint & ((z,s)\cdot e_\perp)\; H_{ss}\; (-H_z D_z v)\, v\, \theta\mu\, dz\, ds + \\
& + 2 \iint ((z,s)\cdot e_\perp)(H_s+1)(-H_z) D_z v\, D_s v\, \theta\mu\, dz\, ds + \\
& + 2 \iint ((z,s)\cdot e_\perp)(H_s+1)(-H_z D_z v) v\, D_s(\theta\mu) dz\, ds + \\
& + e_1 \int (H_s+1)(-H_z) v^2 (\frac{e_1}{e_2} s, s) \theta(\frac{e_1^2}{e_2} s + e_2 s)\mu(0) dz \\
& - e_1 \iint (H_s+1)(-H_{zz}) v^2\, \mu\theta\, dz\, ds \\
& - e_1 \iint H_{sz}(-H_z) v^2\, \theta\mu\, dz\, ds \\
& - e_1 \iint (H_s+1)(-H_z) v^2 D_z(\theta\mu)\, dz\, ds.
\end{aligned}
$$

We now collect the boundary terms (including (3.33)) and combine them using the change of variable: $\frac{e_2}{e_1}\int_{-\infty}^{\infty} f(z, \frac{e_2}{e_1} z) dz = \int_{-\infty}^{\infty} f(\frac{e_1}{e_2} s, s) ds$, and the sum of these terms is equal to

$$
\int v^2 (z, \frac{e_2}{e_1} z)\mu(0)\theta((e_1 + \frac{e_2^2}{e_1} z) E(z, \frac{e_2}{e_1} z) dz, \tag{3.36}
$$

where

$$
E(\cdot) = e_1[H_z^2 - H_z b + \gamma] + \frac{e_2^2}{e_1}(H_s+1)^2 - e_2(H_s+1)(-H_z+b) + e_2(H_s+1)H_z.
$$

Expanding this expression, we find that

$$
\begin{aligned}
e_1 E(\cdot) = & \{e_1^2\gamma + e_2^2 - e_1 e_2 b\} + e_1^2[H_z^2 - bH_z] \\
& + e_2^2 H_s^2 + 2e_2^2 H_s + e_2 e_1\{H_s H_z + 2H_z - bH_s\} \\
& + e_1 e_2 [H_s H_z + H_z].
\end{aligned} \tag{3.37}
$$

The boundary sum (3.36) is clearly bounded from above by a constant times $\frac{1}{e_1}\int v^2(z, \frac{e_2}{e_1})\theta((e_1 + \frac{e_2^2}{e_1})z) dz$, but also from below if $\|\nabla H\|_\infty$ is sufficiently small. For the quantity

$$
\{e_1^2\gamma + e_2^2 - e_1 e_2 b\} = \begin{pmatrix} 1 & b \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} -e_2 \\ e_1 \end{pmatrix} \cdot \begin{pmatrix} -e_2 \\ e_1 \end{pmatrix}
$$

which is bounded from below by $\lambda^{-1}\{e_1^2 + e_2^2\} = \lambda^{-1}$, and so if we choose $\|\varphi'\|_\infty$ sufficiently small, then (3.37) will be bounded from below. We have, so far, an equality of the form

(3.36) $= I + II_1 + III + II_2 +$ the non-boundary integrals of $II_{2,2} + II_{2,3} + II_{2,4}$.

For most of these expressions, the analysis is very similar to that of Lemma 3.8. We make a few further observations however to show the dependence of choice of ϵ' on the δ relating e_1 and e_2. For example, consider term I,

$$I = -e_1 \iint (2H_z H_{zs} - H_{zs} b) v^2(z,s) \theta \mu \, dz \, ds.$$

Since $H(z,s) = \frac{1}{e_1} G(z, e_1 s - e_2 z)$,

$$H_z = e_1^{-1} [D_1 G(z, e_1 s - e_2 z) - e_2 D_2 G(z, e_1 s - e_2 z)]$$

and $H_{sz} = D_1 D_2 G(z, e_1 s - e_2 z) - e_2 D_2 D_2 G(z, e_1 s - e_2 z)$. Obviously, $\|\nabla H\|_\infty \leq \frac{C}{\delta} \|\varphi'\|_\infty \leq \frac{C\epsilon}{\delta}$ so that $\frac{C\epsilon'}{\delta} \leq \frac{C\lambda^{-1}}{100}$ to fulfill the first restriction that (3.36) have a non-negative lower bound. Then term I can be handled exactly as (3.10.1) from the x-graph situation, with the following observation: For $\iint H_{zs} b v^2 \, \theta \mu \, dz \, ds$, we make another change of variable $\rho(x,t) = (x, (t_1 + e_2 x)/e_1) = (z,s)$, so that, if $w(x,t) = v(x, (t_1 + e_2 x)/e_1)$, pulling back to the (z,s) plane introduces quantities depending on δ; $\|N(w)\|_{L^2(dx)} \leq \frac{C}{\delta^2} \|N(v)\|_{L^2(dx)}$. Thus the error terms are multiplied by constants of the form ϵ'/δ^N, for some N, and it suffices to choose ϵ' so small so that this is smaller than ϵ.

