Topics in Number theory

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Contents

Lecture 1: 2010-9-7

Introduction

- Analytic number theory

- Text: Automorphic forms and L-functions for the group $GL(n,\mathbb{R})$

Topics:

1. Selberg class$= \mathcal{S}$, L-functions

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad a(n) \in \mathbb{C}$$

Selberg's axioms:

(1) analytic continuation: $\exists \ell \geqslant 0$ such that $(s-1)^{\ell} L(s)$ is an entire function of $s \in \mathbb{C}$

(2) Euler product:

$$\log L(s) = \sum_{m=1}^{\infty} \frac{b(m)}{m^s}$$

where $b(m) = 0$ unless $b(m) = 0$ unless $m = p^k$ and $|b(m)| \ll m^{\frac{1}{2}}$

(3) functional equation:

$$\Lambda(s) = A^s \prod_{j=1}^{r} \Gamma\left(\lambda_j s + \mu_j\right) L(s) = \overline{\Lambda(1-\overline{s})}$$

for $A \geqslant 1$, $\Re(\lambda_j) > 0,\ \Re(\mu_j) \geqslant 0$

(4) Ramanujan conjecture: $\exists \epsilon > 0$ such that $a(n) \ll n^{\epsilon}$.

Notation: $f(x) \ll g(x) \iff \exists C > 0$ such that $f(x) \leqslant Cg(x)$ for all x.

Conjecture (Selberg): $L(s) \in \mathcal{S} \Rightarrow$R.H. i.e., $L(\rho) = 0$ and $0 < \Re(\rho) < 1 \Rightarrow \Re(\rho) = \frac{1}{2}$

2. $d =$ Degree of $L(s)$ defined

$$d := 2 \cdot \sum_{j=1}^{r} \lambda_j$$

(example) $\zeta(s) = \sum_{n=1}^{\infty}$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

then degree $= 1$.
Conjecture: Degree $=$ integer

3. Selberg Trace Formula (break the group into the conjugacy classes) : Kutznetsov Trace Formula (uses double cosets) which is the special case of Jacquet's relative trace formula

Operator in a Hilbert space: $\lambda_j \in \mathbb{R}$, eigenvalues, for $j = 1, 2, \ldots,$ then

$$\text{Trace} = \sum_{j=1}^{\infty} h(\lambda_j)$$

where $h : \mathbb{R} \to \mathbb{C}$.

Compare:

- Selberg Trace formula: uses conjugacy classes
  → Fundamental lemma (deep connection with algebraic geometry)
- Kuznetsov Trace formula: uses double cosets
  → Kloosterman sum (deep connection with algebraic geometry)

4. Selberg Zeta function: R.H. holds

   Gamma function: $\Gamma(s+1) = s\cdot\Gamma(s)$ and $\Gamma(1) = 1$ + some analytic conditions ⇒ uniquely determined.

   $$\Gamma(s) = \int_0^\infty e^{-u}u^s\,\frac{du}{u}$$

   Selberg zeta function has "double Gamma function"

   $$\Gamma_2(s+1) = \Gamma(s)\Gamma_2(s),\ \Gamma_2(1) = 1$$

   no Euler product but has similar functional equations.
   (Selberg zeta function exists in higher rank? open problem)

5. Bombieri's Theorem (for R.H.) : Bombieri proved that the R.H. holds on average for L-functions on $GL(1)$

Lecture 2: 2010-9-14

1 Selberg Class

1.1 Selberg class $\mathcal{S}$ of L-functions

Axiom 1: $L \in \mathcal{S}$ then

$$L(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}, \qquad a(n) \in \mathbb{C}$$

and this series converges absolutely in $\Re(s) > \sigma$ for some $\sigma \in \mathbb{R}$.

Axiom 2: $\exists m \in \mathbb{Z},\ m \geqslant 0$ such that

$$(s-1)^m L(s)$$

is an entire function of finite order. (Here, this order is the Hadamard order.)

