We define an inner product space and show the following:

1. $\|f+g\|^2 = \|f\|^2 + \|g\|^2$ when $\langle f, g\rangle = 0$.
2. $\|f+g\| \leq \|f\| + \|g\|$ (triangle inequality)
3. $|\langle f, g\rangle| \leq \|f\|\,\|g\|$ (Cauchy-Schwarz)
4. When $g \neq 0$, we have $\langle f - P_g f, g\rangle = 0$ where

$$P_g f = \frac{\langle f, g\rangle}{\langle g, g\rangle} g.$$

Proof. 1 and 4 are direct calculations that are left to the reader. 2 follows from squaring both sides, expanding, and using Cauchy-Schwarz. We observe that 2 implies

$$\|f\|^2 \leq \|f+g\|^2 \text{ when } \langle f, g\rangle = 0,$$

and hence

$$\|f\| \leq \|g\| \text{ when } \langle f, g-f\rangle = 0. \tag{1}$$

Now to prove 3. It is trivial when $g = 0$. Otherwise, we have

$$\langle f - P_g f, P_g f\rangle = 0$$

by 4 and linearity, and hence

$$\|P_g f\|^2 \leq \|f\|^2$$

by (1). This is a restatement of Cauchy-Schwarz. □

Now suppose that E is a finite orthornormal set of vectors. Then $\left\langle f - \sum_{e\in E} P_e f, e_0\right\rangle = 0$ for any $e_0 \in E$, so

$$\left\langle f - \sum_{e\in E} P_e f, \sum_{e\in E} P_e f \right\rangle = 0,$$

and hence $\left\|\sum_{e\in E} P_e f\right\| \leq \|f\|$. Letting c_e be such that $P_e f = c_e e$, we then have

$$\sum c_e^2 = \left\|\sum_{e\in E} c_e e\right\|^2 \leq \|f\|^2 .$$