Finally, the statement of Theorem 3.23 (the inverse inequality) is valid where $\mathscr{O}$ is replaced by the region in Theorem 3.30. This completes Step 1, and we may now state the main theorem for Lipschitz domains with small Lipschitz constant (i.e., Step 2).

Theorem 3.38. *Let* $A(x) = \begin{pmatrix} 1 & b(x) \\ 0 & \gamma(x) \end{pmatrix}$ *be an upper triangular matrix, and* $L = \operatorname{div} A \nabla$ *be the elliptic operator whose matrix is* A*, defined for* $(x,t) \in \mathbb{R}^2$ *with coefficients independent of one of the variables. Then there exists an* $\epsilon_0 > 0$ *sufficiently small such that for any bounded Lipschitz domain* Ω *with Lipschitz character* (ϵ_0, N, c_0) *and any solution* u *to* $Lu = 0$ *in* Ω *we have*

(i) *If* u *is normalized so that* $u(X_0) = 0$ *for some* $X_0 \in \Omega$ *of distance to the boundary at least* $c_0/4$*, then there exists an aperture* a *and a truncation* d *of a regular family of cones, both depending only on* (ϵ_0, N, c_0) *so that if* $N(\cdot)$ *is the non-tangential maximal operator associated to* Ω*, defined for these cones, then*

$$\int_{\partial\Omega} N^2(u) d\sigma \leq C \iint_\Omega d(X) |\nabla u(X)|^2 dX \tag{3.39}$$

where C *depends only on the Lipschitz character of* Ω*.*

(ii) *If* a, d *are as in (i) and the square function* $S(\cdot)$ *is defined with respect to these cones, then without assuming any normalization on* u*, we have,*

$$\int_{\partial\Omega} S^2(u) d\sigma \leq C \int_{\partial\Omega} N^2(u) d\sigma \tag{3.40}$$

where $C = C(\epsilon_0, N, c_0)$*.*

To prove this theorem for general Lipschitz domains, we will use the small constant result (3.38) and a 'build-up' scheme of G. David. This small constant base case is a straightforward result of the previous localized results in the three types of Lipschitz graphs, together with the reduction to graphs of these types.

Briefly, the argument for (3.39) goes as follows (see also [DJK]). If Ω is a domain of Lipschitz character (ϵ_0, N, C_0) then there exists cylinders $\{Z_j\}_{j=1}^N$ with associated Lipschitz functions $\{\varphi_j\}_{j=1}^N$, as in the definition in §1, satisfying $\|\varphi_j'\|_\infty < \epsilon_0$. Given u, with $Lu = 0$ in Ω, we wish to estimate $\int_{\Delta_j} N_{a,d}^2(u)d\sigma$, where $\Delta_j = Z_j \cap \partial\Omega$ and a, d depend on ϵ_0. By our previous results of Steps 1 and 2, we may assume, without loss of generality, that $4Z_j \cap \partial\Omega$ is the intersection of $4Z_j$ with one of three types of domains in Lemma 3.4. We then choose ϵ so small that 3.7 and 3.26 hold. Then choose $\delta = \delta(\epsilon)$ as in Lemma 3.4. For this δ, choose ϵ' as in 3.30. If ϵ_0 is smaller than both $\epsilon/2$ and $\frac{\epsilon'\delta^3}{3}$, we have the conclusion of 3.7, 3.26 and 3.30 available for all graphs, by Lemma 3.4. Then

$$\int_{\Delta_j} N^2(u)d\sigma \le C(\epsilon_0)\iint_{2Z_j\cap\Omega} \delta_j(X)|\nabla u(X)|^2 dX + C(\epsilon_0)\iint_{K_j} u^2 dX$$

for $K_j \subset\subset \Omega$, where $\delta_j(X) = \operatorname{dist}(X, \partial\Omega \cap Z_j)$. Summing on j and observing that $\sum_{j=1}^N \chi_{4\Delta_j}(Q) \le C(N, C_0)$, we obtain inequality (3.39) modulo the factor $C\iint_K u^2 dX$, where $C = C(N, c_0, \epsilon_0)$ and $K \subset\subset \Omega$ is bounded away from $\partial\Omega$ by a constant $C = C(c_0)$. The normalization guarantees that $\iint_K u^2 dX \le C\iint_\Omega \delta(X)|\nabla u(x)|^2$ where $\delta(X) = \operatorname{dist}(X, \partial\Omega)$, by a Poincaré type inequality and the fact that the domain is Lipschitz.

The argument for the converse is much the same and shall be omitted.

