Bounds for bounded primitive renormalization and MLC

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September 11, 2009

Basic Definitions I

1. **Quadratic-like map**: $f \colon U \to V$.
   - f is proper and holomorphic of degree 2.
   - $U \subset\subset V$.

   We let $K_f = \bigcap_{k=0}^{\infty} f^{-k} V$. *We will assume that* $f'(0) = 0$.
2. $f \colon U \to V$ **is** N**-renormalizable**: We can find $U_N \subset U$, $V_N \subset V$ such that $f^N \colon U_N \to V_N$ is quadratic-like and $0 \in K_{f^N}$. *We let* K_N *be* K_{f^N}.
3. $f \colon U \to V$ is **primitively** N-renormalizable: f is N-renormalizable and $f^k(K_N) \cap K_N = \emptyset$ for $0 < k < N$.
4. We say that $f_c(z) = z^2 + c$ is **infinitely renormalizable of** B**-bounded primitive type** if f is primitively $N_0|N_1|N_2|\ldots$-renormalizable (with $N_0 = 1$), and $N_{k+1}/N_k \leq B$ for all $k \geq 0$.

Bounds for bounded-primitive type

Theorem

Suppose that $f(z) = z^2 + c$ is B-bounded infinitely primitively renormalizable. Then for every primitive renormalization time N, we can find U_N, V_N such that $f^N \colon U_N \to V_N$ is an N-renormalization of f and $\operatorname{mod}(V_N, U_N) \geq \epsilon(B)$.

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We say that f has the *a priori* bounds.

Examples and Applications

- Suppose that $f_c(z) = z^2 + c$ is infinitely renormalizable of bounded primitive type.
  1. The Mandelbrot set is locally connected at c.
  2. The quasiconformal map from M to M_Λ for any Mandelbrot-like family Λ is $C^{1+\alpha}$ at c, with conformal derivative.
  3. The rescalings of M around c converge in the Hausdorff topology to $\mathbb{C}$.

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- Suppose that $f_c(z)$ is infinitely renormalizable of constant primitive type. Then the rescalings of the small Mandelbrot sets M_k around c converge in the Hausdorff topology.

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- Suppose that $f_c(z)$ is infinitely renormalizable of constant primitive type. Then the rescalings of the small Mandelbrot sets M_k around c converge in the Hausdorff topology.
- The set of parameter values c such that f_c is infinitely renormalizable of bounded primitive type has measure 0.

Basic Definitions II

Suppose that f is primitively N-renormalizable.

- We let $\mathcal{K}_N = \bigcup_{k=0}^{N-1} f^k(K_N)$ be the union of small Julia sets.
- We let γ_N be the curve in $\mathbb{C} \setminus \mathcal{K}_N$ separating K_N from the other small Julia sets.
- We let $L(\gamma_N; \mathbb{C} \setminus \mathcal{K}_N)$ denote the Poincaré length of the geodesic representative of γ_N in $\mathbb{C} \setminus \mathcal{K}_N$.

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Suppose that f is $N_0|N_1|N_2|\ldots$-primitively renormalizable. For $n > 0$, we let $K_n \equiv K_{N_n}$, $\mathcal{K}_n \equiv \mathcal{K}_{N_n}$, and $\gamma_n \equiv \gamma_{N_n}$ if there is danger of no confusion.