Axiom 3: (Functional equation) $\exists A > 0,\ \lambda_j \in \mathbb{R},\ \mu_j \in \mathbb{C}$ with $\lambda_j > 0,\ \Re(\mu_j) \geqslant 0$ for $j = 1, \ldots, k$ such that

$$\Lambda_L(s) := A^s \prod_{j=1}^{k}\Gamma(\lambda_j(s) + \mu_j)L(s) = \mu\cdot\overline{\Lambda_L(1-\overline{s})}, \quad |\mu| = 1.$$

Remark. Axiom 3 ⇒ $L(s)$ has order = 1, i.e., $|L(s)| \ll e^{|s|^{1+\epsilon}}$.

Axiom 4: (Euler product)

$$\log(L(s)) = \sum_{n=1}^{\infty} b(n)n^{-s}$$

where $b(n) = 0$ if $n \neq p^l$ for some prime p , $l \geqslant 0$ and $|b(n)| \ll n^\theta$ for $\theta < \frac{1}{2}$.

Axiom 5: (Ramanujan bound)

$$|a(n)| \ll n^{\epsilon}$$

for any $\epsilon > 0$ as $n \to \infty$.

Remark. **(Hardmard order)**

$$|f(s)| \ll e^{|s|^{r+\epsilon}}, \ \forall \epsilon > 0$$

as $|s| \to \infty$ we say it has order r. (This r is the smallest one.)

Conjecture. (Selberg)

$$L \in \mathcal{S} \ \Rightarrow \ L \textit{ satisfies the RH}$$

i.e., all non-trivial zero of $L(s)$ *lie on the line* $\Re(s) = \frac{1}{2}$.

Trivial zeros of $L(s)$: $\Gamma(s)$ has poles at $s = 0, -1, -2, \ldots$

$$\Rightarrow \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j)$$

has poles when $\lambda_j s + \mu_j = -m$, for $m = 0, 1, 2, \ldots,$

$$\Rightarrow \ L(s) = 0 \text{ when } s = \frac{-m - \mu_j}{\lambda_j}.$$

Why Selberg chose these axioms

For Axiom 1: for $0 < r_n \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{r_n^s}$$

For Axiom 2: If we allow $L(s)$ to have to have poles at more than one poles we will construct a counter example to RH.

(example) For $0 < \lambda < \frac{1}{2}$, let

$$L(s) = \zeta(s + \lambda)\zeta(s - \lambda)$$

then this function has two poles at $s = \lambda + 1, \ -\lambda + 1$ and satisfies all the other axioms. For Rieman Zeta function $\zeta(s)$, we have

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \Lambda(1 - s)$$

and satisfies the functional equation

$$\pi^{-\frac{s+\lambda}{2}} \Gamma\left(\frac{s + \lambda}{2}\right) \zeta(s + \lambda) = \pi^{-\frac{1-s-\lambda}{2}} \Gamma\left(\frac{1 - s - \lambda}{2}\right) \zeta(1 - s - \lambda)$$

also satisfies other axioms. (we can easily check)

For Axiom 3: Assume that f is a Maass form, which is the counter example for Ramanujan conjecture. (there should be no such form) Then for $L(s, f)$, $\Re(\mu) < 0$.

For Axiom 4: If $\theta > \frac{1}{2}$, then there exists $L \in \mathcal{S}$ which does not satisfy RH.

(example)

$$L(s) = (1-2^{-s})(1-2^{1-s})\zeta(s)$$

$$2^sL(S) = (2^s-1)(1-2^{1-s})\zeta(s),\ \text{and it is invariant under } s \mapsto 1-s$$

$$\Rightarrow\ 2^s\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s) = \Lambda_L(s) = \Lambda_L(1-s)$$

$$\log L(s) = \log(1-2^{-}s) + \log(1-2^{1-s}) + \sum_p\sum_l -\frac{\log p}{p^{ls}} =: \sum\frac{b(n)}{n^s}$$

since $\log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}$,

$$\log(1-2^{-s}) + \log(1-2^{1-s}) + \log(1-2^{-s})^{-1} = -\sum\frac{1}{k2^{ks}} - \sum\frac{2^{(1-s)k}}{k} + \sum_{k=1}^{\infty}\frac{1}{2^{ks}}$$

$$\Rightarrow\ \theta = 1$$

1.2 Degree: $0 \leqslant d \leqslant 1$

$$d := 2\sum_{j=1}^{k}\lambda_j$$

Conjecture. *The degree must be a non-negative integer.*

What are known for this conjecture?