Remark: Once (3.39) and (3.40) are established for one family of regular cones, they follow for any other family, by standard real variable arguments ([D4], for instance).

We begin the arguments for Step 3 in order to establish the following:

Theorem 3.41. *Under the hypotheses of Theorem 3.26, but for Lipschitz domains of* arbitrary *Lipschitz character (M, N, c_0), the inequalities of (3.39) and (3.40) hold.*

In fact, the L^p equivalence between $S(\cdot)$ and $N(\cdot)$ will follow from the proof below. Theorem (3.41) will result from an iterative procedure:

Lemma 3.42. *Suppose that inequalities (3.39) and (3.40) hold for all domains Ω with Lipschitz character $(\frac{19}{20}M, N, c_0)$. Then they hold for all domains with Lipschitz character (M, N, C_0).*

We shall use the one dimensional versions of following lemmas due to G. David,

Lemma 3.43A. [Da]. *Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables (x, y) such that for each y, $x \mapsto F(x, y)$ is Lipschitz with constant M, and for each x, $y \mapsto F(x, y)$ is Lipschitz with constant M_1. Let I, J be compact intervals. Then, there exists a function $G(x, y)$ with the following properties*

(i) *$G(x, y) \geq F(x, y)$ on $I \times J$.*
(ii) *If $E = \{(x, y) \in I \times J : F(x, y) = G(x, y)\}$, then $|E| > \frac{3}{8}|I|\,|J|$.*
(iii) *The function $y \mapsto G(x, y)$ is Lipschitz with constant M_2 and moreover, either $-M \leq \frac{\partial G}{\partial x}(x, y) \leq 4M/5$ for each y, or $-4M/5 \leq \frac{\partial G}{\partial x}(x, y) \leq M$ for each y.*

Lemma 3.43B. [Da]. *Suppose $G(x, y)$ is a function satisfying property (iii) above, but on all of $\mathbb{R} \times \mathbb{R}$. Let Γ denote the graph of G in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then there exists a new orthonormal coordinate system such that*

(i) *Γ is the graph, in these new coordinates, of a function $H(X, Y)$ which is Lipschitz with constant $\frac{9}{10}M$ in X and with constant $C(M)M_1$ in Y. Here $C(M)$ is a function of M which can be chosen bounded by $\frac{5}{4}$.*
(ii) *The change of variables $\rho(x, y) = (X, Y)$ such that $(x, y, G(x, y)) = (X, Y, H(X, Y))$ is bi-Lipschitz with constants bounded by $C(1 + (MM_1)/(1 + M^2))$.*

These two lemmas yield the following, whose proof we will sketch.

Lemma 3.43. *Let F be a Lipschitz function on $\mathbb{R}$ and set $I = [-1, 1]$. Suppose $\|F'\|_\infty \leq M$ and that $F(0) = 0$. For $s > 0$, let $sI = [-s, s]$. Let $C_I(M)$ denote the cylinder $\frac{1}{2}I \times \{|t| \leq M|I|\}$. Then, there exists N, c_0 and α_M and a domain Ω of Lipschitz character $(\frac{9}{10}M, N, c_0)$ such that*

(i) $\Omega \leq C_{\frac{9}{8}I} \cap \{t > F(x)\}$

and (ii) $|\partial\Omega \cap \{(x, F(x)) : x \in I\}| \geq \alpha_M$, where $|E|$ denotes the projected measure, i.e., $|E| = |\{x \in I : (x, F(x)) \in E\}|$, and N and c_0 may depend on M but not on F.

Remark: A dilation argument gives a similar result for intervals of arbitrary length—the scaling is clear. This lemma is one of the main tools in the proof of (3.42). We will establish good-λ inequalities in order to prove the L^p norm equivalence of the non-tangential maximal function and square function. The domain whose existence is guaranteed by Lemma 3.43 will replace certain ‘sawtooth regions’ which arise in the course of proving these inequalities, and which will enable us to carry out an iterative argument.

Proof of (3.43). By Lemma 3.43A, applied to $F(x,y) = F(x)$, we have a $G(x)$ which satisfies, for all x, either $\frac{-4M}{5} \le G' \le M$ or $-M \le G' \le \frac{4M}{5}$ and also $G(x) \ge F(x)$ on I, as well as $|\{x \in I : F(x) = G(x)\}| \ge \frac{3}{8}|I| = \frac{3}{8}$. Let $I' = \frac{7}{8}I$. Then $|\{x \in I' : F(x) = G(x)\}| \ge \frac{2}{7}|I'|$.

Let $m_1 = \max_{x\in I}(G')$, $m_2 = -\min_{x\in I}(G')$. Now construct a $\widetilde{G}$ such that

(i) $\widetilde{G} = G$ on I'
(ii) $\widetilde{G} \ge G$ on I
(iii) $\frac{-4M}{5} \le \widetilde{G}' \le M$, or $-M \le \widetilde{G}' \le \frac{4M}{5}$
(iv) $\widetilde{G}$ is linear on $I\backslash I'$.

