The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function

Martin Dindos*, Stefanie Petermichl†, Jill Pipher‡

October 26, 2006

Abstract

We establish L^p-solvability for $1 < p < \infty$ of the Dirichlet Problem on Lipschitz domains with small Lipschitz constants for elliptic divergence and non-divergence type operators with rough coefficients obeying a certain Carleson condition with small norm.

1 Introduction

This paper continues the study, began in [10], of boundary value problem for second order elliptic operators in either divergence or non-divergence form, when the coefficients satisfy certain natural, minimal smoothness conditions. Specifically, we first consider operators L of divergence form with lower order (drift) terms; that is, $L = \text{div}A\nabla + b.\nabla$ where $b = (b_1, ..., b_n)$ and $A(X) = (A_{ij}(X))$ is strongly elliptic in the sense that there exists a positive constant λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j < \lambda^{-1}|\xi|^2,$$

for all X and all $\vec{\xi} \in \mathbb{R}^n$. The main results of this paper will be established first for $L = \text{div}A\nabla$ and then extended to the full operator with drift terms, $L = \text{div}A\nabla + b.\nabla$, under appropriate conditions on the vector b, via the work of S. Hoffman and J. Lewis[7]. It will then be straightforward to extend these results to non-divergence operators as well. One feature of these theorems is that it is not assumed that the matrix A is symmetric. We shall obtain solvability of the Dirichlet boundary value problem for a class of operators (in both divergence and non-divergence form) when the data is in L^p, for a full range of $1 < p < \infty$.

The operators we consider here have coefficients satisfying a small, or a vanishing Carleson measure condition (see Section 3). The condition on the coefficients is related

*supported by EU FP6 grant # 017407

†Supported by NSF grant # DMS 9729992

‡supported by NSF gran #DMS 0600389

to the condition in [10] as BMO is related to VMO, the space of vanishing mean oscillation. Precise definitions are given in Section 2.

Operators whose coefficients satisfy the vanishing Carleson condition arise in several contexts. For example, consider the Dirichlet boundary value problem associated to the Laplacian in the region above a graph $t = \varphi(x)$. When φ is C^1, it was shown in [5] that the Dirichlet (and Neumann) problems were solvable with data in L^p for $1 < p < \infty$, by the method of layer potentials. Our main theorem will contain and generalize this result: the Dirichlet problem is solvable in this range of p when the boundary of the domain is defined by $t = \varphi(x)$ where $\nabla\varphi \in L^\infty \bigcap VMO$. This corollary can be proven using a change of variable, namely the mapping described below, a variant of Dahlberg's adapted distance function [1].

Let Ω denote the domain in $\mathbb{R}^n_+$ given by $t > \varphi(x)$. Consider the mapping [1] from $\mathbb{R}^n_+$ to Ω of the form

$$\rho(x, t) = (x, ct + \theta_t * \varphi(x)),$$

where c is a constant that depends on $\|\nabla\varphi\|_\infty$ and can be chosen large enough to insure that ρ is one-one. The function $\theta \in C_0^\infty(\mathbb{R}^n)$ is even, and $\theta_t(\cdot) = t^{-n}\theta(\cdot/t)$. The pullback of Δ from Ω to $\mathbb{R}^{n+1}_+$ is also a symmetric elliptic operator, $L = \mathrm{div}A\nabla$, where A possesses the properties:

1. $|\nabla A(x, t)| \le C/t$.

2. $t|\nabla A(x, t)|^2 dxdt$ is a Carleson measure.

(See section 2 for the definition of Carleson measure.)

In 1984, Dahlberg posed two conjectures. The first conjecture concerned perturbation of operators. Suppose that, in the upper half space $\mathbb{R}^{n+1}_+$, one has an elliptic operator $L_0 = \mathrm{div}A_0\nabla$ for which the Dirichlet problem (D_p) with data in $L^p(\mathbb{R}^n, dx)$ is solvable. Now suppose $L_1 = \mathrm{div}A_1\nabla$ is a perturbation of L_0 in the sense that

$$\sup\{|A_1(x', t') - A_0(x', t')|^2 : |x - x'| < t/2\}dx\frac{dt}{t}$$

is a Carleson measure. Then, is the Dirichlet problem D_q for L_1 also solvable, where q may be larger than p? The conjecture has an equivalent formulation: does the measure $d\omega_{L_1}$ belong to the Muckenhoupt class $A_\infty(d\omega_{L_0})$. This conjecture was solved affirmatively in [6], where references to the prior work may also be found. Dahlberg's second conjecture concerned classes of operators whose coefficients satisfy an averaging variant of conditions (1) and (2) above, as opposed to perturbations of reasonable operators. For such operators (see Theorem 2.6 of [10] for precise conditions), it was shown there that the A_∞ condition holds, which means that D_p is solvable for some $p > 1$. Here we consider the related question of what happens if the Carleson condition is replaced by its VMO analog, and we show that the Dirichlet problem, D_p, for such L is solvable for all $p > 1$.

Until recently, most positive results proving A_∞ estimates for a class of elliptic operators relied on L^2 identities, in the spirit of [8], which in turn relied the assumption that the matrix A was both real and symmetric. ([4] is one interesting exception to this.) But there are a variety of reasons for studying the non-symmetric situation. These include the connections with nondivergence form equations, and the broader issue of obtaining estimates on elliptic measure in the absence of special L^2 identities which relate tangential and normal derivatives.

In [9], the study of nonsymmetric divergence form operators with bounded measurable coefficients was initiated. In [10], the methods of [9] were used to prove A_∞ results for elliptic measures of operators satisfying the bounds and (a variant of) the Carleson measure conditions (1) and (2) above. In this paper we develop the A^p results in three contexts: second order divergence form operators whose coefficients satisify gradient conditions, non-divergence form operators whose coefficients satisfy gradient conditions, and divergence form operators whose coefficients satisfy a a Poincaré type condition on differences instead of a gradient condition.

The paper is organized as follows. In Section 2, we give some definitions and state the main results, as well deriving some quick corollaries. Section 3 contains the proofs of several lemmas and in Section 4, we prove the main theorems.

We thank Carlos Kenig and Michael Taylor for helpful conversations and a couple of technical observations.

2 Definitions and Statements of Main Theorems

Let us begin by defining introducing Carleson measures and square functions on domains which are locally given by the graph of a function. We shall therefore assume that our domains are Lipschitz.

Definition 2.1. $\mathbb{Z} \subset \mathbb{R}^n$ *is an M-cylinder of diameter d if there exists a coordinate system (x,t) such that*

$$\mathbb{Z} = \{(x,t) \ : \ |x| \le d, \ -2Md \le t \le 2Md\}$$

and for $s > 0$,

$$s\mathbb{Z} = \{(x,t) \ : \ |x| < sd, -2Md \le t \le 2Md\}.$$

Definition 2.2. $\Omega \subset \mathbb{R}^n$ *is a Lipschitz domain with Lipschitz 'character' (M, N, C_0) if there exists a positive scale r_0 and at most N cylinders $\{Z_j\}_{j=1}^N$ of diameter d, with $\frac{r_0}{c_0} \le d \le c_0 r_0$ such that*

(i) $8Z_j \cap \partial\Omega$ is the graph of a Lipschitz function ϕ_j,

$$\|\phi_j\|_\infty \le M, \ \phi_j(0) = 0,$$

(ii)

$$\partial\Omega = \bigcup_j (Z_j \cap \partial\Omega)$$

(iii)

$$Z_j \cap \Omega \supset \left\{(x,t) \ : \ |x| < d, \ \mathrm{dist}\,((x,t), \partial\Omega) \le \frac{d}{2}\right\}.$$

If $Q \in \partial\Omega$ and

$$B_r(Q) = \{x : |x - Q| \leq r\}$$

then $\Delta_r(Q)$ denotes the surface ball $B_r(Q) \cap \partial\Omega$ and $T(\Delta_r) = \Omega \cap B_r(Q)$ is the called the Carleson region above $\Delta_r(Q)$.

Definition 2.3. *Let $T(\Delta_r)$ be a Carleson region associated to a surface ball Δ_r in $\partial\Omega$.. A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$,*

$$\mu(T(\Delta_r)) \leq C\sigma(\Delta_r).$$

The best possible C is the Carleson norm. When $d\mu$ is Carleson we write $d\mu \in \mathcal{C}$.
If $\lim_{r_0 \to 0} C(r_0) = 0$, then we say that the measure μ satisfies the vanishing Carleson condition, and we denote this by writing $d\mu \in \mathcal{C}_V$.