Theorem 1.1. **(Conrey-Ghosh)***(in this paper, we can find the example above - for Axiom 4) There is no $L \in \mathcal{S}$ with degree d and $0 < d < 1$.*

Theorem 1.2. **(Kaczorowski-Perelli)** *The L-function, $L \in \mathcal{S}$ with degree $d = 1$ if and only if $L =$ L-function of an automorphic function for $GL(1)$, i.e., Hecke L-function.*

Theorem 1.3. **(K-P, to appear in the Annals)** *There are no $L \in \mathcal{S}$ with degree $1 < d < 2$.*

Conjecture.

$$d = 2\ (\text{or } d = n) \iff L \text{ on } GL(2)\ (\text{or } GL(n))$$

Is the degree well defined?

Proposition 1.4. *The degree is well-defined.*

Proof. (Stirling)

$$|\Gamma(s)| = \sqrt{2}\pi\,|t|^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi|t|}{2}}\left(1+\theta\left(\frac{1}{|t|}\right)\right)$$

for $s = \sigma + it$, σ-fixed and $|t| \to \infty$.

Let

$$G_L(s) = A^s\prod_{j=1}^{k}\Gamma(\lambda_j s + \mu_j).$$

Assume we have two different such $G_L^1(s)$ and $G_L^2(s)$. Then function equation

$$\Rightarrow\ \frac{G_L^1(s)}{G_L^2(s)} = \overline{\frac{G_L^1(1-\bar{s})}{G_L^2(1-\bar{s})}} = \frac{G_L^1(1-s)}{G_L^2(1-s)}$$

Assume that degree of G_L^2 is bigger. Then Stirling formula implies that

$$\lim_{|t|\to\infty}\left|\frac{G_L^1(s)}{G_L^2(s)}\right| \to 0 \text{ for } \Re(s) > 1.$$

Then $\frac{G_L^1(s)}{G_L^2(s)}$ is bounded. But by FE, $\frac{G_L^1(s)}{G_L^2(s)}$ is bounded for $\Re(s) < -1$. It is a contradiction. □

Remark (degree= 0). degree $d = 0 \iff k = 0$, so we have an empty product of Gamma functions.

Theorem 1.5. (Conrey-Ghosh)

$$d = 0 \iff L = \textit{constant}$$

Theorem 1.6. (Conrey-Ghosh) *Assume* $\Lambda_L^1(s)$ *and* $\Lambda_L^2(s)$ *are possible* $A^s \prod \Gamma(\lambda_j s + \mu_j)$ *for* $L \in \mathcal{S}$. *Then* $\Lambda_L^1(s) = c \cdot \Lambda_L^2(s)$.

Lecture 3: 2010-9-16

Prove the theorem by Corey and Ghosh.

Theorem 1.7. (Conrey-Ghosh, 1993) *The only element in* $\mathcal{S}$ *of degree* 0 *is the constant function.*

Proof. The proof uses axioms 1, 2, 3, 4 not 5. Without Axiom 4, it is possible to construct elements in the Selberg class of degree 0 which violate RH.

Lemma 1.8. *Assume* $L(s) = \sum \frac{a(n)}{n^s} \in \mathcal{S}$ *with* $d = 0$. *Then for any* $B \gg 1$, $a(n) \ll n^{-B}$ *for all* $n = 1, 2, 3, \ldots$.

Proof. $d = 0 \Rightarrow$ no Gamma in the FE.

$$\Rightarrow\ A^s L(s) = A^{1-s}\overline{L(1-\overline{s})}$$

For $x > 0$ define a function

$$H(x) = \sum_{n=1}^{\infty} a(n) e^{-2\pi n x}$$

By using the Mellin transform,

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^s \, \frac{dx}{x}$$

and using the inverse transform

$$e^{-x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) x^{-s} \, ds$$

$$\begin{aligned}
\Rightarrow\ H(x) &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(s)\Gamma(s)(2\pi x)^{-s} \, ds \\
&= \frac{P_{M-1}(\log x)}{x} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} L(-m)(2\pi x)^m
\end{aligned}$$

where $(s-1)^M L(s)$ is entire. By using FE,

$$L(-m) = A^{1+2m}\overline{L(1+m)}.$$

So $H(x)$ is well defined for all $x \in \mathbb{C}$ but it may have a log-singularity at $x = 1$.