Note that $I' = \left[\frac{-7}{8}, \frac{7}{8}\right]$ and we may define $\widetilde{G}\left(\frac{7}{8}\right) = G\left(\frac{7}{8}\right)$, $\widetilde{G}\left(\frac{-7}{8}\right) = G\left(\frac{-7}{8}\right)$, and $\widetilde{G}'_{(x)} \equiv m$ for $\frac{7}{8} < x < 1$, $\widetilde{G}'(x) = m_2$ for $-1 < x < \frac{-7}{8}$, so that (iii) and (iv) are clear. Then Ω is any domain whose boundary is either smooth or coincides with the graph of $\widetilde{G}$ in I and satisfies (i) of (3.43). (There are many possibilities for Ω.) Then, by (3.43B) there exists a new coordinate system such that, in this coordinate system, the graph of $\widetilde{G}$ coincides with the graph of a function H whose Lipschitz constant is bounded by $\frac{9}{10}M$. The α_M in (ii) comes from the bi-Lipschitz character of ρ in 3.43B. ■

Theorem 3.41 is proven by establishing the induction step (3.42).

Proof of (3.42). We begin by proving (3.39) for domains of Lipschitz character (M, N, C_0), assuming the same for domains of character $(\frac{19}{20}M, N, C_0)$. The strategy is to derive good-λ inequalities, which will result in L^p inequalities for all $0 < p < \infty$. Such good-λ inequalities are fairly standard in this theory and so the details we provide are primarily intended to elucidate the role of the build-up lemmas of David. For further reference, the reader should consult [D4].

Fix then such an Ω and a family of regular cones, truncated at height c_0 and let $N(u)$ be defined relative to these cones.

Let $\{Z_1\}_{j=1}^N$ be cylinders which determine the N coordinate patches of size between $\frac{1}{c_0}$ and c_0, and set $E^j_\lambda = \{Q \in \partial\Omega : N(u)(Q) > \lambda\} \cap Z_j$. Then each E^j_λ has a Whitney decomposition, that is, $E^j_\lambda = \bigcup_i \Delta^j_i(Q_i, r_i)$ where

(i) $Q_i = (x_i, \varphi_j(x_i))$.
(ii) $\Delta_i(Q_i, r_i) = \{(x, \varphi_j(x)) : x \in I_i\}$ where $I_i = \{x : |x - x_i| \le r_i/2\}$.
(iii) the Δ_i's are of two types
 (a) $c_0 \ge r_i \ge \frac{c_0}{4}$.
 (b) $r_i < \frac{c_0}{4}$.
 where, in case (b), there exists a $P_i \in \partial\Omega \cap \partial Z_j$ such that $N(u)(P_i) \le \lambda$ and

such that for all $Q \in \Delta_i(Q_i, r_i)$, $|P_i - Q| \le c_n r_i$.

Fix now a j, and for $Q \in Z_j$, define $S_j u(Q) = (\int_{\Gamma_{\alpha(Q)}} |\nabla u(x)|^2 dX)^{\frac{1}{2}}$ where $\Gamma_\alpha(Q)$ is a right circular cone with vertex at Q in the coordinate system of Z_j, with aperture α chosen so that the slope of this cone is $M(1 + \frac{1}{20})$ and finally, $\Gamma_{\alpha(Q)}$ is assumed to be truncated at height $2Md$, $d =$ diameter(Z_j).

As is usual in this part of the proof of the good-λ inequality, one needs to define for any set $G \subseteq Z_j \cap \partial\Omega$, the set $G^*_\epsilon = \left\{ \sup_{\Delta \ni Q} \left(\frac{|\Delta \cap G|}{|\Delta|} \right) \le \epsilon \right\}$, where $|\cdot|$ denotes, as before, the Lebesgue measure of the projection of the set onto the real axis.

Claim 3.44. *There exists* $\theta_0 < 1$, $\theta = \theta(M, n)$ *such that for any* $\beta > 1$, *there are* $\epsilon > 0$ *and* $\gamma > 0$ *such that*

$$|E_i| \le \theta |\Delta_i| \tag{3.45}$$

where

$$E_i = \Delta_i \cap \{N(u) > \beta\lambda\} \cap \{S(u) \le \gamma\lambda\} \cap \\ \cap \{S(u) > \gamma\lambda\}^*_\epsilon.$$

Let us assume, without loss of generality, that $\partial\Omega \cap Z_j$ coincides with a graph of the form $t = \varphi_j(x)$.

