1. Itineraries

We start with a simple example of a dynamical system obtained by iterating the quadratic polynomial

\[ f_\lambda : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \lambda x (1 - x) \]

where \( \lambda \in [1, 4) \).

Starting with the critical point \( x_0 := 1/2 \), we consider the sequence

\[ P_\lambda := \{ x_0, x_1, x_2, \ldots \}, \]

where \( x_i := f_\lambda(x_{i-1}) = f_\lambda^i(x_0) \). This sequence is called the post-critical set for \( f_\lambda \).

There is a graphical way to construct this sequence, where from the point \((x_{i-1}, x_{i-1})\), we draw a vertical segment to the graph of \( f_\lambda \) ending at \((x_{i-1}, x_i)\), then a horizontal segment to the diagonal line \( y = x \) ending at \((x_i, x_i)\), and so on, as depicted below.

![Figure 1. Post-critical set \( P_\lambda \) for \( \lambda = 3.6275 \)](image)

We say that \( f_\lambda \) is critically periodic if \( x_n = x_0 \) for some \( n > 0 \).

Suppose that you and I each have a critically periodic map and are talking over the phone. How can we figure out if we have the same map?

The first invariant of a critically periodic map \( f_\lambda \) is the period \( n = |P_\lambda| \). The map \( f_\lambda \) permutes the set \( P_\lambda \) cyclically: it sends \( x_i \) to \( x_{i+1} \), with indices taken modulo \( n \). More information is obtained from the ordering that \( P_\lambda \) inherits as a
subset of the real line. We thus relabel the elements of the post-critical set by the order-preserving map \( h : \{1, 2, ..., n\} \rightarrow P_\lambda \). Because \( f_\lambda \) is strictly increasing before \( x_0 \) and then strictly decreasing, the induced permutation \( \sigma := h^{-1} \circ f_\lambda \circ h \) on \( \{1, 2, ..., n\} \) is unimodal, meaning that there is a unique \( i \) such that
\[
\sigma(1) < ... < \sigma(i - 1) < \sigma(i) > \sigma(i + 1) > ... > \sigma(n).
\]
Of course, \( \sigma \) is cyclic as well.

![Graph of the permutation \( \sigma = (1 4 3 5 2 6) \) induced by \( f_{3.6275} \)](image)

It turns out that every such permutation \( \sigma \) arises from a unique \( \lambda \).

**Theorem 1.1** (Milnor-Thurston-Sullivan). Let \( n \geq 1 \) and let \( \sigma \in S_n \) be a unimodal cyclic permutation. Then there exists a unique \( \lambda \in [1, 4) \) such that \( f_\lambda \) is critically periodic and induces the permutation \( \sigma \).

Below, we describe some of the ideas involved in the proof of this theorem.

2. **Uniqueness**

2.1. **Grötzsch’s question.** Let \( 0 < a \leq b \) and let \( g : R_a \rightarrow R_b \) be a diffeomorphism between the rectangles \( R_a = [0, a] \times [0, 1] \) and \( R_b = [0, b] \times [0, 1] \) which takes the left, right, top and bottom sides of \( R_a \) to the corresponding sides of \( R_b \).

![Diagram of rectangles and function g](image)

**Theorem 2.1.** If \( g \) is conformal, then \( a = b \).

**Proof.** One can extend \( g \) to a conformal map from the plane to itself via repeated Schwarz reflections. Therefore, \( g \) is a euclidean similarity. Since we require that it fixes the vertices \((0,0)\) and \((0,1)\), \( g \) is the identity. \( \square \)
If $a < b$, then Grötzsch’s question asks which admissible maps $g : R_a \to R_b$ are the closest to being conformal. This requires the notion of dilatation, a measurement of the failure of a map to be conformal.

For an orientation-preserving diffeomorphism $g : U \to V$ between open subsets of the plane, the derivative $D_z g : \mathbb{R}^2 \to \mathbb{R}^2$ at any point $z \in U$ is a linear map. It thus sends the unit circle onto an ellipse, say with major axis of length $M$ and minor axis of length $m$. The dilatation of $g$ at $z$ is defined as the eccentricity of this ellipse, that is,

$$\text{Dil}_z(g) := M/m.$$ 

If in complex coordinates the derivative is given by $D_z g(w) = Aw + B\overline{w}$, then we have alternatively

$$\text{Dil}_z(g) = \frac{|A| + |B|}{|A| - |B|}.$$ 

The dilatation of $g$ is defined as

$$\text{Dil}(g) := \sup_{z \in U} \text{Dil}_z(g).$$

Grötzsch’s question then becomes: What is $\inf_g \text{Dil}(g)$ over the set of admissible maps $g : R_a \to R_b$? When is it realized?

