The Surface Subgroup Theorem

and

The Ehrenpreis Conjecture

by

Jeremy Kahn

Joint with Vladimir Markovic
Theorem (KM) (Surface subgroup Conjecture)

Let $M$ be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ has a surface subgroup (the isomorphic image of $\pi_1$ of a hyperbolic surface).
Theorem (KM) (Ehrenpreis conjecture)

Let $S$ and $T$ be closed hyperbolic Riemann surfaces. Then $S$ and $T$ have nearby finite covers (in the Teichmüller metric).
2D

Model surface

finite cover

good immersion (cover)

Given surface

3D

Model surface

finite cover

Given surface

3-manifold
Building Covers of the Circle

degree 2

degree 4
We can join immersed 1-manifolds with boundary to form a cover of a closed 1-manifold (a circle).
We can join immersed orientable $n$-manifolds with boundary to form a cover of a closed orientable $n$-manifold.
We can build more than one cover.
We can fail to build a cover
Given a collection $C = \sum n_i x_i$ of 1-manifolds immersed in a circle, we can assemble $C$ to form a cover if and only if

$$\partial C = \sum n_i \partial x_i = 0$$
We can always build a piecewise immersed 1-manifold in a closed 2-manifold if we use the "doubling trick"
A set of points on $s'$ is "evenly distributed to the scale $\varepsilon$" if every $\varepsilon$-interval of $s'$ has the same number of points, up to a factor of $1+\varepsilon$.

\[
\frac{103}{98} < 1 + \varepsilon
\]
We can make the piecewise immersed 1-manifold nearly geodesic (small bending) if the segments are "evenly distributed" around each meeting point:
A (hyperbolic) pair of pants is a hyperbolic 2-manifold with geodesic boundary that is diffeomorphic to the following.
We obtain a closed hyperbolic surface when we join pairs of pants.
Hyperbolic Pairs of Pants

\[ L_3 \quad L_1 \quad L_2 \]

\[ \text{double} \]

\[ L_1 \quad L_2 \quad L_3 \]

Half-lengths \[ L_1, L_2, L_3 \]
Perfect Pants and Perfect Surfaces

\[ \text{hl}(C) = R \]

\[ \text{hl}(C) = R \quad \text{s}(C) = 1 \]

**Theorem**  Any two closed perfect surfaces have a common finite cover.
Good Pants and Good Surfaces

\[ |\ell(c) - R| < \varepsilon \]

\[ |s(c) - 1| < \frac{\varepsilon}{R} \]

**Theorem (KM)**

A good closed panted surface is close to a perfect surface.
Theorem (KM)
For all $\varepsilon < \varepsilon_0$, and $R > R_0$, any $(\varepsilon, R)$ good surface is $10^{12}\varepsilon$-close to an $R$-perfect surface in the Teichmüller metric.
Good Covers $\Rightarrow$ Ehrenpreis
An Immersed Pair of Pants

We can think of the pants as being isometrically immersed and speak of the goodness of pants.
We can identify cuffs of the parametrizing pants whenever the cuffs map to the same geodesics \( y \) and the pants are on opposite sides of \( X \).
Easy Theorem

Let $\mathcal{Q}$ be a finite set of immersed pants $\gamma$ on $S$ such that for every closed geodesic $\gamma$ on $S$ we have the same number of pants in $\mathcal{Q}$ on both sides of $\gamma$. (That number can be zero). Then we can assemble $\mathcal{Q}$ to form a finite cover of $S$. 
Good Pants and Good Curves

We let

\[ \Gamma_{\varepsilon, R} = \left\{ \chi \mid \chi \text{ is a closed geodesic on } S \right\} \]
\[ |hl(\chi) - R| < \varepsilon \]

\[ \Pi_{\varepsilon, R} = \left\{ \mathcal{P} \mid \mathcal{P} \text{ is an immersed pair of pants in } S, \text{ and } \right\} \]
\[ \partial \mathcal{P} \subseteq \Gamma_{\varepsilon, R} \]
The square root of a geodesic
Shear Coordinates and $N^4(\sqrt{y})$
The Equidistribution Theorem

For every good curve \( \gamma \) and large \( \mathcal{R} \)

\[
\left\{ \text{feets } p \mid p \in \Pi_{p, R} \text{ and } \gamma \in \mathcal{P} \right\} \leq N^4(\sqrt{R})
\]

is evenly distributed on \( N^4(\sqrt{R}) \) to the scale \( e^{-qR} \) for \( q = q(\delta, \varepsilon) \).

(We assume that \( R > R_0(\delta, \varepsilon) \).)
Equidistributed & Balanced $\Rightarrow$ Good Cover

If there were exactly the same number of pants on both sides of each geodesic $x$, then we would be able to assemble the pants in $\text{Th}_K$ to form a good cover of $S$. 

