13. \(\Rightarrow\) Suppose that \(X\) is not compact. Then there exist an open cover \(\{U_a\}_{a \in A}\) say, of \(X\), with property that \(X \setminus \bigcup_{i=1}^n U_{x_i} \neq \emptyset\) for any finite selection \(x_1, \ldots, x_n \in A\).

Setting \(F_a = X \setminus U_a\), we thus have a collection \(F = \{F_a\}_{a \in A}\) of closed sets in \(X\) with property:

\[
\bigcap_{i=1}^n F_a = \bigcap_{i=1}^n (X \setminus U_{x_i}) = X \setminus \bigcup_{i=1}^n U_{x_i} \neq \emptyset
\]

for any finite \(x_1, \ldots, x_n \in A\).

That is, \(F\) has the finite intersection property, then we have \(\bigcap_{a \in A} F_a \neq \emptyset\), i.e. \(X \setminus \bigcup_{a \in A} U_a \neq \emptyset\), which contradicts the assumption that \(U\) is an open cover of \(X\).

\[\therefore X \text{ is compact.}\]

\(\Leftarrow\) Assume \(X\) is compact. and suppose (for contradiction) that there is a collection \(F = \{F_a\}_{a \in A}\) of closed subsets of \(X\) with the f.i.p., but \(\bigcap_{a \in A} F_a \neq \emptyset\).

Set \(U_a = X \setminus F_a\), so \(\bigcup_{a \in A} U_a = X \setminus \bigcap_{a \in A} F_a = X\)

i.e. \(\{U_a\}\) is an open cover of \(X\). By compactness, there is a finite subcover. \(X \subseteq \bigcup_{i=1}^n U_{x_i}\), where \(\emptyset = X \setminus \bigcup_{i=1}^n U_{x_i} = \bigcap_{i=1}^n F_{x_i}\) \(\Rightarrow\) contradiction with f.i.p. \(\#\)
Suppose that \( f \) is continuous.

All open sets in \( G(f) \) are of the form \( f(\{x, f(x)\}) \mid x \in U \), \( f(x) \in V \)

for \( U \) open in \( X \), \( V \) open in \( \mathbb{R} \).

This set is equivalent to \( \{ (x, f(x)) \mid x \in U \cap f^{-1}(V) \} \).

Now suppose \( C = \{ O_x \mid O_x \text{ open in } G(f) \} \) is an open cover of \( G(f) \), since \( X \) is compact and all open sets in \( G(f) \) depend only on elements of \( X \), i.e.

\[
C = U \cup O_x = U \cup \{ (x, f(x)) \mid x \in U \cap f^{-1}(V) \}, \quad \text{size } \{ U \cap f^{-1}(V) \}
\]

is an open cover for \( X \), and \( X \) is compact, then we can find a finite subcover, then we can get a finite subcover of \( C \) correspondingly.

\[
\therefore G(f) \text{ is compact.}
\]

Now suppose \( G(f) \) is compact.

Let \( E \subset \mathbb{R} \) closed, the projection mapping \( \pi_2 : G(f) \to \mathbb{R} \) is continuous, so \( \pi_2^{-1}(E) \) is closed in \( G(f) \), we have that \( \pi_2^{-1}(E) \) is a closed subset of a compact set, so it's compact.

We also have \( \pi_1 : G(f) \to X \) continuous, \( \therefore \pi_1(\pi_2^{-1}(E)) = f^{-1}(E) \) is compact in \( X \), and \( X \) is Hausdorff, so \( f^{-1}(E) \) is closed, \( \therefore f \) is continuous.
injective: if \( f(x) = f(y) \), then \( d(f(x), f(y)) = 0 \Rightarrow f(x, y) \),
\[ \Rightarrow x = y \]

surjective: let \( y = f(x) \), if \( y \in X \), i.e., \( x \notin Y \),
let \( x \in (X \setminus Y, f \equiv f(x, y)) \).

\( S = 0 \) then \( x \notin Y \). However, \( f \) is continuous, so \( f(X) \) is compact, but \( X \) is a metric space, so it's Hausdorff.
then all compact subsets are closed.
\[ \therefore Y \text{ is closed} \Rightarrow Y = \overline{Y} \Rightarrow x \in \overline{Y} \text{ contradiction.} \]

\( S > 0 \)

define sequence \( \{ x_n \} \) by \( x_{n+1} = f(x_n) \).
Then for all \( m < n \),
\[ f(x_m, x_n) \neq f(f(x_{m-1}), f(x_{m-1})) = f(f(x_{m-1}, x_{m-1})) \]
\[ \vdots \]
\[ = f(x_{n-m}, x_n) > S \]
\[ \therefore x \text{ is not sequentially compact}. \]
\[ \Rightarrow x \text{ is not compact}. \times \]
\[ \therefore f \text{ is bijective} \]
Let \( X \) be a compact space, and let \( f : X \to (\mathbb{R} \cup \{\pm \infty\}) \) be a u.s.c. function. Let \( A = f(X) \) and \( \forall c \in A \), define \( F_c = \{ x \in X : f(x) > c \} = f^{-1}(\{c\}) \). Since \( (-\infty, c) \) is closed in \( f \) and \( f \) is continuous, \( F_c \) is closed, also, notice that \( F_c \neq \emptyset \).

Now consider any finite subcollection \( \{F_{c_1}, \ldots, F_{c_n}\} \subseteq \{F_c : c \in A\} \), \( \bigcap_{i=1}^{n} F_{c_i} = \{ x \in X : f(x) > \max\{c_1, \ldots, c_n\} \} \).

That is, \( \bigcap_{i=1}^{n} F_{c_i} = F_{c_j} \neq \emptyset \), where \( c_j = \max\{c_1, \ldots, c_n\} \).

Therefore, \( f \) has a finite intersection property.

By problem 13, since \( X \) is compact, we have \( \bigcap_{c \in A} F_c \neq \emptyset \), so exist \( y \in \bigcap_{c \in A} F_c \), such that \( f(y) > c \) for all \( c \in A \).

\[ \therefore f \text{ assumes maximum value at } y. \]