The large balls Δ_i in the Whitney decomposition (those for which $r_i \ge c_o/4$) are handled just as in [D]. Briefly, from the normalization $u(X_0) = 0$ for some $X_0 \in \Omega$ and the a priori finiteness of $\|S(u)\|_p$, for some $p > 0$, one shows, using interior estimates, that for any $K \subset\subset \Omega$,

$$\sup_K |u| \le C(K, \|S(u)\|_p).$$

From this point, the argument is similar to that for small balls, which follows. So we turn to the proof of (3.33), where $r_i < c_0/4$. We are assuming that (3.27) holds for all bounded Ω with Lipschitz character $\left(\frac{19}{20}M, N, C_0\right)$ with a constant $C = C(M, N, c_0)$.

A 'sawtooth region, Ω_i, over E_i' (see [D]) may be constructed so that if $F = \{x \in I_i : (x, \varphi_j(x) \in E\}$ is the projection of E_i onto I_i and if $\psi(x)$ is defined to be $\psi(x) = \varphi_j(x) + \frac{M}{20}\text{dist}\,(x, F)$ for $x \in I_i$, then we set

$$\Omega_i = \{(x,t) : t > \psi(x)\} \cap C^{r_i}_{I_i}(M + \frac{1}{20}M).$$

Here $C^{r_i}_I(\cdot)$ is the rescaled cylinder of Lemma 3.43, viz., $C^r_I(B) = \{(x,t) : |x| \le r, |t| \le Br\}$, and we observe that $\|\psi'\|_\infty \le M + \frac{1}{20}M$. It follows that $\Omega_i \subseteq 2Z_j \cap \Omega$, since $\Delta_i \subseteq Z_j \cap \partial\Omega$ implies that $r_i \le d$ and hence $(M + \frac{1}{20}M)r_i \le 4Md$.

By Lemma 3.43, (or rather the rescaled version), there exists N, c_0 and α_M a domain $\widetilde{\Omega}_i \subseteq \Omega_i$, of Lipschitz character $\left(\frac{9}{10}\widetilde{M}, N, c_0\right)$, where $\widetilde{M} = \left(1 + \frac{1}{20}\right) M$, and such that $|\partial\widetilde{\Omega}_i \cap \{(x, \psi(x)) : x \in I_i\}| \geq r_i \alpha_{\widetilde{M}}$. Let $\theta = 1 - \frac{\alpha_{\widetilde{M}}}{2}$. Choose a point $A_i \in \widetilde{\Omega}_i$ whose distance to $\partial\widetilde{\Omega}_i$ is larger than $\frac{9\widetilde{M}}{10}\frac{c_0 r_i}{4}$. Choose a regular family of cones $\widetilde{\Gamma}(P)$, $P \in \partial\widetilde{\Omega}_i$, for the domain $\widetilde{\Omega}_i$ with the additional property that if $P \in \partial\Omega_i \cap \{(x, \psi(x)) : x \in I_i\}$ then $\widetilde{\Gamma}(P) \supseteq \Gamma(P) \cap B(P, cr_1)$ for suitable c, where the $\{\Gamma(P)\}$ are the regular cones associated to the domain Ω. Let $\widetilde{N}(\cdot)$ denote the non-tangential maximal function for $\widetilde{\Omega}$ defined using the cones $\{\widetilde{\Gamma}(P)\}$. Let $\widetilde{u} = u - u(A_i)$. By standard arguments, (see [D], [DJK] for instance), for γ sufficiently small, $\widetilde{N}(\widetilde{u})(P) \geq \frac{1}{2}(\beta - 1)\lambda$ for $P \in \partial\widetilde{\Omega}_i \cap \{(x, \psi(x)) : x \in F\}$. Suppose that 3.45 failed for this choice of θ, i.e., $|E_i| > \theta|\Delta_i|$. Then, if $G_i = \{(x, \psi(x)) : x \in I_i\}$, we have that $E_i \cap G_i = E_i$ and so

$$\begin{aligned} \alpha_{\widetilde{M}}|G_i| = \alpha_{\widetilde{M}}|\Delta_i| &\leq |\partial\widetilde{\Omega}_i \cap G_i| \\ &= |\partial\widetilde{\Omega}_i \cap E_i \cap G_i| + |\partial\widetilde{\Omega}_i \cap G_i \backslash E_i| \\ &\leq |\partial\widetilde{\Omega}_i \cap E_i| + |G_i \backslash E_i|. \end{aligned}$$

Therefore, since $|G_i| = |G_i \cap E_i| + |G_i \backslash E_i| \geq \theta|G_i| + |G_i \backslash E_i|$, then $\alpha_{\widetilde{M}}|G_i| \leq |\partial\widetilde{\Omega}_i \cap E_i| + (1 - \theta)|G_i|$ and if $1 - \theta < \alpha_{\widetilde{M}}/2$, this implies $|\partial\widetilde{\Omega}_i \cap E_i| \geq \alpha M/2|I_i|$.