Definition 2.4. *A cone of aperture a is a non-tangential approach region for $Q \in \partial\Omega$ of the form*

$$\Gamma_a(Q) = \{X \in \Omega : |X - Q| \leq a \ \ dist(X, \partial\Omega)\}.$$

Sometimes it is necessary to truncate the height of Γ by h. Then $\Gamma_{a,h}(Q) = \Gamma_a(Q) \cap B_h(Q)$.

When $p = 2$, the square function appearing below is the classical square function for a Lipschitz domain, as in [2] for example.

Definition 2.5. *If $\Omega \subset \mathbb{R}^n$, the p-adapted square function in $Q \in \partial\Omega$ relative to a family of cones Γ is*

$$S_p u(Q) = \left(\int_{\Gamma(Q)} |\nabla u(X)|^2 |u|^{p-2}(X) dist(X, \partial\Omega)^{2-n} dX \right)^{1/p}.$$

and the non-tangential maximal function at Q relative to Γ is

$$Nu(Q) = \sup\{|u(X)| : X \in \Gamma(Q)\}.$$

There are several remarks in order here, since the solution u is not assumed to be positive. Even in the case of harmonic functions, it is not obvious that the expressions appearing in the integral are locally integrable. In fact, the following Cacciopoli type inequality holds for $|u|^{p-2}|\nabla u|^2$:

Proposition 2.1. *Suppose $\Delta u = 0$ in Ω, B_r is a ball of radius r such that B_{2r} is compactly contained in Ω, then, for $p > 1$,*

$$\int_{B_r} |u|^{p-2} |\nabla u|^2 dx \leq C_p \frac{1}{r^2} \int_{B_{2r} \setminus B_r} |u|^p dx$$

Proof. Set $u_\epsilon = \sqrt{u^2 + \epsilon^2}$ and observe that

$$|\nabla u_\epsilon|^2 = \frac{u^2 |\nabla u|^2}{u^2 + \epsilon^2}.$$

Therefore, $|u_\epsilon|^{p-2}|\nabla u_\epsilon|^2 \to |u|^{p-2}|\nabla u|^2$ as $\epsilon \to 0$.

By Fatou, it suffices to control the lim inf of $\int_{B_r} |u_\epsilon|^{p-2}|\nabla u_\epsilon|^2 dx$. Compute the Laplacian of u_ϵ:

$$\Delta u_\epsilon = \frac{|\nabla u|^2 \epsilon^2}{(u^2 + \epsilon^2)^{3/2}},$$

which is non-negative.

If we now compute the Laplacian of u_ϵ^p, we have that

$$\Delta u_\epsilon^p = p u_\epsilon^{p-1} \Delta u_\epsilon + p(p-1) u_\epsilon^{p-2} |\nabla u_\epsilon|^2. \tag{2.1}$$

Let η be a C^∞ function identically 1 on B_r and supported in $B_2 r$. The first term in 2.1 is positive, and so

$$\int |u_\epsilon|^{p-2}|\nabla u_\epsilon|^2 \eta^2 dx < \frac{1}{p(p-1)} \int \Delta u_\epsilon^p \eta^2 dx.$$

Integration by parts and Cauchy-Schwarz gives that this is in turn bounded by

$$C_p \int p|u_\epsilon|^{p-1}|\nabla u_\epsilon| \eta |\nabla \eta| dx \leq C_p (\int |u_\epsilon|^{p-2}\eta^2 dx)^{1/2} (\int |u_\epsilon|^{p-2}|u_\epsilon|^2 |\nabla \eta|^2 dx)^{1/2},$$

where $C_p \sim \frac{1}{p-1}$.

The limit as $\epsilon \to 0$ gives the inequality

$$\int |u|^{p-2}|\nabla u|^2 \eta^2 dx \leq C \int |u|^p |\nabla \eta|^2 dx.$$

□

The argument is perfectly general, and works for solutions u of $Lu = 0$ when $L = divA\nabla$ is elliptic and A is bounded and measurable.

Thus we will use the fact that Proposition 2.1 holds for solutions $Lu = 0$ also.

This local integrabilty justifies an apriori assumption of finiteness of the p-adapted square function and the integration by parts.

Definition 2.6. *The Dirichlet problem with data in $L^p(\partial\Omega, d\sigma)$ is solvable for L if the solution u for continuous boundary data f satisfies the estimate*

$$\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)}.$$

The implied constant depends only the ellipticity of the operator, the p, and the Lipschitz constant of the domain as measured by the triple of Definition 2.2.

We now state our main Theorems, and some corollaries.

Theorem 2.2. *Let $1 < p < \infty$. Let $L = divA\nabla$ be an elliptic operator and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with small Lipschitz constant M. Let $\delta(X) = dist(X, \partial\Omega)$ and suppose that $A = (a_{ij})$ has distributional derivatives so that*

$$\sup\{\delta(X)|\nabla a_{ij}(X)|^2 \;:\; X \in B_{\delta(Z)/2}(Z)\} \tag{2.2}$$

is the density of a Carleson measure in Ω with norm C. Then there exists $\varepsilon(p) > 0$ such that if $C < \varepsilon(p)$ and $M < \varepsilon(p)$, then the L^p Dirichlet problem for the operator L is solvable.
In particular, if the domain Ω is C^1 and $A = (a_{ij})$ satisfies the vanishing Carleson condition, then the Dirichlet problem is solvable for all $1 < p < \infty$. More generally, the conclusion of the theorem holds in domains whose boundary is locally given by a function ϕ such that $\nabla\phi$ belongs to $L^\infty \cap VMO$.

The proof of Theorem 2.2 uses the the assumption that the expression in 2.2 is small when the Carleson region is also small.

There is a reformulation of the gradient condition in terms of differences of values which follows as a corollary. If Z is a point in Ω, let $avg(a(Z))$ denote the average of the function a over the interior ball $B_{\delta(Z)/2}(Z)$.

Corollary 2.3. *The conclusion of 2.2 holds if the coefficients of A satisfy the following.*

$$\sup\{(\delta(X))^{-1}|a_{ij}(X) - avg(a_{ij}(X))|^2 \;:\; X \in B_{\delta(Z)/2}(Z)\} \tag{2.3}$$

is a sufficiently small, or vanishing, Carleson measure in Ω.

Proof. We prove the Corollary when the domain is flat. The general result will follow from a change of variables, as in the argument for Theorem 2.2 below.

Let's fix the notation in this case, dropping the subscripts on the matrix coefficients when no confusion arises. The expression $avg(a)$ at a point (y, s) is the average of a over the ball $B_{s/2}(y, s)$ centered at (y,s) of radius $s/2$. Given a matrix coefficient $a(x, t)$ in $\mathbb{R}^n_+$, set $\widetilde{a}(x, t) = \int a(u, s)\phi_t(x - u, s - t)dsdu$ where ϕ is a smooth bump function supported in the ball of radius $1/2$ and $\phi_t(y, s) = t^{-n}\phi(y/t, s/t)$.

We are assuming that

$$(\sup\{|a(y, , t) - \text{avg}(a(y, t))|^2 : (y, s) \in B_{t/2}(x, t)\})\frac{dxdt}{t} \tag{2.4}$$

is a Carleson measure with small norm.

We aim to establish two facts:

$$t|\nabla\widetilde{a}(x, t)|^2 dxdt \tag{2.5}$$

is a Carleson measure, with small norm, and

$$(\sup\{|a(y, , t) - \widetilde{a}(y, t)|^2 : (y, s) \in B_{t/2}(x, t)\})\frac{dxdt}{t} \tag{2.6}$$

satisfies the hypothesis of Theorem 1 of [Dahlberg] (In fact it does not satisfy one condition, namely the vanishing Carleson norm of the difference, however as was shown in [6] Theorem 2.18 where Dahlberg's result is reproven, the small Carleson norm suffices.)

From 2.5 we use Theorem 2.2 to conclude solvability of the L^p Dirichlet problem for the operators whose matrix is $\widetilde{a}_{ij}$. From 2.6 we use Dahlberg's theorem to draw the same conclusion for the operator with coefficients a_{ij} since this is a small Carleson perturbation of the $\widetilde{a}_{ij}$.

That 2.5 follows from the hypotheses is a straightforward computation. Apply the gradient to $\phi_t(y,s)$, and subtract a constant from the a_{ij} inside the integrand to see that

$$|\nabla\widetilde{a}(x,t)| \leq Ct^{-1}(\sup\{|a(y,,t) - \text{avg}(a(y,t))| : (y,s) \in B_{t/2}(x,t)\}).$$

The proof of 2.6 is equally straightforward: add and subtract the constant $avg(a(y,t))$ inside the difference.