**Theorem 2.2** (Grötzsch). *We have*

$$\text{Dil}(g) \geq \frac{b}{a}$$

*with equality only for the stretching map*

$$(x, y) \overset{g}{\mapsto} (bx/a, y).$$

### 2.2. Teichmüller’s question.

Let $X, Y$ be finite subsets of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with the same cardinality and let $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be an orientation-preserving homeomorphism with $h(X) = Y$.

Call a homeomorphism $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ admissible if $g|_X = h|_X$, $g$ is homotopic to $h$ rel $X$, and $g$ is a diffeomorphism off of a finite set. Teichmüller’s question is: What is the infimum of $\text{Dil}(g)$ over admissible maps $g$ and when is it realized?

**Theorem 2.3** (Teichmüller). *There exists a unique admissible map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, called the Teichmüller map, having minimal dilatation. Let $K := \text{Dil}(g)$. If $K = 1$, then $g$ is a Möbius transformation. If $K > 1$, then for every point $p \in \hat{\mathbb{C}}$ at which $g$ is non-singular, there is a conformal chart $\varphi$ about $p$ and a conformal chart $\psi$ about $g(p)$ such that*

$$\psi \circ g \circ \varphi^{-1}(x, y) = (Kx, y).$$
Note how similar the answers to Teichmüller’s question and Grötzsch’s question are. One can indeed think of a Teichmüller map as several Grötzsch’s stretch maps glued together.

The charts mentioned in the above theorem are unique up to translation and rotation by angle $\pi$. This implies that one can form a foliation on the complement in $\hat{\mathbb{C}}$ of the singular set of $g$, the leaves of which are the curves along which $g$ is stretching maximally, by pulling-back horizontal lines under $\varphi$-charts. The singularities of this foliation are pronged singularities, which locally look like some number of rectangles glued around a point along horizontal edges. 1-pronged singularities can only occur on the set $X$.

2.3. Idea for uniqueness. Here is how one can apply Teichmüller’s theorem to proving the uniqueness part of the Milnor-Thurston-Sullivan theorem. Suppose that $f_\lambda$ and $f_\eta$ are critically periodic and induce the same permutation $\sigma$.

Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism such that $h(P_\lambda) = P_\eta$. We can take $h$ to be piecewise linear for example. Then extend $h$ to $\mathbb{R}^2$ by setting $h(x, y) = (h(x), y)$. Let $X := P_\lambda \cup \{\infty\}$ and let $Y := P_\eta \cup \{\infty\}$.

Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the map such that $g|_X = h|_X$ and $g$ is homotopic to $h$ rel $X$ with minimal dilatation. Since the map $z \mapsto g(\overline{z})$ is also admissible and has the same dilatation as $g$, it is equal to $g$ by Teichmüller’s uniqueness theorem. In particular, $g(\mathbb{R}) = \mathbb{R}$.

Note that $g$ conjugates the actions of $f_\lambda$ and $f_\eta$ on their respective critical sets, since the two maps induce the same permutation $\sigma$ by hypothesis. It follows that we can lift $g$ to a homeomorphism $\tilde{g} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the diagram

\[
\begin{array}{ccc}
(C, P_\lambda) & \xrightarrow{\tilde{g}} & (C, P_\eta) \\
\downarrow f_\lambda & & \downarrow f_\eta \\
(C, P_\lambda) & \xrightarrow{g} & (C, P_\eta)
\end{array}
\]

commutes.

If $g$ has dilatation greater than 1, then it has a finite positive number of singularities. A combinatorial argument shows that the stretching foliation for $\tilde{g}$ must have more singularities counting multiplicities than the stretching foliation for $g$. This is because $f_\lambda$ has degree two, so a typical singularity has two preimages, although a 1-prong singularity at the critical value of $f_\lambda$ would unfold to a regular point.
Since \( f_\lambda \) and \( f_\eta \) are locally conformal on \( \mathbb{C} \setminus P_\lambda \) and \( \mathbb{C} \setminus P_\eta \) respectively, the dilatation of \( \tilde{g} \) is the same as the dilatation of \( g \). Moreover, \( \tilde{g} \) is admissible for the same problem as \( g \), since it maps \((\mathbb{C}, \mathbb{R}, P_\lambda)\) to \((\mathbb{C}, \mathbb{R}, P_\eta)\) and is increasing on \( \mathbb{R} \). By Teichmüller’s uniqueness theorem we must have \( \tilde{g} = g \), a contradiction.