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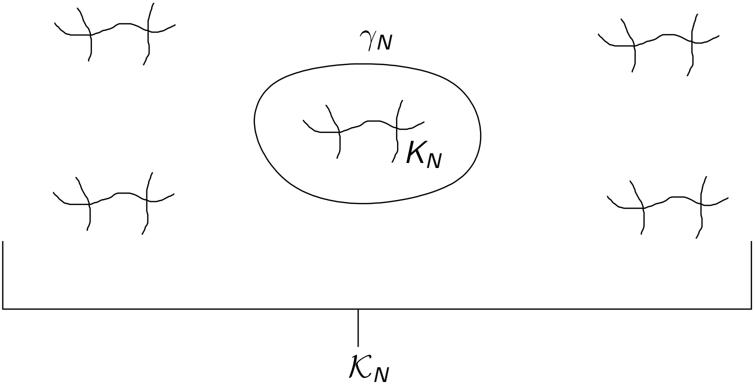
Suppose that f is $N_0|N_1|N_2|\ldots$-primitively renormalizable. For $n > 0$, we let $K_n \equiv K_{N_n}$, $\mathcal{K}_n \equiv \mathcal{K}_{N_n}$, and $\gamma_n \equiv \gamma_{N_n}$ if there is danger of no confusion. We will prove the following well-known theorem as part of our theory:

Theorem

Suppose that f is as above, and $L(\gamma_n; \mathbb{C} \setminus \mathcal{K}_n) \leq L_0(f)$ for $n \in \mathbb{Z}^+$. Then f has the a priori bounds.

Illustration of K_N, $\mathcal{K}_N$, and γ_N for $N = 5$



If it's bad now, it was worse earlier

We will prove the following:

Theorem

Suppose that f is B-bounded $N_0|N_1|N_2|\ldots$-primitively renormalizable. Then

$$L(\gamma_{n-12};\mathbb{C}\setminus\mathcal{K}_{n-12}) \geq L(\gamma_{n-12};\mathbb{C}\setminus\mathcal{K}_n) \geq 2L(\gamma_n;\mathbb{C}\setminus\mathcal{K}_n)$$

whenever $L(\gamma_n;\mathbb{C}\setminus\mathcal{K}_n) \geq L_0(B)$.

This implies that $L(\gamma_n;\mathbb{C}\setminus\mathcal{K}_n) \leq L_0(B)$ for all n, and hence the *a priori* bounds.

Pseudo-quadratic-like maps

Pseudo-quadratic-like map $(i, f)\colon U \to V$:

- Simply connected Riemann surfaces U and V.
- Non-degenerate compact full continua $K_U \subset U$ and $K_V \subset V$.
- A holomorphic immersion $i\colon U \to V$ such that $i^{-1}(K_V) = K_U$, and $i\colon K_U \to K_V$ is a bijection.
- A proper degree 2 holomorphic map $f\colon U \to V$ such that $f^{-1}(K_V) = K_U$.

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Theorem

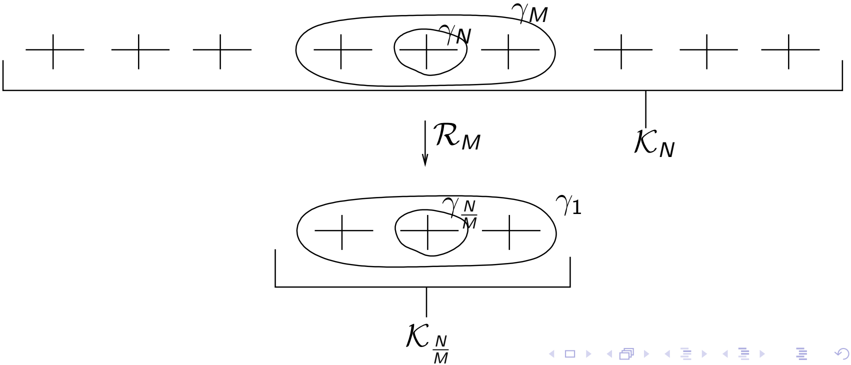
Suppose that $(i, f): U \to V$ is a pseudo-quadratic-like map. Then we can find $U' \subset U$ and $V' \subset V$ such that $(i, f): U' \to V'$ is quadratic-like. Moreover, we can make $\mathrm{mod}(V', i(U')) \geq \epsilon(mod(V, K_V))$.