□

We prove two results which show that we can add drift terms which satisfy a vanishing Carleson condition and get solvability of the L^p Dirichlet problem. Both of these results rely on a main lemma, which is stated and proved in the next section. Our proof is perturbative and so we find we still need some bound on the Carleson density. Thus for example, this theorem does not prove that drift terms can be added to operators whose coefficients satisfy instead the averaging condition in 2.3.

Theorem 2.4. *Let $1 < p < \infty$. Let $L = \text{div} A\nabla + b.\nabla$ be an elliptic operator for which the L^p Dirichlet problems is solvable. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with small Lipschitz constant M and let $\delta(X) = dist(X, \partial\Omega)$ and suppose that $A = (a_{ij})$ and $b = (b_i)$ have distributional derivatives satisfying*

$$\sup\{\delta(X)|\nabla a_{ij}(X)|^2 \ : \ X \in B_{\delta(Z)/2}(Z)\} \tag{2.7}$$

is the density of a Carleson measure in Ω with norm C_1. and

$$\sup\{\delta(X)|b_i(X)|^2 \ : \ X \in B_{\delta(Z)/2}(Z)\} \tag{2.8}$$

is the density of a Carleson measure in Ω with norm C_2. Then there exists $\varepsilon(p) > 0$ such that if $C_2 < \varepsilon(p)$ and $M < \varepsilon(p)$, then the L^p Dirichlet problem for the operator L is solvable.

Corollary 2.5. *The conclusion of Theorem 2.2 holds when the coefficients a_{ij} satisfy the hypotheses, but where $L = a_{ij}D_iD_j$ is a non-divergence elliptic operator.*

Proof. If $L = a_{ij}D_iD_j$, then $L = \text{div}(a_{ij})\nabla$ plus lower order terms of the form $b.\nabla$ where the b_i satisfy the vanishing Carleson condition of the main Theorem. □

We start with the following key lemma. Here we assume that the boundary $\partial\Omega$ is a smooth set - n-dimension compact manifold. We will see later that all other cases can be reduced to this one.

Lemma 2.6. *Let $1 < p < \infty$ be given. Assume that Ω be a bounded domain with smooth boundary. let $Lu = divA\nabla u + B.\nabla u$ be an elliptic differential operator with bounded coefficients satisfying*

$$\sup\{\delta(X)|\nabla a_{ij}(X)|^2 \ : \ X \in B_{\delta(Z)/2}(Z)\} \tag{2.9}$$

is the density of a Carleson measure on all Carleson boxes of size $\leq r_0$ with norm C, and similarly for

$$\sup\{\delta(X)|b_i(X)|^2 \ : \ X \in B_{\delta(Z)/2}(Z)\} \tag{2.10}$$

Then, given $\varepsilon > 0$ there exists $r_1 > 0$ depending only on p, ε and the geometry of the domain Ω such that for all $0 < r \leq \min\{r_0, r_1\}$, if u is a bounded nonnegative solution to $Lu = 0$ in the domain Ω then

$$\int_{\Omega_{r/2}} |u|^{p-2}|\nabla u|^2 \, dist(X, \partial\Omega) dX \leq C_1 \int_{\partial\Omega} |u|^p dX + \varepsilon \int_{\partial\Omega} N_r(u)^p d\sigma, \tag{2.11}$$

provided the Carleson norm $C = C(\varepsilon, p) > 0$ is sufficiently small. Here $\Omega_r = \{X \in \Omega; dist(X, \partial\Omega) < r\}$ and $N_r(u)$ is the standard nontangential maximal function truncated at height r, that is computed only for $X \in \Omega_r$.

Hence, by combining Lemma 2.6 and Proposition 3.2 we will be able to show the following:

Corollary 2.7. *Let $1 < p \leq 2$. Consider any operator L of the form $Lu = \mathrm{div} A\nabla u$ on a Lipschitz domain Ω with bounded and strongly elliptic coefficients A such that (2.9) is a Carleson measure. Then for any solution $Lu = 0$ in Ω*

$$\|N(u)\|_{L^p(\partial\Omega)} \approx \|S_p(u)\|_{L^p(\partial\Omega)} + \left| \int_{\partial\Omega} u \, d\sigma \right|. \tag{2.12}$$

We start by proving the Lemma 2.6.

Proof. Note that (2.11) is a statement about what happens near the boundary of Ω. For this reason we introduce a convenient parametrization of points near $\partial\Omega$.

We want to write any point $X \in \Omega$ near $\partial\Omega$ as $X = (x, t)$ where $x \in \partial\Omega$ and $t > 0$. The boundary $\partial\Omega$ itself then will be the set $\{(x, 0); x \in \partial\Omega\}$. One way to get such a parametrization is to consider the inner normal N to the boundary $\partial\Omega$. The assumption that $\partial\Omega$ is smooth implies smoothness of N. On Ω we have a smooth underlying metric (most likely just Euclidean metric in $\mathbb{R}^n$ if $\Omega \subset \mathbb{R}^n$). We consider the geodesic flow $\mathcal{F}_t$ in this metric starting at any point $x \in \partial\Omega$ in the direction $N(x)$. We assign to a point $X \in \Omega$ coordinates (x, t) if $X = \mathcal{F}_t x$, that is starting at $x \in \partial\Omega$ it takes time t for the flow to get to X. It's an easy exercise that the map $(x, t) \mapsto X = \mathcal{F}_t x$ is a smooth diffeomorphism for small $t \leq t_0$. Using this parametrization we consider the set $\Omega_{t_0} = \{(x, t); x \in \partial\Omega \text{ and } 0 < t < t_0\}$.

Let us now deal with the issue of the metric. We want to work with the simplest possible metric on Ω available. Since we only work on Ω_{t_0} we take our metric tensor there to be a product $d\sigma \otimes dt$ where $d\sigma$ is the original metric tensor on Ω restricted to $\partial\Omega$. The product metric $d\sigma \otimes dt$ is different that the original metric on Ω, but they are both smooth and comparable, that is the distances between points are comparable. Now we express the operator L in this new metric.

Since $\partial\Omega$ itself is a smooth compact manifold of dimension $n-1$ we can find a finite collection of open sets $U_1, U_2, \dots, U_m$ of in $\mathbb{R}^{n-1}$ and smooth diffeomorphisms $\varphi_i : U_i \to \partial\Omega$ such that $\bigcup_i \varphi_i(U_i)$ covers $\partial\Omega$. From now on we will work on one such open set $U = U_i$ with corresponding map $\varphi = \varphi_i$. We can now consider the operator L as being defined on an open subset $U \times (0, t_0)$ of $\mathbb{R}^n_+$, where $\partial\Omega$ corresponds to the hyperplane $\{(x,0); x \in U\}$. We achieve this by pulling back the coefficients of L from Ω_{t_0} to $U \times (0, t_0)$ using the smooth map $\Phi : (x,t) \mapsto (\varphi(x), t)$. Hence from now on, we consider L as being given on an open set $U \times (0, t_0) \subset \mathbb{R}^n_+$. At this stage we also pull back the product metric $d\sigma \otimes dt$ from Ω_{t_0} to $U \times (0, t_0)$ and we get another product metric $d\sigma' \otimes dt$ on $U \times (0, t_0)$.

We note that under this pullback the new coefficients of our operator are going to satisfy the same Carleson condition as the original coefficients with Carleson norm comparable to the original.

Let $0 < r \le r_0$ be fixed. Consider an arbitrary open set $B \subset U \subset \mathbb{R}^{n-1}$ of diameter $\operatorname{diam}(B) \in (r/2, r)$. Let $\widetilde{B} = \{x \in \mathbb{R}^{n-1}; \operatorname{dist}(x, B) < 2r\}$, i.e. $\widetilde{B}$ is a set of diameter $\le 3r$ containing $\overline{B}$.

By $T_r(B)$ we denote the Carleson-like region in $\mathbb{R}^n_+$

$$T_r(B) = \{(x,t); x \in B \text{ and } 0 < t < r\}.$$

Let $\phi(x,t) = \phi(x)$ be any smooth function defined on $\mathbb{R}^n_+$, independent of t variable such that

$$0 \le \phi(x,t) \le 1, \quad \phi > 1/2 \text{ on } T_r(B), \quad \operatorname{supp} \phi \subset \widetilde{B} \times \mathbb{R}.$$

The computation below, which results in (2.24) does not require the assumption that u is non-negative. From (2.24) the bound $\|S_p(u)\|_p \le C\|N(u)\|_p$ follows. The opposite inequality is part (c) of Proposition 3.2.

