HWS 5 solution.

1. Proof

   need to show for E closed under F, \( \Rightarrow \) E closed under T.

   Since all closed subset of compact space are compact, so E is compact under F.

   \( \Rightarrow \) TCF, any open cover of E under T is also an open cover of E under F.

   E is compact under F, so we can find a finite subcover in F, then it's also a finite subcover in T. (since we first suppose it's open cover under T)

   \( \Rightarrow \) E is compact under T.

   \( \Rightarrow \) T is Hausdorff.

   \( \therefore \) E is closed under T

   \( \therefore \) FCT

   Therefore, \( F = T \)
2. Let \( D = \{(x, x) : x \in X\} \).

Only need to show \( D^c = \{(x, y) : x \neq y\} \) is open.

For \((x, y) \in D^c\), \( x \neq y \),
then since \( X \) Hausdorff, exist disjunct open sets
\( U, V, x \in U, y \in V, U \cap V = \emptyset \)
then we have \( U \cap V \subseteq D^c \) and \( U \cap V \) is open in \( X \times X \).

\( \therefore D^c \) is open.

Then we know \( D \) is closed. \( \# \)

3. Let \( C \subseteq G \) be a component of \( G \).

If \( x \in C \), since \( G \) is locally connected, there exists
an open connected neighborhood \( U_x \) of \( x \).

Now since \( U_x \) is connected and \( x \in U_x \), \( \therefore U_x \subseteq C \).

\( \therefore C = \bigcup_{x \in C} x = \bigcup_{x \in C} U_x \) union of open sets

\( \therefore C \) is open. \( \# \)
22. Let \( U \) be open subset of \( \mathbb{R} \).

Since \( \mathbb{R} \) is connected, then it's also locally connected.

\( \therefore \) From 21, we know every component of \( \mathbb{R} \) is open and \( \mathbb{R} = \mathbb{R} \) or \( \mathbb{R} = \mathbb{R} \).

\( \therefore U = \bigcup C \) must be union of disjoint component.

Also, we know every component is an open interval (since it's open).

\( \therefore U \) is the union of disjoint sequence of open interval.

23. (i) Let \( F \) be a set of convex sets in \( \mathbb{R}^d \), \( D = \bigcap_{C \in F} C \).

\( \forall x, y \in D, \quad x, y \in C \) for all \( C \in F \)

and since each \( F \) convex, \( \therefore t \in (0,1) \quad y \in C \) for all \( C \in F \).

\( \therefore t x + (1 - t) y \in D \) for all \( t \in (0,1) \).

\( \therefore D \) is convex.

(ii) Now let \( C \subset \mathbb{R}^d \) convex. For \( x, y \in C \).

Define \( \delta_{xy} : [0,1] \to C \) as

\( \delta_{xy}(t) = ty + (1-t)x, \quad \forall C \in D \). (since \( C \) convex)

\( \delta_{xy}(0) = x, \quad \delta_{xy}(1) = y \)

\( \therefore \) when it's easy to see that \( C \) is connected.

\( \therefore \)
$\forall i \in \{x, y\}, \text{ either } x \text{ or } y \text{ irrational}\}$

First, show that if $(x, y) \in X$, then there is a continuous path to $(\bar{x}, \bar{y}) \in X$.

Since $(x, y) \in X$, w.l.o.g., suppose $x \in \mathbb{Q}$.

Construct a path as in graph.

1. $(x, y) \rightarrow (x, \bar{y})$

   $y(t) = (x, t\bar{y} + (1-t)y)$ \quad \text{ if } \quad y(t) = (x, t\bar{y} + (1-t)y) \in X$

2. $(x, \bar{y}) \rightarrow (\bar{x}, \bar{y})$

   $x(t) = (t\bar{x} + (1-t)x, \bar{y}) \in X$

   \[\therefore (x_1, y_1), (x_2, y_2) \in X.\]

   Can construct a continuous path via $(\bar{x}, \bar{y})$.

   $(x_1, y_1) \rightarrow (\bar{x}, \bar{y}) \rightarrow (x_2, y_2)$

   \[\therefore X \text{ is connected}\]