This latter inequality is not possible for the right choice of γ and ϵ. To see this, observe that

$$|\partial\widetilde{\Omega}_i \cap E_i| \leq \left[\frac{2}{(\beta - 1)\lambda}\right]^2 \int_{\partial\widetilde{\Omega}_i \cap E_i} \widetilde{N}^2(\widetilde{u})dx$$

where $dx(E) = |E|$, the Lebesgue measure of the projection of E onto $\mathbb{R}$. If we integrate $\widetilde{N}^2(\widetilde{u})$ over the larger set $\partial\widetilde{\Omega}_i$, we can use inequality (3.39) for the domain $\widetilde{\Omega}_i$ and hence, by the induction step,

$$\begin{aligned} |\partial\widetilde{\Omega}_i \cap E_i| &\leq C(M)\frac{4}{(\beta - 1)^2\lambda^2} \iint_{\widetilde{\Omega}_i} d_{\widetilde{\Omega}_i}(X)|\nabla u(x)|^2 dx \\ &\leq C(M)\frac{4}{(\beta - 1)\lambda^2} \iint_{\Omega_i} d_{\Omega_i}(X)|\nabla u(x)|^2 dx \\ &\leq \frac{C'(M, \epsilon)}{(\beta - 1)^2\lambda^2} \int_{\partial\Omega_i \cap E_i} S^2(u)dx, \text{ for } \epsilon \text{ small,} \\ &\leq \frac{C'(M, \epsilon)}{(\beta - 1)^2\lambda^2}\gamma^2\lambda^2|I_i|. \end{aligned}$$

In this inequality, β is a fixed constant less than 1, chosen small enough so that $\beta^p\theta < 1/2$, which means that the good-λ inequality has the L^p norm inequality as a consequence. The choice of θ has been determined by the proportionality constant $\alpha_{\widetilde{M}}$. So, in the above expression, once ϵ is chosen we are free to choose γ so that $C'(M,\epsilon)\frac{\gamma^2}{(\beta-1)^2} < \frac{\alpha_{\widetilde{M}}}{2}$, contradicting the earlier assumption that $|E_i| > \theta|\Delta_i|$.

The proof of (3.41) is completed by showing the converse inequality—the analog of (3.40). Here, the build-up lemma of G. David enters in the same way, in the course of proving a good-λ inequality. Of course, no normalization is needed for this direction—the details are repetitive, and hence omitted.

Step 4 is preliminary to removing the restriction that the matrix A of $L = \operatorname{div} A\nabla$ be upper triangular. Essentially we need the statement (3.41) for A triangular, but for Lipschitz graphs of arbitrary Lipschitz constant. This is accomplished by proving a good-λ inequality, using the previous result for bounded domains.

Lemma 3.46. *Let $\Omega_{\vec{e},\varphi}$ be a domain above the graph of a Lipschitz function φ with respect to the direction $\vec{e}$. Let $A(x) = \begin{pmatrix} 1 & b(x) \\ 0 & \gamma(x) \end{pmatrix}$ and suppose $Lu = \operatorname{div} A\nabla u = 0$ in $\Omega = \Omega_{\vec{e},\varphi}$. Then inequalities (3.39) and (3.40) (with appropriate normalizations) hold for u and for any $0 < p < \infty$ as well as $p = 2$.*

Proof. It suffices to prove the good-λ inequality (3.45), and its analog for the square function. As in [D] and [DJK] and as we did for 3.42, one constructs sawtooth regions, which are themselves bounded Lipschitz domains, and then invokes the L^2-norm inequality on the bounded domains for the non-tangential maximal function and the square function. This lemma is therefore a standard consequence of Theorem (3.41) for bounded domains.

Step 5 in the outline is the removal of the restriction that the matrix A be triangular—that is, the L^p equivalence between N and S holds for general A on graphs in any direction with small Lipschitz constant. The next lemma shows how the restriction is removed via a change of variable and the proof of the lemma explains the restriction that the graph have small Lipschitz constant. In order to prove for example that $\|N(u)\|_{L^p(\partial\Omega,d\sigma)} \le C\|S(u)\|_{L^p(\partial\Omega,d\sigma)}$ for solutions in Ω to $\operatorname{div} A\nabla u = 0$ (A arbitrary), we use Lemma 3.35 below to transfer the inequality to another domain for a solution to $\operatorname{div} B\nabla(\cdot) = 0$, where B is now triangular.

Lemma 3.35. *Let $\Omega_{\vec{e},\varphi}$ $(= \Omega)$ be the domain above a Lipschitz graph φ in direction $\vec{e}$ with $\|\varphi'\|_\infty \le \epsilon$. Let A be any elliptic matrix $\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$, and suppose that $Lu = \operatorname{div} A\nabla u = 0$ in Ω. Then there exists, for sufficiently small ϵ depending on ellipticity a change of variables $\phi : \widetilde{\Omega} \to \Omega$ such that*

(i) *If $v(z,s) = u \circ \phi(z,s)$ then $\operatorname{div} B\nabla v = 0$ in $\widetilde{\Omega}$, where B is upper triangular and* *independent of the s-variable, of form* $\begin{pmatrix} 1 & b(z) \\ 0 & \gamma(z) \end{pmatrix}$.