Note that $\operatorname{dist}(X, \partial\Omega)$ for a point $X = (x,t)$ is now exactly equal to t, so instead of the lefthand side of (2.11) (by the ellipticity of the coefficients) we are going to estimate the comparable expression

$$\int_{T_r(\widetilde{B})} |u|^{p-2} \frac{a_{ij}}{a_{nn}} (\partial_i u)(\partial_j u) \phi t \, d\sigma' dt.$$

Here and below we use the summation convention and think about variable t as the n-th variable. The important aspect is that this expression is independent of particular choice of coordinates on $\partial\Omega$. Hence if for some $i \neq j$ we have that $\widetilde{B} \subset \varphi_i(U_i) \cap \varphi_j(U_j)$, where φ_i and φ_j are two coordinate maps, then in both coordinates (i and j) the value of this expression is same. This is in part due to the fact that the last n-th coordinate (the t-variable) does not change when we make a choice of coordinates on $\partial\Omega$. We begin by integrating by parts

$$(p-1)\int_{T_r(\widetilde{B})}|u|^{p-2}\frac{a_{ij}}{a_{nn}}(\partial_i u)(\partial_j u)\phi t\,d\sigma'\,dt=\frac{1}{p}\int_{\widetilde{B}^r}\partial_j(|u|^p)\frac{a_{ij}}{a_{nn}}\phi t\nu_i\,d\sigma'+ \tag{2.13}$$
$$-\int_{T_r(\widetilde{B})}\frac{1}{a_{nn}}|u|^{p-2}u\partial_i(a_{ij}\partial_j u)\phi t\,d\sigma'\,dt-\int_{T_r(\widetilde{B})}|u|^{p-2}u(\partial_j u)a_{ij}\partial_i\left(\frac{\phi t}{a_{nn}}\right)d\sigma'\,dt.$$

Here we introduce the notation

$$\widetilde{B}^s=\{(x,s)\in\mathbb{R}^n;x\in\widetilde{B}\}\quad\text{for } s\in\mathbb{R}.$$

ν_i is the i-th component of the outer normal ν, which on the upper part of the box $T_r(\widetilde{B})$ is just the vector e_n. Hence the first term is non-vanishing only for $i=n$. We work on the last term, as it is the most complicated. This one splits into three new terms, one when the derivative hits t (only term with $i=n$ will remain) and another two when it hits ϕ and $1/a_{nn}$:

$$-\int_{T_r(\widetilde{B})}|u|^{p-2}u(\partial_j u)\frac{a_{nj}}{a_{nn}}\phi\,d\sigma'\,dt\;-\;\int_{T_r(\widetilde{B})}|u|^{p-2}u(\partial_j u)\frac{a_{ij}}{a_{nn}}(\partial_i\phi)t\,d\sigma'\,dt \tag{2.14}$$
$$+\int_{T_r(\widetilde{B})}|u|^{p-2}u(\partial_j u)\frac{a_{ij}}{a_{nn}^2}(\partial_i a_{nn})\phi t\,d\sigma'\,dt.$$

Consider now the first term of (2.14). For $j=n$ as ϕ is independent of $x_n=t$ we only get

$$-\frac{1}{p}\int_{T_r(\widetilde{B})}\partial_n(|u|^p\phi)\,d\sigma'\,dt=\frac{1}{p}\int_{\widetilde{B}}|u|^p\phi\,d\sigma'-\frac{1}{p}\int_{\widetilde{B}^r}|u|^p\phi\,d\sigma' \tag{2.15}$$

For $j<n$ the first term of (2.14) is handled as follows. We introduce an artificial one into it by putting $\partial_n t$ inside the integral. After integration by parts we get

$$\begin{aligned}&-\frac{1}{p}\int_{T_r(\widetilde{B})}\partial_j(|u|^p)\frac{a_{nj}}{a_{nn}}\phi\partial_n t\,d\sigma'\,dt=-\frac{1}{p}\int_{\widetilde{B}^r}\partial_j(|u|^p)\frac{a_{nj}}{a_{nn}}\phi t\,d\sigma'\\&+\int_{T_r(\widetilde{B})}\partial_n(\partial_j(|u|^p)\frac{a_{nj}}{a_{nn}}\phi)t\,d\sigma'dt=-\frac{1}{p}\int_{\widetilde{B}^r}\partial_j(|u|^p)\frac{a_{nj}}{a_{nn}}\phi t\,d\sigma'\\&+\int_{T_r(\widetilde{B})}\partial_j\partial_n(|u|^p)\frac{a_{nj}}{a_{nn}}\phi t\,d\sigma'dt+\int_{T_r(\widetilde{B})}\partial_j(|u|^p)\partial_n\left(\frac{a_{nj}}{a_{nn}}\right)\phi t\,d\sigma'dt.\end{aligned} \tag{2.16}$$

The first term here gets completely cancelled out by the first term of (2.13) as they have opposite signs. The second term can be further integrated by parts and we obtain

$$\begin{aligned}\int_{T_r(\widetilde{B})}\partial_j\partial_n(|u|^p)\frac{a_{nj}}{a_{nn}}\phi t\,d\sigma'dt&=-\int_{T_r(\widetilde{B})}\partial_n(|u|^p)\partial_j\left(\frac{a_{nj}}{a_{nn}}\right)\phi t\,d\sigma'dt\\&-\int_{T_r(\widetilde{B})}\partial_n(|u|^p)\frac{a_{nj}}{a_{nn}}(\partial_j\phi)t\,d\sigma'dt\end{aligned} \tag{2.17}$$

Notice that the third term on the righthand side of (2.16) and the first on the righthand side of (2.17) are of same type. We handle them now. First,

$$\left|\int_{T_r(\widetilde{B})} \nabla(|u|^p)\nabla\left(\frac{a_{nj}}{a_{nn}}\right)\phi t\, d\sigma' dt\right| \leq C\int_{T_r(\widetilde{B})} |u|^{p-1}|\nabla u||\nabla a|\phi t\, d\sigma' dt. \tag{2.18}$$

Here ∇A stands for either ∇a_{nj} or ∇a_{nn}. Notice also the the last term of (2.14) is also of this type, as well as, the second term of (2.13). To see this we use the fact that $Lu = -B.\nabla u$ to get that

$$\left|-\int_{T_r(\widetilde{B})} \frac{1}{a_{nn}}|u|^{p-2}u\partial_i(a_{ij}\partial_j u)\phi t\, d\sigma'\right| \leq \int_{T_r(\widetilde{B})} |u|^{p-1}|\nabla u||B|\phi t\, d\sigma' dt. \tag{2.19}$$

Since ∇A and B satisfy the same type of Carleson condition, we treat them together. By Cauchy-Schwarz we get that the righthand sides of (2.18) and (2.19) are less than

$$C\left(\int_{T_r(\widetilde{B})} |u|^p(|\nabla A|^2 + |B|^2)\phi t\, d\sigma' dt\right)^{1/2}\left(\int_{T_r(\widetilde{B})} |u|^{p-2}|\nabla u|^2\phi t\, d\sigma' dt\right)^{1/2}. \tag{2.20}$$

Using the Carleson condition on the coefficients, and the fact that the Carleson constant is less than ε we get that this can be further written as

$$C\varepsilon\left(\int_{\widetilde{B}} N_r(u)^p dy\right)^{1/2}\left(\int_{T_r(\widetilde{B})} |u|^{p-2}|\nabla u|^2\phi t\, d\sigma'\right)^{1/2}. \tag{2.21}$$

This is a good term, since using $ab \leq \frac{1}{2}(a^2+b^2)$ we see that the first term is on the righthand side of (2.11) whereas the second term due to the small constant which can be incorporated in the lefthand side of (2.13).

We summarize our computations. For some constant C depending only on p and the ellipticity of coefficients we have that

$$\begin{aligned} C\int_{T_r(\widetilde{B})} |u|^{p-2}|\nabla u|^2\phi t\, d\sigma'\, dt \quad &\leq \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.22) \\ \int_{\widetilde{B}} |u|^p\phi\, d\sigma' - \int_{\widetilde{B}^r} |u|^p\phi\, d\sigma' \quad &+ \quad \int_{\widetilde{B}^r} \partial_n(|u|^p)\phi t\, d\sigma' + \varepsilon\int_{\widetilde{B}} N_r(u)^p\, d\sigma' + \text{error terms} \\ = \int_{\widetilde{B}} |u|^p\phi\, d\sigma' - 2\int_{\widetilde{B}^r} |u|^p\phi\, d\sigma' \quad &+ \quad \int_{\widetilde{B}^r} \partial_n(|u|^p t)\phi\, d\sigma' + \varepsilon\int_{\widetilde{B}} N_r(u)^p\, d\sigma' + \text{error terms.} \end{aligned}$$

The third term on the righthand side is the first term of (2.13) for $i = j = n$. We call "the error terms" the second term of (2.14) and the second term on the righthand side of (2.17). Both terms are of same type and contain $\partial_i\phi$ for $i < n$. (Recall that $\partial_n\phi = 0$).