$$\lim_{x \to 0} \left[\left(\frac{d}{dy} \right)^B \int_0^1 H(x+iy) e^{-2\pi i n y} \, dh \right] = \lim_{x \to 0} \left(a(n) e^{-2\pi n x} (-2\pi n)^B \right)$$

Then

$$a(n) \left| 2\pi n \right|^B = O(1), \ \forall \text{ integer } B.$$

□

Lemma 1.8 implies $L(s) =$ finite dirichlet series. Let's look at the functional equation.

$$\sum \frac{a(n)}{n^s} = \sum \overline{a(n)} \frac{A^{1-2s}}{n^{1-s}}$$

$$\sum \frac{a(n)}{n^s} = A \sum \frac{\overline{a(n)}}{n} \left(\frac{n}{A^2}\right)^s \;\Rightarrow\; A^2 \in \mathbb{Z}, \text{ and } n \mid A^2$$

$$\Rightarrow\; L(s) = \sum_{n \mid A^2} \frac{a(n)}{n^s}$$

Normalize by $a(1) = 1$. Assume that $A = p$.

$$L(s) = 1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} = \left[\frac{a(p^2)}{p^2} + \frac{\overline{a(p)}/p}{p^s} + \frac{1}{p^2 s}\right] p$$

$$\Rightarrow\; a(p^2) = p \;\Rightarrow\; \theta = \frac{1}{2}$$

and it is a contradiction to Axiom 4. ☐

Remark. **(Mellin Transform)** For a function

$$f : \mathbb{R} \to \mathbb{C}$$

define the Mellin transform

$$\tilde{f}(s) := \int_0^\infty f(x) x^s \, \frac{dx}{x}$$

when this integral is convergent for some $\Re(s) > \sigma$ and $s \in \mathbb{C}$. We have the inverse Mellin transform

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{f}(s) x^{-s} \, ds.$$

Theorem 1.9. (Conrey-Ghosh, Richert 1956) *There are no elements of degree* $0 < d < 1$ *in* $\mathcal{S}$.

Proof. We only use Axioms 1, 2 and 3.

Lemma 1.10. *Let* $L(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \in \mathcal{S}$ *then for any* $B \gg 1$ *we must have* $|a(n)| \ll \frac{1}{n^B}$.

Proof.

$$H(x) = \sum_1^\infty a(n) e^{-2\pi n x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(s) \Gamma(s) (2\pi x)^{-s} \, ds$$

$$H(x) = \frac{P_{M-1}(\log x)}{x} + \sum_{m=0}^\infty \frac{(-1)^m}{m!} L(m) (2\pi x)^m$$

$$A^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) L(s) = A^{1-s} \prod_1^r \Gamma(\lambda_j (1-s) + \mu_j) \overline{L(1 - \overline{s})}$$

$$L(-m) = A^{1+2m} \prod_1^r \frac{\Gamma(-\lambda_j (m+1) + \mu_j)}{\Gamma(\lambda_j m + \mu_j)} \overline{L(1+m)}$$

(little different when $m = 0$)

Stirlings

$$\Gamma(x) = x^{x - \frac{1}{2}} e^{-x} (1 + O(\frac{1}{x})) \text{ for } x \to \infty,\; x = x_0 + i\alpha,\; x_0 \to \infty, \alpha \text{ fixed}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$
$$\Gamma(-x) = \frac{\pi}{\sin(-\pi x)\Gamma(1+x)}$$

Claim:

$$\prod_1^r \left| \frac{\Gamma(\lambda_j(m+1)+\mu_j)}{\Gamma(-\lambda_j m+\mu_j)} \right| \ll m! \cdot m^{m(d-1)}$$

$$\Rightarrow\ H(x) = \frac{P(\log x)}{x} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \prod \frac{\Gamma(\lambda_j(m+1)+\mu_j)}{\Gamma(-\lambda_j m+\mu_j)} \overline{L(1+m)}(2\pi x)^m$$