The Canonical Renormalization

Theorem

Suppose that $f_c(z) = z^2 + c$ is primitively $M|N$-renormalizable. Then we can find a pseudo-quadratic-like M-renormalization $(i_M, f_M)\colon U_M \to V_M$ such that

- $L(\gamma_1; V_M \setminus \mathcal{K}_{\frac{N}{M}}) = L(\gamma_M; \mathbb{C} \setminus \mathcal{K}_N)$.
- $L(\gamma_{\frac{N}{M}}; V_M \setminus \mathcal{K}_{\frac{N}{M}}) = L(\gamma_N; \mathbb{C} \setminus \mathcal{K}_N)$



The Main Local Theorem

Theorem

Suppose that the pseudo-quadratic-like map $(i, f)\colon U \to V$ *is* N*-renormalizable. Then*

$$L(\gamma_1; V \setminus \mathcal{K}_N) \geq 2^{-18} \cdot N \cdot L(\gamma_N; V \setminus \mathcal{K}_N) - C(N).$$

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$$L(\gamma_{n-12}; \mathbb{C} \setminus \mathcal{K}_n) \geq 2L(\gamma_n; \mathbb{C} \setminus \mathcal{K}_n)$$

whenever $L(\gamma_n; \mathbb{C} \setminus \mathcal{K}_n) \geq L_0(B)$.

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Let $N \equiv \frac{N_n}{N_{n-12}}$: then $N \leq B^{12}$ and $2^{-18}N \geq 2^{-18}3^{12} > 2$.
(Let $(i, f)\colon U \to V$ be the canonical N_{n-12}-renormalization of f_c).

The Extremal Width "Functor"

For any path family Γ we let

$$\mathcal{W}(\Gamma) = \inf \left\{ \int \rho^2 \mid L_\rho(\gamma) \geq 1 \text{ for all } \gamma \in \Gamma \right\}$$

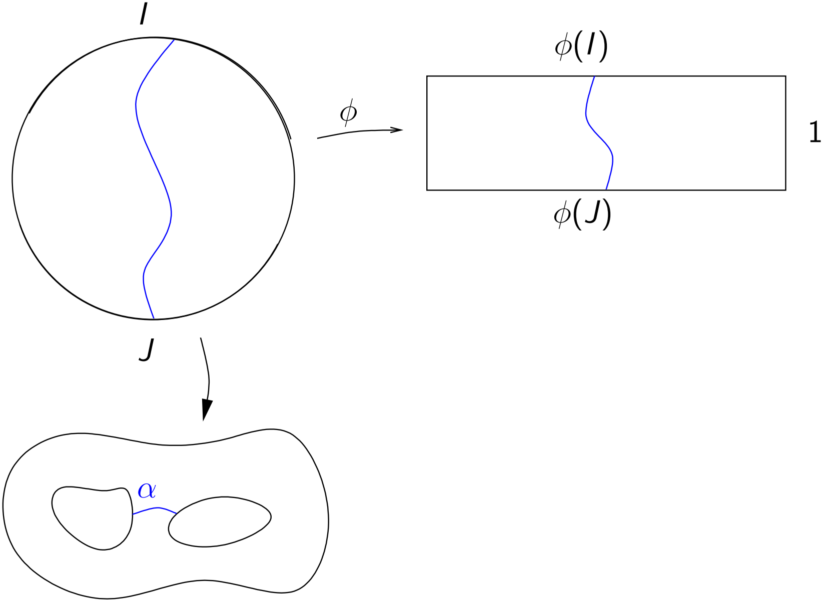
Let S be a compact Riemann surface with boundary. We let $\mathcal{A}(S)$ denote the space of arcs on S, and $\mathcal{WA}(S)$ denote the space of formal sums of disjoint arcs on S. Let $\mathcal{F}$ be a partial proper foliation on S. Then we let

$$\mathcal{W}(\mathcal{F}) = \sum_{\alpha \in \mathcal{A}(S)} \mathcal{W}(\mathcal{F}|_\alpha)\alpha,$$