Now we use (2.22) as follows. We write $\partial\Omega$ as a disjoint union of sets $B^1, B^2, \ldots, B^k$ all of approximately same diameter r. We can also arrange that each such set has approximately same number of neighbors. For each set B^i we consider the corresponding set $\widetilde{B^i}$ defined above of diameter approximately $3r$. When we do this, we make sure that each $\widetilde{B^i}$ belongs to at least one of the charts $\varphi_j(U_j)$ for some $j = 1, 2, \ldots, m$. Consider a partition of unity (ϕ^i) on $(\widetilde{B^i})$. As each $\widetilde{B^i}$ belongs to at least one chart $\phi_j(U_j)$ we get (2.22) for $\widetilde{B} = \varphi_j^{-1}(\widetilde{B^i})$ and $\phi = \varphi_j^{-1} \circ \phi^i$.

Now we sum over all i for $\widetilde{B^i}$ belonging to one chart. We claim that the "error terms" completely disappear, as the "error terms" for neighboring $\widetilde{B^i}$ look same and each contain term $\partial_j \phi$. Since $\sum_i \phi^i = 1$ we get that $\sum_i (\partial_j \phi^i) = 0$. That means that summing over j these terms cancel out. This cancellation does happen even if we have two neighboring $\widetilde{B^i}$, $\widetilde{B^j}$ that belong to different coordinate charts, since (2.22) as we pointed out earlier does not depend on choice of coordinates. Having taken care of the "error terms" we finally get from (2.22) after summing over all $\widetilde{B^i}$:

$$\begin{aligned} C \quad & \int_{\Omega_r} |u|^{p-2} |\nabla u|^2 t(X) dX \leq \int_{\partial\Omega} |u|^p d\sigma \\ -2 \quad & \int_{\partial\Omega_r \setminus \partial\Omega} |u|^p d\sigma + \int_{\partial\Omega_r \setminus \partial\Omega} \partial_t (|u|^p t(X)) d\sigma(X) + \varepsilon \int_{\partial\Omega} N_r(u)^p d\sigma. \end{aligned} \tag{2.23}$$

Recall that $\Omega_r = \{X = (x,t) \in \Omega; t < r\}$, hence in the second and third term we integrate over the $n-1$ dimensional set $\{(x,r); x \in \partial\Omega\}$. Here $t = t(X)$ is the t-th coordinate of a point $X = (x,t) \in \Omega$, which is well defined near $\partial\Omega$ and comparable to $\mathrm{dist}(X, \partial\Omega)$.

The second term here is not pleasant - we get rid of it by integrating both sides of (2.23) over an interval $(0, r_0)$ and dividing by r_0. This also leads to introduction of some harmless weight terms. We get after setting $r = r_0$:

$$C \int_{\Omega_r} |u|^{p-2} |\nabla u|^2 (t - \tfrac{t^2}{r}) dX + \frac{2}{r} \int_{\Omega_r} |u|^p dX \leq \int_{\partial\Omega} |u|^p d\sigma + \int_{\partial\Omega_r \setminus \partial\Omega} |u|^p \, d\sigma + \varepsilon \int_{\partial\Omega} N_r(u)^p d\sigma. \tag{2.24}$$

From this (2.11) follows provided we prove an estimate

$$\int_{\partial\Omega_r \setminus \partial\Omega} |u|^p d\sigma \leq \frac{2+\varepsilon}{r} \int_{\Omega_r} |u|^p dX \tag{2.25}$$

Indeed, if for given $p > 1$, such an estimate holds then we only have to bound from above $\frac{\varepsilon}{r} \int_{\Omega_r} |u|^p \, dX$ which can be done by $\varepsilon \int_{\partial\Omega} N_r(u)^p \, d\sigma$. This introduces 2ε into (2.11) instead of ε but that's a detail.

We first observe that when $p = \infty$ this estimate does indeed hold. Indeed, (2.25) is equivalent to

$$\|u\|_{L^p(\partial\Omega_r \setminus \partial\Omega)} \leq \left(\frac{2+\varepsilon}{r} \right)^{1/p} \|u\|_{L^p(\Omega_r)} \tag{2.26}$$

which by limiting $p \to \infty$ gives us $\|u\|_{L^\infty(\partial\Omega_r \setminus \partial\Omega)} \leq \|u\|_{L^\infty(\Omega_r)}$ which holds by the maximum principle. If we establish an L^1 version of this result, the rest follows by the interpolation for all p. To get the L^1 result we need to use the assumption that $u \geq 0$.

Recall what Ω_r is - it essentially a collar neighborhood of the boundary $\partial\Omega$ of width r. We will make our final choice of r in a while. As we have an ε to work with, we make one important simplification and prove instead the inequality

$$\int_{\partial\Omega_r\setminus\partial\Omega} u\,d\sigma \le \frac{2+\varepsilon/2}{r(1-\delta)}\int_{\Omega_{\delta r,(1-\delta)r}} u\,dX, \tag{2.27}$$

where

$$\Omega_{\delta r,(1-\delta)r} = \{(x,t)\in\Omega;\, \delta r < t < (1-\delta)r\}$$

and $\delta > 0$ is very small (depending on ε). More precisely we pick $\delta = \delta(\varepsilon)$ so that

$$\frac{2+\varepsilon/2}{r(1-2\delta)} \le \frac{2+\varepsilon}{r}$$

The main reason we introduce δ is to avoid completely the two boundaries of Ω_r. So, $\Omega_{\delta r,(1-\delta)r}$ this is a strip of width (essentially r) but of distance δr from both boundaries.

To prove the inequality we return to our partitioning of $\partial\Omega$ and local coordinates. Since this part was detailed above we skip the details. Let us therefore assume we are in the situation we are on one Carleson box in $\mathbb{R}^n_+$ which now we choose to look like

$$T_r(\widetilde{B}) = \{(x,t)\in\mathbb{R}^{n-1}\times\mathbb{R};\, |x| < mr, 0 < t < r\},$$

where $B = \{|x| < r\}$ and $\widetilde{B} = \{|x| < mr\}$. Here m is some fixed (large) positive integer, to be determined later. We now assume such overlapping boxes (even for $m = 1$) are covering Ω_r, that is that collection of all sets B cover $\partial\Omega$. We will establish a local version of the estimate (2.27) then put all the pieces together via a partition of unity.

Assume therefore that we have a nonnegative solution u of our equation $Lu = 0$ in $T_r(\widetilde{B})$. Using a partition of unity we may assume that u at the top portion $\widetilde{B}^r$ of the boundary $\partial T_r(\widetilde{B})$ is only supported on the set B^r and similarly, at the bottom portion $\widetilde{B}^0$ the support of u is the set B^0. (Recall that $B^s = \{(x,s); x\in B\}$ and $\widetilde{B}^s = \{(x,s); x\in\widetilde{B}\}$).

The Carleson conditions in (2.9) and (2.10) imply that the coefficients a_{ij}, b_i satisfy on

$$T_{\delta r,r}(\widetilde{B}) = \{(x,t)\in\mathbb{R}^{n-1}\times\mathbb{R};\, |x| < mr, \delta r < t < r\}$$

the condition

$$|\nabla a_{ij}|, |b_i| \le \frac{\sqrt{C}}{\delta r},$$

where C is the Carleson constant for $T_r(\widetilde{B})$. Hence, since $\delta > 0$ was already chosen we may choose the Carleson constant $C(\varepsilon,p) > 0$ in the statement of Lemma 2.6 so small so that $\sqrt{C}/(\delta r) \le K/r$, where K will be specified a bit later. The inequality we want to prove is invariant under rescaling in the variable r, hence we may re-scale everything to a box of size $r = 1$. Our goal therefore is to prove that in a box $T_{\delta,1}(\widetilde{B})$ where the coefficients of the operator a_{ij} are essentially constant ($|\nabla a_{ij}| \le K$) and b_i very small ($|b_i| \le K$), we have for $\widetilde{B} = \{|x| < m\}$:

$$\int_{B^1} u\, d\sigma' \le \frac{2+\varepsilon/2}{1-2\delta} \int_{T_{\delta,1-\delta}(\widetilde{B})} u\, dX \tag{2.28}$$