Therefore, \( g \) has dilatation 1 and is thus conformal, hence a euclidean similarity.

3. Existence

Given a unimodal cyclic permutation \( \sigma \in S_n \), we have to find a set of \( n \) points \( X_\sigma \subset \mathbb{R} \) and a quadratic polynomial \( f \) inducing the permutation \( \sigma \) on \( X_\sigma \). It is then a simple matter to conjugate \( f \) by an affine map to a polynomial of the form \( f_\lambda(x) = \lambda x(1-x) \).

To do this, we start with any set \( X_0 \subset \mathbb{R} \) of cardinality \( n \) and apply the following procedure.

Given the set \( X_{i-1} \), let \( h : \{1, 2, \ldots, n\} \rightarrow X_{i-1} \) be the order-preserving bijection, \( m := \max(X_{i-1}) = h(n) \) and \( \tau := h \circ \sigma \circ h^{-1} \) the permutation induced by \( \sigma \) on \( X_{i-1} \). For \( x \in X_{i-1} \), let

\[
g(x) := \pm \sqrt{m - x},
\]

with positive sign if \( \tau^{-1}(x) \geq \tau^{-1}(m) \) and negative sign otherwise. Then define \( X_i := g(X_{i-1}) \) and repeat.

By construction, the polynomial \( f(y) = m - y^2 \) maps the \( j \)-th element of \( X_i \) to the \( \sigma(j) \)-th element of \( X_{i-1} \). As we iterate this process, the sets \( X_i \) will converge, up to translation and rescaling, to a set \( X_\sigma \) solving the initial problem.

![Figure 4. Pull-back map for \( \sigma = (1\ 4\ 3\ 5\ 2\ 6) \) applied to the set \( \{1, 2, 3, 4, 5, 6\} \)](image-url)
Let us make this more precise. Define
\[ F_n := \{ X \subset \mathbb{R} : |X| = n \}/\sim, \]
where \( X \sim Y \) if \( Y = aX + b \) for some \( a > 0 \) and \( b \in \mathbb{R} \). The process described above yields a map
\[ F_\sigma : F_n \to F_n, \]
called the pull-back map. For \([X],[Y] \in F_n\), let \( h : \mathbb{C} \to \mathbb{C} \) be an orientation-preserving homeomorphism such that the restriction \( h : X \to Y \) is order-preserving. The Teichmüller distance between \([X],[Y]\) defined as
\[ d([X],[Y]) := \log \inf \{ \text{Dil}(g) \mid g \text{ is admissible with respect to } h \}. \]

**Theorem 3.1.** There exists an \([X_\sigma] \in F_n\) such that
\[ F_\sigma([X_\sigma]) = [X_\sigma] \]
and
\[ F_\sigma^k([X]) \to [X_\sigma] \]
as \( k \to \infty \) for all \([X] \in F_n\). Moreover,
\[ d(F_\sigma([X]), F_\sigma([Y])) < d([X],[Y]) \]
for all \([X],[Y] \in F_n\) with \([X] \neq [Y]\). In particular, the fixed-point \([X_\sigma]\) is unique.

We discuss briefly how the contraction property (2) can be used to prove (1). If \( F_\sigma \) was a uniform contraction, then the existence of a fixed-point would follow from the fact that the Teichmüller metric is complete. In order to prove (2), we show that \( F_\sigma \) is smooth and its derivative has norm less than 1 on every compact subset of \( F_n \) with respect to the Teichmüller metric. It matters then to understand the behavior of \( F_\sigma \) near infinity. Note that a sequence \([X_k]\) in \( F_n \) goes to infinity (i.e. escapes every compact set) if the points of \( X_k \) are clumping together. If for some \([X] \in F_n\) the sequence \( F_\sigma^k([X]) \) was going to infinity, this would mean that a certain clumping phenomenon is invariant under the dynamics, but we will see that this is impossible.
Figure 5. Second iteration of the pull-back map