or

$$\mathcal{W}(\mathcal{F})(\alpha) = \mathcal{W}(\mathcal{F}|_\alpha).$$

The canonical foliation: we illustrate $\mathcal{F}_{\mathsf{can}}|_{\alpha}$



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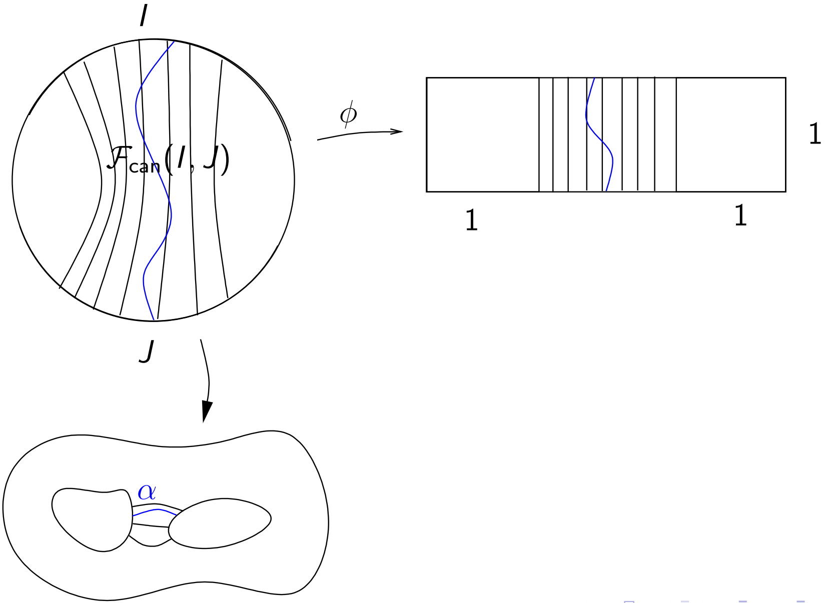
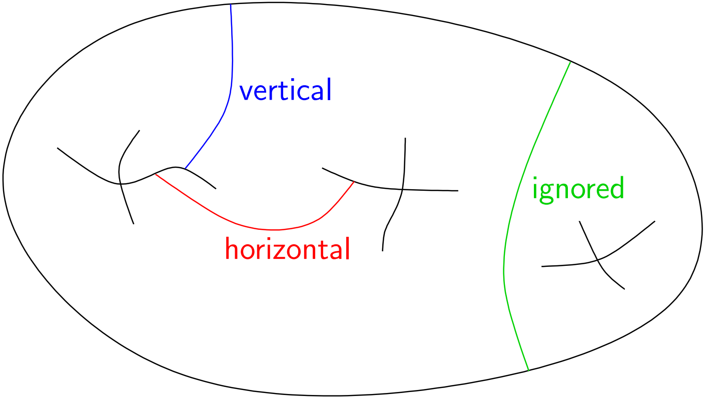


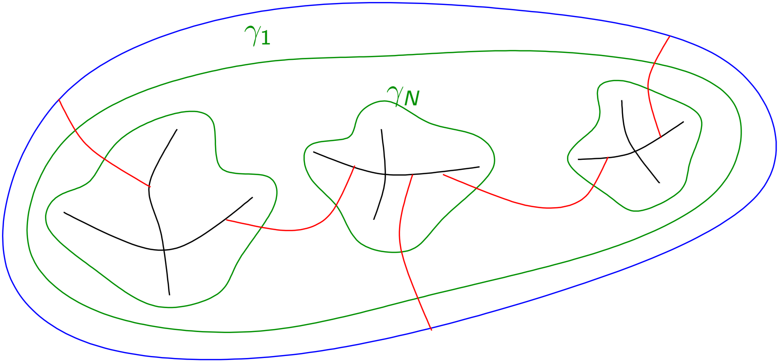
We let $W_{\mathsf{can}}(S) = \mathcal{W}(\mathcal{F}_{\mathsf{can}}(S))$.