![Diagram](attached_diagram)
Equidistributed $\implies$ Nearly balanced

It follows from equidistribution that there are nearly the same number of pants in $\mathbb{H}_r \setminus \mathbb{H}_r'$ on either side of a good geodesic $\gamma$.
An Interlude in Three Dimensions

In a hyperbolic 3-manifold \( N^4(\mathbb{R}) \) (and \( N^4(\sqrt{2}) \)) is connected, so we can use the "doubling trick" to build a nearly geodesic immersed surface (which is therefore essential).

\[ \pi_1(f):\pi_1(S) \to \pi_1(M^3) \text{ injective} \]
A skew pair of pants

\[ \gamma_i = f(C_i) \text{ is a closed geodesic} \]

\[ \eta_i = f(h_i) \text{ is a geodesic segment orthogonal to } f(\gamma_{i+1}). \]
The feet of a skew pair of pants

\[ N^1(\gamma_c) \cong \mathbb{C}/2\pi i \mathbb{Z} + h\ell(x) \mathbb{Z} \]

\[ N^1(\sqrt{\delta_c}) \]
Two and Three Dimensions

2D

\[ N^4(\gamma) \cong \frac{\mathbb{R}}{\ell(\gamma) \cdot \mathbb{Z}} \times \mathbb{Z} \cong \mathbb{Z}^* \]

\[ \cong \mathbb{R}^*/\langle x \mapsto e^{\ell(\gamma)x} \rangle \]

3D

\[ N^4(\gamma) \cong \frac{\mathcal{C}}{\ell(\gamma) \cdot \mathbb{Z}^2 + 2\pi i \cdot \mathbb{Z}} \cong \mathcal{C}^* / \langle z \mapsto e^{\ell(\gamma)z} \rangle \]
The meaning of equidistribution

The feet \( \{ \text{feet } \{ x \in \mathbb{R} \} \} \) are \( e^{-2\pi} \) - evenly distributed as points on \( N^1(\sqrt{8}) \)

\[ X \mapsto X + i\pi + 1 \]
The "doubling trick" revisited

We take

\[ A_x = \{ \text{foot } P : x \in \partial P, \ P \in \Pi_{\varepsilon, R} \} \]

We can find

\[ \sigma : A_x \to A_x \] a permutation

such that

(\text{for all } x \in A_x)

\[ |\sigma(x) - x - i \pi - 1| < \frac{\varepsilon}{R} \]

and then define

\[ T : A_x^+ \sqcup A_x^- \to A_x^+ \sqcup A_x^- \]
The Idea of Self-Correction

We find \( g : \mathbb{D} \Gamma_{\varepsilon, R} \rightarrow \mathbb{D} \Pi_{\varepsilon, R} \) such that

1. \( \partial g(\lambda) = \lambda \) whenever \( \lambda = 2\pi \)

2. \( \| g(\lambda) \|_\infty \leq P(R) e^{-R/\|\lambda\|_\infty} \) for all \( \lambda \).

(Where \( P(R) = CR^N \) is a polynomial in \( R \))
Counting Good Pants and Good Curves

\# \Pi_{\varepsilon, R} \leq e^{3R}

\# \Gamma_{\varepsilon, R} \leq e^{2r/R}

\# \{ p \in \Pi_{\varepsilon, R} | y \in \partial P \} \leq e^{r/R}

for each \( y \in \Gamma_{\varepsilon, R} \)
Then let \( \Pi = \sum_{P \in \Pi_{\Sigma R}} P \), and \( \alpha = \partial \Pi \).

Then

\[ \| \alpha \|_{\infty} \leq e^{(1-\varepsilon)R} \text{ by equidistribution} \]

So

\[ \| g(\omega) \|_{\infty} \leq p(R)e^{-\varepsilon R} \ll 1. \]

So \( \Pi - g(\alpha) \) is positive and equidistributed.

And \( \partial(\Pi - g(\omega)) = \partial \Pi - \partial g(\partial \Pi) = 2 \Pi - 2 \Pi = 0 \).

After clearing denominators, we can assemble the pants of \( \Pi - g(\alpha) \) to form a good cover!

😊
The Relative Surface Subgroup Theorem

Let $M$ be a closed hyperbolic 3-manifold, and let $\gamma$ be a closed geodesic in $M$ of length $l(\gamma) > R_0(M, \varepsilon)$. (and suppose $\gamma$ bounds a singular chain). Then we can find a compact hyperbolic surface $S$ with boundary and $f : S \rightarrow M$ that is $\varepsilon$-close to being an isometric immersion, and for which $f(\gamma) = \gamma$ for every component $\gamma$ of $\partial S$. 
The Good Pants Homology

We let \( H^e_1(S; \mathbb{Q}) = \mathcal{P} \mathcal{T} / 2 \mathcal{Q} \mathcal{T} \).

We find a series of identities leading to \( H^e_1(S; \mathbb{Q}) = H_1(S; \mathbb{Q}) \).
We let $H_1^{\varepsilon, \mathbb{R}}(S; \mathbb{Q}) = \mathbb{Q} \Gamma_{\varepsilon, \mathbb{R}} \setminus \mathbb{Q} \mathbb{T}_{\varepsilon, \mathbb{R}}$.

We find a series of identities leading to $H_1^{\varepsilon, \mathbb{R}}(S; \mathbb{Q}) = H_1(S; \mathbb{Q})$.

Then we observe that $g: \mathbb{Q} \Gamma_{\varepsilon, \mathbb{R}} \to \mathbb{Q} \mathbb{T}_{\varepsilon, \mathbb{R}}$ has been implicitly defined, such that $dg(\alpha) = \alpha$ when $\alpha = 0$ in $H_1$. 
The Algebraic Square Lemma

Under conditions of reasonable geometry,

\[ \sum_{i, j = 0, 1} (-1)^{i+j}[A_i U B_j V] = 0 \]

in \( H_{\mathbb{R}}^1 \)

(Where \([X]\) denotes the closed freely geodesic homotopic to \( X \), for \( X \in \pi_1(S, x) \)).
We then define

\[ A_T = \frac{1}{2} [T \bar{A} T B] - [T \bar{A} T B] \]

(which is independent of the choice of \( B \) in \( \mathcal{H}_1^{\mathbb{E}, \mathbb{R}} \))

and prove

\[ (X Y)_T = X_T + Y_T \]

in \( \mathcal{H}_1^{\mathbb{E}, \mathbb{R}} \).