Once we have this, we rescale back, add up all boxes and get (2.27). Let us denote by v the solution of our equation $Lv = 0$ in the box $T_{\delta,1}(\widetilde{B})$ which is equal to u on $B^1 = \{(x,1); |x| < 1\}$ and vanishing on all other parts of $\partial T_{\delta,1}(\widetilde{B})$. It follows that v is a subsolution of u and if we prove that

$$\int_{\partial T_{\delta,1}(\widetilde{B})} v\, d\sigma' = \int_{B^1} v\, d\sigma' \le \frac{2+\varepsilon/2}{1-2\delta} \int_{T_{\delta,1-\delta}(\widetilde{B})} v\, dX, \tag{2.29}$$

then (2.28) follows. So why is (2.29) true? Let us pretend for a second that in fact L is a constant coefficient operator on $T_{\delta,1-\delta}(\widetilde{B})$. Then (if also $m = \infty$ and $d\sigma' = dx$) this is a classical result for a constant coefficient operator with no drift term - on an infinite strip in $\mathbb{R}^n$ the average integral of a harmonic function inside the strip is just the boundary integral of that function. Hence in such a case (2.29) holds exactly with a constant $\frac{2-2\delta}{1-2\delta}$. Since we have $\frac{2\delta}{1-2\delta}$ to spare and we know that v is supported on the boundary only on B^1 we now find m sufficiently large for which (2.29) does hold with constant $\frac{2-\delta}{1-2\delta}$ that is

$$\int_{B^1} v\, dx \le \frac{2-\delta}{1-2\delta} \int_{T_{\delta,1-\delta}(\widetilde{B})} v\, dxdt, \tag{2.30}$$

Notice that the measure in (2.30) is dx not $d\sigma'$, similarly instead of the measure dX we have the product measure $dxdt$. But that is not a problem. By making r_1 smaller if necessary, we can ensure that for $r < r_1$ in the Carleson box $T_r(\widetilde{B})$ the metric tensors are almost constant (recall that the box is very small). This might introduce an additional error in the inequality, hence we get

$$\int_{B^1} v\, d\sigma' \le \frac{2-\delta}{1-2\delta} \int_{T_{\delta,1}(\widetilde{B})} v\, dX. \tag{2.31}$$

Note that we have also replaced the set $T_{\delta,1-\delta}(\widetilde{B})$ by a larger set $T_{\delta,1}(\widetilde{B})$ so the inequality will still hold. Next we interpolate between this L^1 estimate and the L^∞ result. We obtain

$$\int_{B^1} v^p\, d\sigma' \le \frac{2}{1-2\delta} \int_{T_{\delta,1}(\widetilde{B})} v^p\, dX, \tag{2.32}$$

Remember that we have (2.32) for a constant coefficient equation. Now we consider the variable coefficient case. Assume that $\widetilde{v}$ is a solution with the same boundary conditions as v, but for a variable coefficient operator $\widetilde{L} = \mathrm{div} A\nabla + B.\nabla$ on a box $T_{\delta,1}(\widetilde{B})$ on which we have $|\nabla a_{ij}|, |b_i| \le K$. Say v is a solution for the constant coefficient operator L with coefficients taken from the middle of the box $T_{\delta,1}(\widetilde{B})$. How do v and $\widetilde{v}$ compare? In both cases for any $1 < p < \infty$ the solvability of the Dirichlet problem is assured by layer potential techniques - done for the variable coefficient case with Lipschitz continuous coefficients in [11] and [12], for example. Moreover this result gives us that both v and

$\widetilde{v}$ belong to the Sobolev space $L^p_{p+1/p}(T_{\delta,1}(\widetilde{B}))$ and for some constant $C(p) > 0$ we have an estimate

$$\|v\|_{L^p_{p+1/p}(T_{\delta,1}(\widetilde{B}))} + \|\widetilde{v}\|_{L^p_{p+1/p}(T_{\delta,1}(\widetilde{B}))} \leq C(p)\|v\|_{L^p(\partial T_{\delta,1}(\widetilde{B}))} = C(p)\|\widetilde{v}\|_{L^p(B^1)}.$$

But this is not the end of the story. We also know that L and $\widetilde{L}$ have coefficients that are close in the Lipschitz norm. Hence one can show (e.g. [3]) that in such a case the corresponding layer potential operators are also close and how close they are only depend on K that measures $|\nabla a_{ij}|, |b_i|$ on $T_{\delta,1}(\widetilde{B})$. From this an estimate

$$\|v - \widetilde{v}\|_{L^p_{p+1/p}(T_{\delta,1}(\widetilde{B}))} \leq C(p,K)\|v\|_{L^p(\partial T_{\delta,1}(\widetilde{B}))} = C(p,K)\|\widetilde{v}\|_{L^p(B^1)}$$

follows. Here $C(p,K) > 0$ depends only on p and K and $C(p,K) \to 0$ as $K \to 0$. This is the final missing ingredient. Using this estimate and the fact that the L^p norm is weaker that $L^p_{1/p}$ norm we get for $\widetilde{v}$:

$$\begin{aligned}\int_{B^1} \widetilde{v}^p \, d\sigma' &= \int_{B^1} v^p \, d\sigma' \leq \frac{2}{1-2\delta} \int_{T_{\delta,1}(\widetilde{B})} v^p \, dX \\ &\leq \frac{2}{1-2\delta} \left(\int_{T_{\delta,1}(\widetilde{B})} \widetilde{v}^p \, dX + C^p(p,K) \int_{B^1} \widetilde{v}^p \, d\sigma' \right). \end{aligned} \tag{2.33}$$

So finally, we can select the last undetermined constant K. Given $1 < p < \infty$ we choose K so small so that $C^p(p,K)$ is small enough such that $(1 - \frac{2C^p(p,K)}{1-2\delta})^{-1}$ is less than $\frac{2+\varepsilon/2}{1-2\delta}$ and hence (2.28) holds for $\widetilde{v}$. This concludes the proof. □

Now we are ready to prove the Main theorem 2.2.

Proof. To keep matters simple let us first consider the case when $\partial\Omega$ is smooth. In this case Lemma 2.6 applies directly. Let $1 < p < \infty$ be given and let us assume a function f in $L^p(\partial\Omega)$ is given. We split f into a positive and negative part f^+, f^- and consider the corresponding solutions u^+, u^- to the Dirichlet problem for the operator L. Or goal is to prove the estimates

$$\|N(u^\pm)\|_{L^p(\partial\Omega)} \leq C\|f^\pm\|_{L^p(\partial\Omega)},$$

from which the result follows as $N(u) \leq N(u^+) + N(u^-)$. We only present the proof for u^+ as the argument remains same for u^-. We simplify our notation and use $u = u^+$ and $f = f^+$.

Proposition 3.2 implies that there exists $C(p) > 0$ such on each Carleson box $T_r(B)$ we have an estimate on comparability of truncated nontangential maximal function and the square functions:

$$\|N_r(u)\|_{L^p(B)} \leq C\|S_p^{2r}(u)\|_{L^p(\widetilde{B})} + C_2 \left| \int_B u \, d\sigma \right|. \tag{2.34}$$

As in the proof of previous lemma, if we now consider a partition of $\partial\Omega$ into sets $B_1, B_2, \ldots, B_m$ of approximate diameter r and the corresponding sets $\widetilde{B_1}, \widetilde{B_2}, \ldots, \widetilde{B_m}$. The geometry of domain $\partial\Omega$ implies that these set can be chosen so that there exists an integer K independent of r such that each point $X \in \partial\Omega$ belongs to at most K of sets $\widetilde{B_i}$.

