1. Let $X$ be an $n$-point set of $d$-length sequences $\{x_k\}$ of 0s and 1s, and equip this set with the $l_1$ size: that is, the distance between two sequences $\{x_k\}$ and $\{y_k\}$ is

$$\sum_{k=1}^{d} |x_k - y_k|. $$

Let’s name this metric space $l_d^1$. Find an isometric embedding of $l_d^1$ into the sequence space $l_{\infty}'$, where $d' = 2^d$ and the distance between $\{w_j\}$ and $\{z_j\}$ is the sup$|w_j - z_j|$. Hint: Define, for every choice of $\sigma \in \{+1, -1\}^d$, a mapping $f$ into the space of sequences indexed by $\sigma$.


3. Let $p > 1$. If $X$ is the space of all sequences $x = \{x_k\}$ such that

$$\|x_k\|_p := \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty,$$

then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$  

There were two hints in the lecture: one involved the splitting

$$|x_k + y_k|^p \leq |x_k|(|x_k + y_k|)^{p-1} + |y_k|(|x_k + y_k|)^{p-1},$$

and the other involved using the inequality

$$ab \leq a^p/p + b^q/q,$$

for $q = p/(p - 1)$ and $a, b > 0$. The remaining trick involves finding the right normalization...

4. In a metric space $(X, d)$, define the ball centered at $x$ of radius $r$ by $B(x, r) := \{y : d(x, y) < r\}$. A subset $A$ of a metric space is said to be open if for all $x \in A$, there exists an $r$ such that $B(x, r) \subset A$.

(i) Show that these balls are in fact open sets.

(ii) Show that a countable union of open sets is open.

(iii) Show that a finite intersection of open sets is open, but a countable intersection may fail to be open.

5. A sequence in a metric space is called Cauchy if for all $n, m$, the distance $d(x_n, x_m) \to 0$ as $n, m \to \infty$. A metric space is called complete if every Cauchy sequence converges. By Theorem 2.19, $\mathbb{R}$ is complete.

(i) Show that the sequence space $l_d^1$ of problem (1) is complete.

(ii) Show that the space of all bounded functions $f : X \to \mathbb{R}$, where $X$ is some set, with the metric $d(f, g) := \sup\{|f(x) - g(x)| : x \in X\}$ is complete.