The Mandelbrot Set is Connected: a Topological Proof

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Given $c \in \mathbb{C}$, let $f_c : \mathbb{C} \to \mathbb{C}$ be defined by $f_c(z) = z^2 + c$, and let f_c^n be the n^{th} iterate of f_c. The Mandelbrot set M is defined by

$$M = \{c \mid f_c^n(0) \not\to \infty\}.$$

It is easy to show that the Mandelbrot set is compact: one shows that if $|z| > \max(|c|, 2)$, then $|f_c^n(z)| > |z|$ and $f_c^n(z) \to \infty$; therefore

$$M = \{c \mid |f_c^n(0)| \leq 2 \text{ for all } n \geq 1\}.$$

Douady and Hubbard[1] showed that the Mandlebrot set is connected. Their proof was analytic in that it directly constructed the Riemann map from the complement of the Mandelbrot set to the complement of the disk.

We give here a topological proof that M is connected. For each $n \geq 1$, let $P_n(c) = f_c^n(0)$. Choose $R > 2$, and let D_R denote the (open) disk of radius R, and C_R its boundary. It is convenient to choose R transcendental, so that it is generic for our purposes. Let $\overline{X}$ denote the closure of a set $X \subset \mathbb{C}$.

We have that

$$M = \bigcap_{n=1}^{\infty} P_n^{-1}(\overline{D_R})$$

is a nested intersection of compact sets, so we need only show that $P_n^{-1}(\overline{D_R})$ is connected for all $n \geq 1$. So assume that for some $n \geq 1$, $P_n^{-1}(\overline{D_R})$ is disconnected. Let U be a component of $P_n^{-1}(D_R)$ such that $0 \notin \overline{U}$. We claim that, for $1 \leq k \leq n$,

$$(*) \qquad 0 \notin P_k(\overline{U}).$$

But P_n is a branched cover, so $P_n(U) = D_R$, which contradicts the claim.

So all that remains is to prove $(*)$ above. This in turn will follow from the "principle of correspondence of winding number", namely

1 Proposition

Let S be a smooth Jordan curve, and $(C_s)_{s\in S}$ be a smoothly varying family of smooth Jordan curves, and for $s \in S$, let D_s be the Jordan domain such that $C_s = \partial D_s$. Now let $a, b, f : S \to \mathbb{C}$ be smooth functions such that for all $s \in S$, $a(s), b(s) \in D(s)$, and $f(s) \in C(s)$. Then

$$\#_{s\in S}(f(s) - a(s)) = \#_{s\in S}(f(s) - b(s))$$

where $\#$ denotes the winding number around 0.

Given this proposition, we can prove $(*)$ by induction on k. For $k = 1$, equation $(*)$ is just our assumption that $0 \notin \overline{U}$. Now assume $(*)$ for some $k \in [1, n)$; we will prove it for $k+1$. We have

$$f_c^k(0) \neq 0 \text{ for } c \in \overline{U}$$

and therefore

$$f_c^{k+1}(0) \neq c \text{ for } c \in \overline{U}.$$

Now if g is an analytic function on a neighborhood of $\overline{U}$, then

$$0 \notin g(\overline{U})$$

if and only if

$$\#_{c\in U}\; g(c) \;=\; 0.$$

Thus we have

$$\#_{c\in U}(f_c^{k+1}(0) - c) = 0$$

and need only show that

$$\#_{c\in U}(f_c^{k+1}(0) - 0) = 0.$$

This follows immediately from the principle of correspondence of winding number, because for $c \in \partial U$,

$$f_c^{k+1}(0) \in f_c^{k+1-n}(C_R),$$

which is a smoothly varying family of smooth Jordan curves, and

$$0, c \in f_c^{k+1-n}(D_R),$$

because $f_c^{n-1}(c) \in C_R$, so $f_c^m(c) \in D_R$ for $m < n-1$.