Hence using (2.34) on each B_i we get an estimate for all $r > 0$

$$\|N_r(u)\|_{L^p(\partial\Omega)} \leq CK\|S_p^{2r}(u)\|_{L^p(\partial\Omega)} + C_2 \int_{\partial\Omega} |u| d\sigma. \tag{2.35}$$

The crucial point is that the constants C and K do not depend on r. Having this we now choose the $\varepsilon > 0$ to be used in Lemma 2.6. We first find a third constant $M > 0$ (again independent of r) such that

$$N_{4r}(u) \leq MN_r(u) \tag{2.36}$$

for any $u \geq 0$ and any $r > 0$. The existence of such an M is consequence of the Harnack inequality. Finally, we take $\varepsilon > 0$ such that $CKM\varepsilon = 1/2$ and find $r_1 > 0$ such that (2.11) holds for all operators L with coefficients that have sufficiently small Carleson norm on boxes of size at most r_0. We now pick $r > 0$ such that $4r < \min\{r_0, r_1\}$. Combining (2.11), (2.35) and (2.36) we get that

$$\begin{aligned} \|N_r(u)\|_{L^p(\partial\Omega)} &\leq CK\|S_p^{2r}(u)\|_{L^p(\partial\Omega)} + \int_{\partial\Omega} u\, d\sigma \\ &\leq C_1CK\|f\|_{L^p(\partial\Omega)} + CK\varepsilon\|N_{4r}(u)\|_{L^p(\partial\Omega)} + C_2\int_{\partial\Omega} u\, d\sigma \\ &\leq C_1CK\|f\|_{L^p(\partial\Omega)} + CKM\varepsilon\|N_r(u)\|_{L^p(\partial\Omega)} + C_2\int_{\partial\Omega} u\, d\sigma. \end{aligned} \tag{2.37}$$

Now we use the fact that $CKM\varepsilon = 1/2$, hence we can hide $CKM\varepsilon\|N_r(u)\|_{L^p(\partial\Omega)}$ on the lefthand side. Also

$$\int_{\partial\Omega} u\, d\sigma \leq C_3\|f\|_{L^p(\partial\Omega)}$$

for any $p \geq 1$. So finally we get that

$$\frac{1}{2}\|N_r(u)\|_{L^p(\partial\Omega)} \leq (C_1CK + C_2C_3)\|f\|_{L^p(\partial\Omega)}. \tag{2.38}$$

From this the result follows as $N(u) \lesssim N_r(u)$.

Now we deal with the more general case, when Ω has a Lipschitz boundary with sufficiently small Lipschitz constant L. This case also includes the C^1 boundary as in such case L can be taken arbitrary small.

The crucial point is that the proofs of Lemma 2.6 and the main Theorem 2.2 in the smooth case are based on local estimates such as (2.22) and (2.34). Hence we can again reduce the situation to local coordinate patches where we want to establish out estimates. This means we can reduce the matter to a situation where we have U - an open set in $\mathbb{R}^n$ and a Lipschitz function ϕ with Lipschitz constant L such that in U the set Ω looks like $\{(x,t) \in \mathbb{R}^n; t > \phi(x)\}$.

Now, let θ_t be a family of mollifiers as in [10]. As observed there, the map $\Phi : (x,t) \mapsto (x, (\theta_t * \phi)(x) + ct)$ for any $c > L$ is then a bijection between the sets $\mathbb{R}^n_+$ and $\{(x,t) \in \mathbb{R}^n; t > \phi(x)\}$. In fact if $c > \ell$ then the map Φ is a local bijection,, where $\ell = \|\nabla\phi\|_{BMO}$. Hence by pulling back everything (metric, coefficicients) using Φ we are left with proving local estimates like (2.22) and (2.34) on a subset of $\mathbb{R}^n_+$. However, this is exactly what we did above. We only have to be careful about how much the Carleson constant of the coefficients changes when we move from the set $\{(x,t) \in \mathbb{R}^n; t > \phi(x)\}$ to $\mathbb{R}^n_+$. A computation gives us that if the original constant was C, the new constant on $\mathbb{R}^n_+$ will be $C + C(\ell)$ where $C(\ell)$ is an increasing function in ℓ such that $\lim_{\ell\to 0+} C(\ell) = 0$. From this the claim follows, as this implies that $C + C(\ell)$ will be small as long as both C and ℓ are small enough. So we get solvability on domains with small Lipschitz constant, as well as on domains whose boundaries are given locally by functions in VMO.

□

3 The Nontangential Maximal Function and a p-Adapted Square Function

In this section we recall a lemma proven in [10].

Lemma 3.1. *Let (2.9) be a Carleson measure for the operator $L = divA\nabla$ with bounded and strongly elliptic coefficients. Let u be a solution to $Lu = 0$ on a bounded Lipschitz domain Ω, normalized so that $u(P) = 0$ for some $P \in \Omega$. Then*

$$\|N(u)\|_{L^p} \lesssim \|S_2(u)\|_{L^p}$$

for any $1 < p < \infty$.

From the lemma we get the following:

Proposition 3.2. *Let $1 < p \le 2$ and let $\mathcal{H}^p$ be a space of solutions to $Lu = divA\nabla u = 0$ such that $N(u) \in L^p(\partial\Omega)$. For any $r > 0$ and any $x \in \partial\Omega$ let*

$$\begin{aligned} B &= \{y \in \partial\Omega; dist(x,y) < r\} \\ \widetilde{B} &= \{y \in \partial\Omega; dist(x,y) < 2r\}. \end{aligned}$$

Let $P \in \Omega$ be a point in Ω whose distances to x and $\partial\Omega$ are both approximately r. Then for $u \in \mathcal{H}^p$ and for any positive ε :

(a) $\|N_r(u)\|_{L^p(B)} \le C(\|S_p^{2r}(u)\|_{L^p(\widetilde{B})} + r^{n/p}|u(P)|) + \varepsilon\|N_{2r}(u)\|_{L^p(B)}$.

(b) $\|N_r(u)\|_{L^p(B)} \le C(\|S_p^{2r}(u)\|_{L^p(\widetilde{B})} + r^{n(1/p-1)}\left|\int_B u\, d\sigma\right|) + \varepsilon\|N_{2r}(u)\|_{L^p(B)}$. Here σ is the standard surface measure on $\partial\Omega$.

The estimates above have a global counterpart:

(c) $\|N(u)\|_{L^p(\partial\Omega)} \leq C(\|S_p(u)\|_{L^p(\partial\Omega)} + \left|\int_{\partial\Omega} u\, d\sigma\right|)$

Moreover, in both (a) and (b) the constant C in the estimates depend only on the size of the Carleson constant of the coefficients (2.9), p and ε, but not on $x \in \partial\Omega$ or $r > 0$. Here $N_r(u)$ denotes the truncated nontangential maximal function of height r, similarly, S_p^{2r} denotes the truncated S_p function for cones of height $2r$. Additionally, ε can be taken to be zero in the case that u is a non-negative solution.

Proof. The global inequality part(c) follows from (b): when $B = \partial\Omega$, it is easy to see that $\|N_{2r}(u)\|_{L^p(\partial\Omega)} \leq C\|N_r(u)\|_{L^p(\partial\Omega)}$. The constant C will depend only on the Lipschitz constant if r is chosen sufficiently large. In this case, one uses the solvability of the L^p Dirichlet problem for smooth domains (interior to Ω) for this operator. We turn the proofs of (a) and (b).

We first establish that

$$\|N_r(u)\|_{L^p(B)} \leq C(\|S_2^{2r}(u)\|_{L^p(\widetilde{B})} + r^{n/p}|u(P)|), \tag{3.39}$$

$$\|N_r(u)\|_{L^p(B)} \leq C\left(\|S_2^{2r}(u)\|_{L^p(\widetilde{B})} + r^{n(1/p-1)}\left|\int_B u\, d\sigma\right|\right). \tag{3.40}$$

Then we will give a simple argument proving that for $1 < p < 2$ and any $\varepsilon > 0$ we have an estimate

$$\|S_2^r(u)\|_{L^p} \leq C_\varepsilon\|S_p^r(u)\|_{L^p} + \varepsilon\|N_r(u)\|_{L^p}. \tag{3.41}$$

Combining (3.39) and (3.41), part (a) follows and, similarly, from (3.40) and (3.41), part (b) follows. From parts (a) and (b), the Harnack inequality for non-negative solutions shows that $N_{2r}(u) \leq CN_r(u)$ for a fixed $C > 0$ independent of $r > 0$. Thus, for non-negative solutions, the ε may be taken to be zero in both (a) and (b). Finally, part (c) is proved by

Clearly (3.39) is just a local version of Lemma 3.1, applied to a function $u - u(P)$. The term $r^{n/p}$ next to $|u(P)|$ is to obtain the correct scaling. When we rescale both $N_r(u)$ and $S_2^{2r}(u)$ from sets of diameter approximately r to sets of diameter 1 we get that both these will scale like $r^{n/p}$. Hence if (3.39) works for $r = 1$ it works for all $r > 0$ by the scaling argument.