(ii) *the domain* $\widetilde{\Omega}$ *is the domain above the graph of a Lipschitz function.*

Proof: Consider the transformation $\phi(z,s) = (f(z), s + g(z))$, $\alpha \neq 0$ with $f' > 0$. We shall first place restrictions on f and g so that condition (i) above is met. The new matrix $B(z,s)$ in the equation that $v = u \circ \phi$ satisfies is

$$(\phi')^{-1,t} A \circ \phi (\phi')^{-1} J\phi = \frac{1}{f'(z)} \begin{pmatrix} 1 & 0 \\ -g'(z) & f'(z) \end{pmatrix} A \circ \phi \begin{pmatrix} 1 & -g'(z) \\ 0 & f'(z) \end{pmatrix}$$
$$= \frac{1}{\alpha f'(z)} \begin{pmatrix} a \circ f & -a \circ f g' + b \circ f f' \\ (-g' a \circ f + f' c \circ f) & (g')^2 a \circ f - g' f' c \circ f - g' b \circ f f' + (f')^2 d \circ f \end{pmatrix}.$$

We want to choose f and g so that $a \circ f(z) = f'(z)$ and so that $g' a \circ f = f' c \circ f$. By ellipticity, the coefficient a is bounded below by λ^{-1}. If $a(z) = f' \circ f^{-1}(z) = \frac{1}{(f^{-1})'}$, then f^{-1} can be defined as a primitive of $1/a$ and the bounds on $a(z)$ guarantee that f is increasing and Lipschitz. We then choose g so that $c \circ f(z) = g'(z)$ and g is also Lipschitz. By Lemma 3.4 it suffices to consider three types of domains $\Omega_{\vec{e},\varphi}$. If $\Omega_{\vec{e},\varphi} = \Omega_{(1,0),\varphi} = \{(x,t) : t > \varphi(x)\}$ then $\widetilde{\Omega}$ has the form $\{(z,s) : s > \psi(z)\}$. Here ψ is defined by $\psi(z) = \varphi \circ f(z) - g(z)$, which implies $s = \psi(z)$ if and only if $s + g(z) = \varphi \circ f(z)$, or $t = \varphi(x)$. If $\Omega_{\vec{e},\varphi} = \Omega_{(0,1),\varphi}$ then we shall be able to choose ψ such that $\widetilde{\Omega} = \{(z,s) : z > \psi(s)\}$ if $\|\varphi'\|_\infty$ is sufficiently small. For the boundaries of $\widetilde{\Omega}$ and Ω to be in correspondence we need $z = \psi(s)$ if and only if $f(z) = \varphi(s + g(z))$. Let $z' = f(z)$ (recall that f^{-1} exists) and thus we need $z' = f \circ \psi(s)$ if and only if $z' = \varphi(s + g \circ f^{-1}(z'))$. That is,

$$\begin{aligned} f \circ \psi(s) &= \varphi(s + g \circ f^{-1}(z)) \\ &= \varphi(s + g \circ \psi(s)). \end{aligned}$$

Set $h(s) = \alpha s + g \circ \psi(s)$. Then

$$s = h^{-1}(s) + g \circ \psi \circ h^{-1}(s) = h^{-1}(s) + g \circ f^{-1} \circ \varphi(s),$$

since $\psi \circ h^{-1}(s) = f^{-1} \circ \varphi(s)$. Solving for $h^{-1}(s) = f^{-1} \circ \varphi(s)$. Solving for h^{-1} gives $h^{-1}(s) = s - g \circ f^{-1} \circ \varphi(s)$, which is invertible as long as $\|(g \circ f^{-1} \circ \varphi)'\|_\infty < 1$. So if we choose $\|\varphi'\|_\infty$ sufficiently small, depending on λ, we may solve for h^{-1} and hence ψ. Finally, suppose that $\Omega_{\vec{e},\varphi} = \{(x,t) : e_1 t > e_2 x + \varphi(x)\}$ where $e_1 \geq \delta e_2$ and $e_2 \geq \delta e_1$. We claim that $\widetilde{\Omega} = \{(z,s) : s > z + \psi(z)\}$. Here we need $s = z + \psi(z)$ if and only if $e_1(\alpha s + g(z)) = e_2 f(z) + \varphi \circ f(z)$, i.e., $s = -g(z) + \frac{e_2}{e_1} f(z) + \frac{1}{e_1} \varphi \circ f(z) = z + \psi(z)$. Solving for ψ gives $\psi(z) = -z + \frac{e_2}{e_1} f(z) - g(z) + \frac{1}{e_1} \varphi \circ f(z)$.