1 Dotting the i's and crossing the t's

In order to complete our argument, we should

1. define a smoothly varying family of curves,

2. prove proposition 1, and

3. prove the italicized statement in the previous section.

We say that a family of Jordan curves J_λ, for $\lambda \in \Lambda \subset \mathbb{C}$, varies smoothly if there is a diffeomorphism

$$\mathbf{J} : \Lambda \times C_1 \to \{(\lambda, z) : \lambda \in \Lambda \text{ and } z \in J_\lambda\}$$

that preserves the first coordinate. Thus a smoothly varying family is a smooth isotopy, and we can extend it by the ambient isotopy theorem to

$$J : \Lambda \times \mathbb{C} \to \Lambda \times \mathbb{C}$$

that is again a diffeomorphism preserving the first coordinate.

Now returning to the notation of proposition 1, we have by the above a diffeomorphism

$$\mathbf{C} : S \times \mathbb{C} \to S \times \mathbb{C}$$

such that for each s there exists $\mathbf{C}_s : \mathbb{C} \to \mathbb{C}$ such that

$$\mathbf{C}(s, z) = (s, \mathbf{C}_s(z)),$$

with

$$\mathbf{C}_s(C_1) = C_s,$$

and

$$\mathbf{C}_s(D_1) = D_s.$$

Then we define, for $t \in [0, 1]$,

$$e_t(s) = \mathbf{C}_s((1-t)\mathbf{C}_s^{-1}(a(s)) + t\mathbf{C}_s^{-1}(b(s))),$$

so that $e_t(s) \in D_s$ for all s, t, and hence $e_t(s) \neq f(s)$ for any s, t. It follows that

$$\#_{s \in C_s}(f(s) - e_t(s))$$

is constant (in t), and the proposition follows.

We now show

2 Lemma
Let $V = P_m^{-1}(D_R)$, for $R \geq 2$. Then $(f_c^{-m}(C_R))_{c \in V}$ is a smoothly varying family of smooth Jordan curves.

This implies, in the notation of the previous section, that for $k \geq 1$, $f_c^{k+1-n}(C_R)$ is a smoothly varying family of smooth Jordan curves over $\partial U \subset P_{n-1}^{-1}(D_R)$. This is the italicized statement of the previous section.

The lemma is more or less obvious, given the standard (topological) fact that $f_c^{-m}(C_R)$ is a smooth Jordan curve provided that $f_c^m(0) \in D_R$.

We can give a cute proof of the lemma that reproves this standard fact, on the assumption that $P_m^{-1}(D_R)$ is connected. This is of course what we are trying to prove as the main theorem, but if we assume in the previous section that n is the *least* value for which $P_n^{-1}(D_R)$ is disconnected, then $P_{n-1}^{-1}(D_R)$ is connected, and we need the lemma only for $m = n - 1$.

Accordingly, let $V = P_m^{-1}(D_R)$, and assume that V is connected. Then V is simply connected by the maximum modulus principle. Let $\mathbf{f}(c, z) = (c, f_c(z))$; then

$$\mathbf{f}^n : \{(c, z) : c \in V, f_c^n(z) \in C_R\} \to V \times C_R$$

is a covering map that preserves the first coordinate. Now let $F_0 = f_0^{-n}(C_R)$; then the map

$$V \times F_0 \to V \times C_R$$

given by $(c, z) \mapsto (c, f_0^n(z))$ is also a covering that preserves the first coordinate, and it has the same image in π_1 as $\mathbf{f}^n$ above. Therefore there is a diffeomorphism

$$V \times F_0 \to \{(c, z) : c \in V, f_c^n(z) \in C_R\}$$

that preserves the first coordinate; and F_0 is a round circle, so $(f_c^{-m}(C_R))_{c \in V}$ is a smoothly varying family of smooth Jordan curves.

References

[1] Adrien Douady and John Hamal Hubbard. Itération des polynômes quadratiques complexes. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(3):123–126, 1982.