1. If $f\colon S \to T$ is a finite cover, then $W_{\mathsf{can}}(S) = f^* W_{\mathsf{can}}(T)$.
2. If $\mathcal{F}$ is any proper partial foliation on S, then
   $W_{\mathsf{can}}(S) \geq \mathcal{W}(\mathcal{F}) - 2$
   $(W_{\mathsf{can}}(S)(\alpha) \geq \mathcal{W}(\mathcal{F})(\alpha) - 2)$.
3. If γ is a peripheral closed Poincaré geodesic on S, then

$$L(\gamma; S) = \pi \langle \gamma, \mathcal{W}_{\mathsf{can}}(S) \rangle + O(1; \chi(S)).$$

Horizontal and vertical arcs





γ_1
γ_N

The $\mathcal{W}_{\text{can}}$ version of the Main Local Theorem

We let $\mathcal{W}^{h}_{\text{can}}(V \setminus \mathcal{K}_N)$ denote $\mathcal{W}_{\text{can}}(V \setminus \mathcal{K}_N)$ restricted to the horizontal arcs of $\mathcal{W}_{\text{can}}(V \setminus \mathcal{K}_N)$, and likewise define $\mathcal{W}^{v}_{\text{can}}$.

Theorem

$$\|\mathcal{W}^{v}_{\text{can}}(V \setminus \mathcal{K}_n)\|_1 \geq 2^{-16}\|\mathcal{W}^{h+v}_{\text{can}}(V \setminus \mathcal{K}_n)\|_1 - C(N).$$

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By property 3,

$$L(\gamma_1; V \setminus \mathcal{K}_n) = \pi \langle \gamma_1, V \setminus \mathcal{K}_N \rangle + O(1; N) \geq \pi \|\mathcal{W}^v_{\mathrm{can}}\|_1 - C(N)$$

and

$$\sum_{k=0}^{N-1} L(f^{-k*}\gamma_N; V \setminus \mathcal{K}_N) = \pi \|\mathcal{W}^{v+2h}_{\mathrm{can}}\|_1 + O(1; N) \leq 2\pi \|\mathcal{W}^{v+h}_{\mathrm{can}}\|_1 + C(N).$$

Moreover $L(\gamma_1) \leq 2L(f^{-k*}(\gamma_1))$ for $k = 1, \ldots, N-1$. So Theorem 8 implies the Main Local Theorem.

Three steps to prove the Main Local Theorem

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1. $\|\mathcal{W}^h_{\text{can}}(V^{7N} \setminus \mathcal{K}_N)\|_1 \leq \frac{1}{2}\|\mathcal{W}^h_{\text{can}}(V^0 \setminus \mathcal{K}_N)\|_1 + C(N).$

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3. $$\|\mathcal{W}^v_{\text{can}}(V^{rN} \setminus \mathcal{K}_n)\|_1^2 \leq 2 \cdot 2^{2r}\|\mathcal{W}^v_{\text{can}}(V_0 \setminus \mathcal{K}_N)\|_1 \|\mathcal{W}^{v+h}_{\text{can}}(V_0 \setminus \mathcal{K}_N)\|_1 + C(r, N).$$

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3. $\|\mathcal{W}^v_{\text{can}}(V^{rN} \setminus \mathcal{K}_n)\|_1^2 \le$
   $2 \cdot 2^{2r}\|\mathcal{W}^v_{\text{can}}(V_0 \setminus \mathcal{K}_N)\|_1 \|\mathcal{W}^{v+h}_{\text{can}}(V_0 \setminus \mathcal{K}_N)\|_1$
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$$1\&2 \implies \|\mathcal{W}^v_{\text{can}}(V^{7N} \setminus \mathcal{K}_n)\|_1 \ge \|\mathcal{W}^{h+v}_{\text{can}}(V^0 \setminus \mathcal{K}_N)\|_1 - C(N)$$