Our strategy now is to pass from the graph situation, for general matrices A, to the case of bounded domains.

Step 6 has, essentially, been carried out in the proof of Theorem 3.18. We need to use the results of Step 5 in order to verify the hypotheses, namely that $N(\cdot)$ and $S(\cdot)$ have comparable L^p norms on small constant graphs.

Finally, the reduction to arbitrary bounded Lipschitz domains, in the situation of general matrices (Step 7), is also a repetition of earlier arguments for Steps 2 and 3. This completes the proof of Theorem 3.1.

References

[Ca] L. Carleson, *On the existence of boundary values of harmonic functions of several variables*, Arkiv Math. **4** (1962), 339–393.

[Ca2] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. **76** (1962), 547–559.

[CFK] L. Caffarelli, E. Fabes and C. Kenig, *Completely singular elliptic-harmonic measures*, Indiana U. Math. J. **30** (1981), 189–213.

[CFMS] L. Caffarelli, E. Fabes, S. Mortola and S. Salsa, *Boundary behavior of non-negative solutions of elliptic operators in divergence form*, Ind. U. Math. J. **30** (1981), 621–640.

[D1] B. Dahlberg, *On estimates for harmonic measure*, Arch. Rat. Mech. Anal. **56** (1977), 272–288.

[D2] B. Dahlberg, *On the Poisson integral for Lipschitz and C^1 domains*, Studia Math. **66** (1979), 13–24.

[D3] B. Dahlberg, *Poisson Semigroups and Singular Integrals*, PAMS, **97** (1986), 41–48.

[D4] B. Dahlberg, *Weighted norm inequalities for the Lusin area integral and the nontangential maximal function fro functions harmonic in Lipschitz domains*, Studia Math. **65** (1980), 297–314.

[D5] B. Dahlberg, *Approximation of harmonic functions*, Ann. Inst. Fourier (Grenoble) **30** (1980), 97–107.

[Da] G. David, *Integrales Singulieres sur des courbes du plan*, Annales Scientifiques de L'Ecole Norm. Sup. **17** (1984), 157–189.

[DeGi] E. De Giorgi, *Sulla differenziabilita e analiticita delle estremali degli integrali multipli regolari*, Mem. Acad. Sci. Torino **3** (1957), 25–43.

[DJK] B. Dahlberg, D. Jerison, and C. Kenig, *Area integral estimates for elliptic differential operators with nonsmooth coefficients*, Arkiv Mat. **22** (1984), 97–108.

[DKPV] B. Dahlberg, C. Kenig, J. Pipher, and G. Verchota, *Area integral estimates for higher order elliptic equations and systems*, Ann. Inst. Fourier (Grenoble) **47** (1997), 1425–1461.

[F-St] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[FKP] R. Fefferman, C. E. Kenig, J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Annals of Math. **134** (1991), 65–124.

[G] J. Garnett, *Bounded Analytic Functions*, vol. 96, Academic Press, 1981.

[Gi] M. Giaquinta *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, vol. 105, Annals of Math. Studies, Princeton U. Press, 1983.

[GW] M. Gruter and K. Widman, *The Green function for uniformly elliptic equations*, Manus. Math. **37** (1982), 303–342.

[G-C,RdeF] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Math. Studies 1996, North Holland, 1985.

[H-W] R. Hunt and R. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **147** (1970), 507–527.

[JK] D. Jerison and C. Kenig, *The Dirichlet problem on non-smooth domains*, Annals of Math. **113** (1981), 367–382.

[K] C. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Regional Conference Series*, AMS, 1994.

[KM] C. Kenig and Y. Meyer, *The Cauchy integral on Lipschitz curves and the square root of second order differential equations are the same*, Recent Progress in Fourier Analysis, North Holland Math. Studies **111** (1985), 123–145.

[KP] C. Kenig and J. Pipher, *The Dirichlet problem for elliptic equations with drift terms*, in preparation.

[KP1] C. Kenig and J. Pipher, *The Neumann problem for elliptic equations with non-smooth coefficients*, Invent. Math. **113** (1993), 447–509.

[LH] J. Lewis and S. Hofmann, *The Dirichlet problem for parabolic operators with singular drift terms*, preprint.

[LSW] W. Littman, G. Stampacchia and H. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Sc. Norm. Sup. Pisa **17** (1963), 45–79.

[Mor] C. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, 1966.

[Mo] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

[Na] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. of Math. **80** (1958), 931–954.

[P] J. Pipher, *Littlewood-Paley estimates: some applications to elliptic boundary value problems*, CRM Proceedings and Lecture Notes **12** (1997).

[St] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[V] N. Varopoulos, *A remark on BMO and bounded harmonic functions*, Princeton University Press, 1970.

[VV] G. Verchota and A. Vogel, *Nonsymmetric systems on nonsmooth planar domains*, TAMS **349**, No. 11, (1997), 4501–4535.