For $0 < d < 1$, the sum converges, since

$$\ll \sum_{m=0}^{\infty} m^{m(d-1)}(2\pi x)^m.$$

(When $d \geqslant 1$, this argument doesn't work) Once we know that the series

$$H(x) = \frac{P_{M-1}(\log x)}{x} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} L(-m)(2\pi x)^m$$

converges absolutely for all complex $x \neq 1$ the same argument as before. Shows

$$|a(n)| \ll n^{-B}$$

since

$$\lim_{x \to 0} \left(\frac{d}{dx}\right)^B \int H(x+iy)e^{-2\pi i n y}\, dy = O(1)$$

☐

Now, we want to prove $L \in \mathcal{S}$ with $0 < d < 1$ can't exist. It is easy to get a contradiction because $a(n) \ll \frac{1}{n^B}$ implies $|L(s+it)| = O(1)$ for $\Re(s) > -B$ for any $B \gg 1$.

On the other hand,

$$L(-2+it) \text{ will blow up as } t \to \infty$$

by FE and Stirlings (cirtical use that $0 < d < 1$).

($L(-2+it) = \prod$ Gammas $\cdot\, L(3-it)$ and $L(3-it)$ bounded.)

☐

Go to the case when $d = 1$

Remark. $\zeta(s+i\alpha)$ with $\alpha \in \mathbb{R}$ is not in $\mathcal{S}$. But $L(s+i\alpha,\chi) \in \mathcal{S}$ where $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ and $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.

$$L(s+i\alpha,\chi) = \sum \frac{\chi(n)|n|^{-i\alpha}}{n^s}$$

The FE, (we need $\chi(n)|n|^{-i\alpha}$ is periodic in q)

$$\left(\frac{\pi}{\sqrt{q}}\right)^{\frac{s+i\alpha}{2}} \Gamma\left(\frac{s+i\alpha}{2}\right) L(s+i\alpha,\chi) = \left(\frac{\pi}{\sqrt{q}}\right)^{\frac{1-s-i\alpha}{2}} \Gamma\left(\frac{1-s-i\alpha}{2}\right) L(1-s-i\alpha,\overline{\chi})$$

Theorem 1.11. **(K-P)** *Assume that $L \in \mathcal{S}$ with $d = 1$. Then*

$$L(s) = \begin{cases} \zeta(s) \\ L(s+i\alpha,\chi) \end{cases}$$

where $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$, primitive and $\xi = \chi \cdot |\ |^{-i\alpha}$ should be periodic with integral period. (This corresponds to $GL(1,\mathbb{Q})$ L-functions)

Lecture 4: 2010-9-21

Reference: Sound, Degree one elements in the Selberg class, Expos. Math. (2005)

The proof in Sound is much simpler than in K-P.

Theorem 1.12. *The only elements in the Selberg class $\mathcal{S}$ of degree one are $\zeta(s)$ or $L(s+ia,\chi)$ with $a \in \mathbb{R}$ and primitive character $\chi : (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ of mod q.*

This theorem is first proved by K-P, and later by Sound, in simpler form.

Proof. Let $L \in \mathcal{S}$,

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad d = 1.$$

Definition 1.13. *Fix $\alpha \in \mathbb{R}$, $T \gg 1$.*

$$\mathcal{L}(\alpha, T) := \frac{1}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} L\left(\frac{1}{2} + it\right) \exp\left(it + \log\left(\frac{1}{2\pi e\alpha}\right) - \frac{i\pi}{4}\right)\, dt$$

We will prove the theorem by considering

$$\lim_{T\to\infty} \left(\frac{\mathcal{L}(\alpha, T)}{T^{1+ia}}\right)$$

for some $a \in \mathbb{R}$.

Lemma 1.14. *For all $t \in \mathbb{R}$ and any $X \gg 1$,*

$$L\left(\frac{1}{2} + it\right) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\frac{1}{2}+it}} e^{-\frac{n}{X}} + O\left((1+|t|)^{1+\epsilon}\right) X^{-1+\epsilon} + O\left(X^{\frac{1}{2}+\epsilon} e^{-|t|}\right)$$

Remark. We will apply this with $X = t^{5/4}$.