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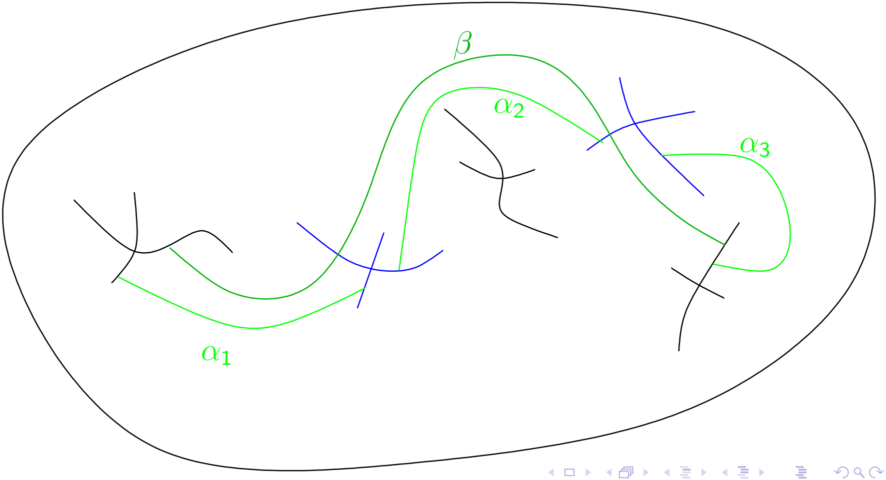
The Arrow Relation

Suppose that $U \subset V$, and $\beta \in \mathcal{A}(V)$, and $\alpha_i \in \mathcal{A}(U)$. We say that

$$(\alpha_i) \to \beta$$

if β has a representative b that restricts to (a_i) representing (α_i).



The Art of Domination

Suppose that $U \subset V$, and $X \in \mathcal{WA}(U)$, $Y \in \mathcal{WA}(V)$. We say that $X \multimap Y$ if

$$X = \sum w_{ij}\alpha_{ij}$$

and

$$Y \leq \sum v_i\beta_i$$

and

$$\forall i \quad (\alpha_{ij})_j \to \beta_i$$

$$\forall i \quad \sum w_{ij}^{-1} \leq v_i^{-1}.$$

Domination and Restriction

Suppose that $U \subset V$ (and $\pi_1(U) \to \pi_1(V)$ is surjective), and $\mathcal{F}$ is a proper partial foliation of V. Then

$$\mathcal{W}(\mathcal{F}|_U) \multimap \mathcal{W}(\mathcal{F}).$$

Corollary

$$\mathcal{W}_{\mathrm{can}}(U) \multimap \mathcal{W}_{\mathrm{can}}(V) - 6|\chi(U)|.$$

The Pullback Diagram

Let $X^k = \mathcal{W}^h_{\text{can}}(V^k - \mathcal{K}_N) - g_N(k)$, where $g_N(0) = 0$ and $g_N(k) = 3N(g_N(k) + 2)$. Then the following diagram holds:

$$
\begin{tikzcd}
 & X^3 & \leq & X^2 & \leq & X^1 & \leq & X^0 \\
\cdots & f^*X^2 \arrow[u, dash] \arrow[urr] & & f^*X^1 \arrow[u, dash] \arrow[urr] & & f^*X \arrow[u, dash] \arrow[urr] & &
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\end{tikzcd}$$

We will show that this diagram implies that

$$X^{7N} < \frac{1}{2}X^0.$$

From horizontal arcs to the Hubbard tree

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Then for every vertical $\gamma \in \mathcal{A}(\mathbb{C} - \mathcal{K}_N)$,

$$\left\langle \gamma, \operatorname{supp} X^k \right\rangle = 0 \implies \left\langle f^*\gamma, \operatorname{supp} X^k \right\rangle = 0.$$

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Let $H_* \subset \mathcal{A}^h(\mathbb{C} - \mathcal{K}_N)$ be those arcs that do not intersect any external ray from $\mathcal{K}_N$. Then $\operatorname{supp} X^{3N} \subset X^k \subset H_*$.