Next we look at (3.40). This is an important step as it allows us to replace the value of u at a point P (essentially the elliptic measure) with the (better controllable) surface measure. We claim that to prove (3.40) it suffices to show that on the subspace of $\mathcal{H}^p$

$$\mathcal{H}^p_{av} = \{u \in \mathcal{H}^p; \textstyle\int_B u\, dA = 0\}$$

we have for all $u \in \mathcal{H}^p_{av}$

$$\|N_r(u)\|_{L^p(B)} \leq C\|S_2^{2r}(u)\|_{L^p(\widetilde{B})}, \tag{3.42}$$

with C depending only on the Carleson constant of the coefficients. The rest is just a scaling argument. Hence, consider just when $r = 1$. For this reason from now on we drop the sub(super)script r. Assume that (3.42) is false. Then we can find a sequence of solutions $u_1, u_2, u_3 \dots$ of equations $L^k u_k = 0$ (for $L^k = \partial_i(a^k_{ij}\partial_j)$) such that

$$\|N(u_k)\|_{L^p(B)} = 1, \quad \|S_2(u_k)\|_{L^p(\widetilde{B})} \le 1/k, \quad \fint_B u_k \, d\sigma = 0. \tag{3.43}$$

Here, for each operator L^k we assume that the coefficients a^k_{ij} are uniformly elliptic with same constant for all k and also satisfy the Carleson condition for coefficients (2.9) with same constant. This however implies that on any compact subset of Ω, both sequences a^k_{ij} and ∇a^k_{ij} are uniformly bounded for all k. Thus for a subsequence in k we get that $a^k_{ij} \to a_{ij}$ for some a_{ij} in any C^α, $\alpha < 1$. Moreover, a_{ij} is also uniformly elliptic and locally Lipschitz. By repeating this argument on any compact subset of Ω and diagonalization, we may assume that the sequence $(a^k_{ij})_{k\in\mathbb{N}}$ is such that $a^k_{ij} \to a_{ij}$ locally uniformly in any C^α, $\alpha < 1$. Denote the operator that corresponds to coefficients a_{ij} by L.

Let P be the point in Ω given in (3.39). By (3.39) we have that a for large k

$$1 = \|N(u_k)\|_{L^p} \approx |u_k(P)|,$$

hence the sequence (u_k) is bounded from above and below at the point P. Naturally, the exact constant will depend on the position of the point P with respect to the boundary. In fact, for any compact $K \subset\subset \bigcup_{x\in\widetilde{B}} \Gamma(X)$ (cones Γ of height 2) we can find $c(K), C(K) > 0$ such that for large k:

$$c(K) \le |u_k(P)| \le C(K), \qquad \text{for all } P \in K.$$

It follows that the sequence (u_k) is bounded on such set K. As all u_k are also solutions of $L^k u_k = 0$ and $L^k \to L$, we get that $\{u_k\big|_K\}$ is a precompact set, hence we can find a locally uniformly convergent subsequence. Repeating this argument on any compact subset K and diagonalization then implies that there exists a function u solving $Lu = 0$ such that a subsequence (u_{k_n}) (and its derivative) converges to u (∇u respectively) locally uniformly in $\bigcup_{x\in\widetilde{B}} \Gamma(X)$. What is u? Fix a compact set $K \subset \bigcup_{x\in\widetilde{B}} \Gamma(X)$. Then for any $Q \in K$ let us denote by $S^K_2 u(Q)$

$$S^K_2 u(Q) = \left(\int_{\Gamma(Q)\cap K} |\nabla u(X)|^2 dist(X, \partial\Omega)^{2-n} d\sigma(X) \right)^{1/2}.$$

The uniform convergence on K implies that

$$\|S^K_2 u\|_{L^p} = \lim_{n\to\infty} \|S^K_2 u_{k_n}\|_{L^p} \le \lim_{n\to\infty} \|S_2 u_{k_n}\|_{L^p} = 0.$$

From this we get that $|\nabla u| = 0$ on K that is u is constant. Hence we get that $u \equiv c_0 \ne 0$ in $\bigcup_{x\in\widetilde{B}} \Gamma(X)$. What can be said about $N(u - u_{k_n})$? Using again part (3.39) we get that

$$\|N(u - u_{k_n})\|_{L^p} \lesssim \|S_2(u - u_{k_n})\|_{L^p} + |u(P) - u_{k_n}(P)|$$

The first term on the righthand side has an estimate

$$\|S_2(u - u_{k_n})\|_{L^p} \lesssim \|S_2(u)\|_{L^p} + \|S_2(u_{k_n})\|_{L^p} \to 0,$$

the second term goes to zero trivially. So $N(u-u_{k_n}) \to 0$ in L^p. But $|u\big|_B - u_{k_n}\big|_B| \leq N(u-u_{k_n})$, hence $u_{k_n}\big|_B \to u\big|_B = c_0$ in $L^p(B)$, $p>1$, therefore also in $L^1(B)$. However, as

$$0 = \int_B u_{k_n}\, dA \to \int_B u\, dA = \int_B c_0\, dA \neq 0$$

we get a contradiction. Therefore the estimate (3.42) holds and (3.40) is true.

Finally, we establish (3.41) for $1<p<2$. W now drop the index r, as the following does not depend on r in any way. We have (using $|u|^{2-p} \leq |N(u)|^{2-p}$)

$$\begin{aligned}
\|S_2(u)\|^p_{L^p} &= \int_{\partial\Omega}\left(\int_{\Gamma(x)} |\nabla u(y)|^2 |u(y)|^{p-2}|u(y)|^{2-p}\mathrm{dist}(y,\partial\Omega)^{2-n}dy\right)^{p/2} d\sigma(x)\\
&\leq \int_{\partial\Omega} N(u)^{p(2-p)/2}\left(\int_{\Gamma(x)} |\nabla u(y)|^2 |u(y)|^{p-2}\mathrm{dist}(y,\partial\Omega)^{2-n}dy\right)^{p/2} d\sigma(x)\\
&\leq \int_{\partial\Omega} N(u)^{p(2-p)/2}[S_p(u)]^{p^2/2}d\sigma \qquad (3.44)\\
&\leq \left(\int_{\partial\Omega} N(u)^p d\sigma\right)^{(2-p)/2}\left(\int_{\partial\Omega} S_p(u)^p d\sigma\right)^{p/2} = \|N(u)\|^{p(2-p)/2}_{L^p}\|S_p(u)\|^{p^2/2}_{L^p}.
\end{aligned}$$

In the last step we used the Hölder inequality. From this our claim follows as for any $r, r' > 1$ and $1/r + 1/r' = 1$ we have that $a^{1/r}b^{1/r'} \leq ra + r'b$. □

Concluding Remarks. Several questions remain open, and research continues in these areas. First, it should be possible to develop a Hardy space theory, in particular, the endpoint atomic result. See [3], where results for p near 1 on C^1 domains and manifolds are treated by the method of layer potentials. Second, it remains to prove the results for operators satisfying the averaging condition in (2.3) in the presence of drift terms, or similarly, and for a related reason, results for nondivergence operators whose coefficients satisfy this averaging condition. Some partial progress in this direction follows from some perturbation results in [14] (see also [13].)

References

[1] B.E.J. Dahlberg *Poisson semigroups and singular integrals*, PAMS **97**, no. 1 (1976), 41–48.

[2] B.E.J. Dahlberg *Weighted norm inequalities for the Luisin area integral and the nontangential maximal function for functions harmonic in a Lipschitz domain*, Studia Math. **67**, no. 3 (1980), 297–314.

[3] M.Dindos, *Hardy spaces and layer potentials on C^1 domains in Riemannian manifolds*, Memoirs of AMS, to appear.

[4] E. Fabes, D. Jerison and C. Kenig, *Multilinear Littlewood-Paley estimates with applications to partial differential equations*, Proc. Natl. Acad. Sci. U.S.A. **79(18)** (1982), 5746–5750.

[5] E. Fabes, M. Jodeit and N. Rivière *Potential techniques for boundary value problems on C^1 domains*, Acta Math. **141** (1978), 165–186.

[6] R. Fefferman, C. Kenig and J. Pipher *The theory of weights and the Dirichet problem for elliptic equations*, Ann. Math, 2nd Ser., **131**, no. 1 (1991), 65–121.

[7] S. Hofmann and J. Lewis *The Dirichlet problem for parabolic operators with singular drift terms*, Memoirs of AMS **719** (2001).

[8] D. Jerison and C. Kenig *The Dirichlet problem in nonsmooth domains*, Ann. Math. (2), **113**, no. 2 (1981), 367–382.

[9] C. Kenig, H. Koch, J. Pipher and T. Toro *A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations*, Adv. in Math., **153**, no. 2 (2000), 231–298.

[10] C. Kenig and J. Pipher, *The Dirichlet problem for elliptic equations with drift terms*, Publ. Mat. **45** no. 1 (2001), 199–217.

[11] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy, and Hölder space results*, Communications in Analysis and Geometry, 9 (2001), 369–421.

[12] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*, J. Funct. Anal., 176 (2000), 1–79.

[13] C. Rios *The L^p Dirichlet problem and nondivergence harmonic measure*, TAMS **355**, no. 2 (2002), 665–687.

[14] C. Rios *L^p regularity for the Dirichlet problem for elliptic equations with singular drift*, preprint.