Proof. Consider

$$\begin{aligned} &\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L\left(\frac{1}{2} + it + w\right) X^w \Gamma(w)\, dw \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum \frac{a(n)}{n^{\frac{1}{2}+it}} \left(\frac{X}{n}\right)^w \Gamma(w)\, dw \\ &= \sum \frac{a(n)}{n} e^{-\frac{n}{X}}. \end{aligned}$$

(since Γ functions has pole at 0, $L(a+w)\Gamma(w)$ has residue $L(a)$ at $w = 0$)

We shift the line of integration from $\int_{2-i\infty}^{2+i\infty}$ to $\int_{-1+\epsilon-i\infty}^{-1+\epsilon+i\infty}$. We encounter poles at

- $w = \frac{1}{2} - it$ (possible pole if $L(1)$ has a pole)
- $w = 0$ (from $\Gamma(w)$)

$$\Rightarrow \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L\left(\frac{1}{2}+it+w\right) X^{w}\Gamma(w)\,dw = \underbrace{O\left(X^{\frac{1}{2}+\epsilon}e^{-|t|}\right)}_{\text{contribution from the possible pole if } L(1) \text{ has a pole}}$$

$$+ \underbrace{L\left(\frac{1}{2}+it\right)}_{\text{contributions from } \Gamma} + \int_{-1+\epsilon+i\infty}^{-1+\epsilon-\infty} L\left(\frac{1}{2}+it+w\right) X^{w}\Gamma(w)\,dw$$

It remains to show that

$$\frac{1}{2\pi i}\int_{-1+\epsilon-i\infty}^{1+\epsilon+i\infty} L\left(\frac{1}{2}+it+w\right) X^{-w}\Gamma(w)\,dw = O\left((1+|t|)^{1+\epsilon}X^{-1+\epsilon}\right)$$

Use the functional equation,

$$L\left(\frac{1}{2}+it+w\right) = A^{-2it-2w}\prod\frac{\Gamma\left(\lambda_j\left(\frac{1}{2}-it-2\right)+\overline{\mu_j}\right)}{\Gamma\left(\lambda_j\left(\frac{1}{2}+it+w\right)+\mu_j\right)}\cdot\overline{L\left(\frac{1}{2}-it-\overline{w}\right)}$$

Stirling:

$$|\Gamma(\sigma+it)| \cong \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}$$

as $|t|\to\infty$, σ fixed.

Since $L\left(\frac{1}{2}-it-\overline{w}\right)=O(1)$, we get the lemma. ☐

In C-G, they use $\sum\frac{a(n)}{n^{\frac{1}{2}}}e^{-nX}$ with $a(n)\to 0$ as $n\to\infty$. Sound use $\sum\frac{a(n)}{n^{\frac{1}{2}}}e^{-\frac{n}{X}}$.

Choose $X=T^{\frac{5}{4}}$ and insert Lemma into the definition of $\mathcal{L}(\alpha,T)$.

$$\begin{aligned}\mathcal{L}(\alpha,T) &= \frac{1}{\sqrt{\alpha}}\int_{\alpha T}^{2\alpha T} L\left(\frac{1}{2}+it\right)\exp\left(it+\log\frac{1}{2\pi e\alpha}-\frac{i\pi}{4}\right)\,dt\\ &= \frac{\mu}{\sqrt{\alpha}}\int_{\alpha T}^{2\alpha T}\overline{L\left(\frac{1}{2}+it\right)}A^{-2it}\pi\frac{\Gamma(\,)}{\Gamma(\,)}\exp\left(it+\log\frac{1}{2\pi e\alpha}-\frac{i\pi}{4}\right)\,dt\end{aligned}$$

with $|\mu|=1$.

Stirling degree $=1$

$$\pi\frac{\Gamma(\,)}{\Gamma(\,)} = e^{-it\log\frac{1}{2e}+\frac{i\pi}{4}+ib}t^{ia}c^{-it}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

for some $a,b,c\in\mathbb{R}$.

$$\begin{aligned}\Rightarrow\mathcal{L}(\alpha,T) &= \frac{\mu}{\sqrt{\alpha}}e^{ib}\int_{\alpha T}^{2\alpha T}\overline{L\left(\frac{1}{2}+it\right)}\left(\pi cA^{2}\alpha\right)^{-it}t^{ia}\left(1+O\left(\frac{1}{|t|}\right)\right)\,dt\\ &= \frac{\mu e^{ib}}{\sqrt{\alpha}}\sum_{m=1}^{\infty}\frac{\overline{a(m)}}{\sqrt{m}}e^{-\frac{m}{X}}\int_{\alpha T}^{2\alpha T}\left(\frac{m}{\pi cA^{2}\alpha}\right)^{it}t^{ia}\,dt+O\left(T^{\frac{2}{3}+\epsilon}\right)\end{aligned}$$