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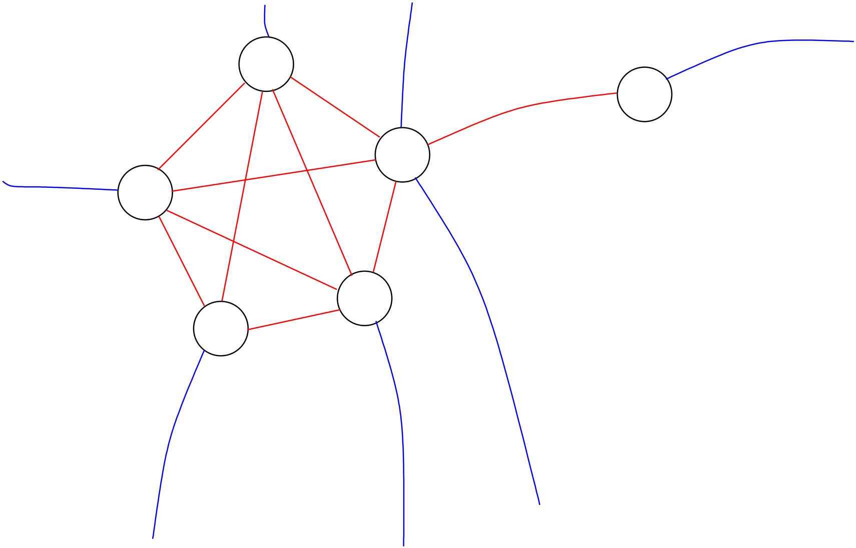
$$\left\langle \gamma, \operatorname{supp} X^k \right\rangle = 0 \implies \left\langle f^* \gamma, \operatorname{supp} X^k \right\rangle = 0.$$

Then $\langle \eta, \operatorname{supp} X^k \rangle = 0$ for every external ray η to $\mathcal{K}_N$.
Let $H_* \subset \mathcal{A}^h(\mathbb{C} - \mathcal{K}_N)$ be those arcs that do not intersect any external ray from $\mathcal{K}_N$. Then $\operatorname{supp} X^{3N} \subset X^k \subset H_*$.
We say that $\operatorname{supp} X^{3N}$ is *aligned with the Hubbard tree*.

Arcs “aligned with the Hubbard tree”



The straight-arrow relation

For $\beta \in H_*$, we say that $(\alpha_i) \twoheadrightarrow \beta$ if (α_i) is the shortest sequence in $f^{k*}H_*$ such that $(\alpha_i) \rightarrow \beta$. For each β and k there is a unique such sequence (α_i).

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Theorem

If $(\alpha_i) \twoheadrightarrow \beta$ and the $\alpha_i \in f^{kN*}H_*$ then $\#(\alpha_i) \geq 2^k$.

Weighted arc diagrams restricted to a single $K_N(i)$.

Let

$$X|_{D_i} = \sum_{D_i \in \partial\alpha} X(\alpha).$$

Then

$$\sup_{D_i} X^{4N} \leq 2 \inf_{D_i} X^{3N}.$$

The Fundamental Theorem

Suppose $(\alpha_i) \twoheadrightarrow \beta$ where the $\alpha_i \in f^{rN*}H_*$ and $\beta \in H_*$.

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Corollary

We have

$$\|X^{7N}\|_1 \leq 2^{-1}\|X^{3N}\|_1 \quad (\leq 2^{-1}\|X^0\|_1).$$

(because $f^{3N*}X^{4N} \multimap X^{7n}$))

The picture for the fundamental theorem