There are 2 possibilities: either $\frac{m}{\pi cA^2\alpha}=1$ or $\frac{m}{\pi cA^2\alpha}\neq 1$. If $\frac{m}{\pi cA^2\alpha}=1$, the integral becomes

$$\int_{\alpha T}^{2\alpha T}1^{it}t^{ia}\,dt=\frac{(2\alpha T)^{1+ia}-(i\alpha T)^{1+ia}}{1+ia}.$$

If $\frac{m}{\pi c A^2 \alpha} \neq 1$, then

$$\int_{\alpha T}^{2\alpha T} \left(\frac{m}{\pi^2 c A^2 \alpha}\right)^{it} t^{ia}\, dt = O\left(\frac{1}{\log\left(\frac{m}{\pi^2 c A^2 \alpha}\right)}\right)$$

Let

$$\mathcal{L}(\alpha) = \lim_{T \to \infty} \frac{\mathcal{L}(\alpha, T)}{T^{1+ia}}.$$

Only $\frac{m}{\pi^2 c A^2 \alpha} = 1$ summaries the limit above.

$$\Rightarrow\ \mathcal{L}(\alpha) = \mu e^{ib} \delta(\pi c A^2 \alpha) \frac{\overline{a(\pi c A^2 \alpha)\alpha^{ia}}}{\sqrt{\pi c} A} \frac{2^{1+ia} - 1}{1 + ia}$$

where $\delta(b) = \left\{ \begin{array}{ll} 1, & b \in \mathbb{Z},\ b \geqslant 0 \\ 0, & b \notin \mathbb{Z} \end{array} \right.$.

Next Sound evaluates $\mathcal{L}(\alpha, T)$ and $\mathcal{L}(\alpha)$ by another method and proves

$$\mathcal{L}(\alpha + 1) = \mathcal{L}(\alpha)$$

i.e., it is periodic.

$$\Rightarrow\ \pi c A^2 = q \in \mathbb{Z}$$

$$\Rightarrow\ \overline{a(n)} n^{ia} \text{ is periodic}$$

because $\delta\left(\pi c A^2 \alpha\right) \alpha^{ia} = \delta(\pi c A^2 \alpha)\alpha^{ia} = \delta(\pi c A^2 (\alpha + 1))(\alpha + 1)^{ia}$. We also know $\overline{a(n)} n^{ia}$ is multiplicative. So $L(s) = L(s + ia, \chi)$. □

Notation For functions $F(t), G(t) : \mathbb{R} \to \mathbb{C}$,

$$\begin{aligned} F(t) = O(G(t)) &\iff \exists c > 0,\ |F(t)| \leqslant c \cdot |G(t)|, \forall t \in \mathbb{R} \\ &\iff F(t) \ll G(t) \end{aligned}$$

Lecture 5: 2010-9-23

1.3 Degree: Kaczorowski and Perelli's result (no elements for $1 < d < 2$)

There are no elements in the Selberg class of degree $1 < d < 2$.

Nonlinear Twist $L(s) = \sum_1^\infty \frac{a(n)}{n^s} \in \mathcal{S}$ with degree d. Let $\alpha \in \mathbb{R}$.

$$L(s, \alpha) := \sum \frac{a(n)}{n^s} e^{-2\pi i n^{\frac{1}{d}} \alpha}$$

When $d = 1$ this is a linear twist

$$\sum \frac{a(n)}{n^s} e^{-2\pi i n \alpha}$$

and this type has been studied before. But using $\frac{1}{d}$ is a new idea.

By Mellin transform,

$$L(s, \alpha) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L\left(s + \frac{w}{d}\right) \Gamma(w) (2\pi i \alpha)^w\, dw.$$

Define

$$n_\alpha := \left(\frac{2\pi\alpha}{d}\right)^d A^2 \prod_1^r \gamma_j^{2\gamma_j}$$

where

$$\Lambda_L(s) = A^s \prod_1^r \Gamma(\gamma_j s + \mu_j) L(s) = \mu \overline{\Lambda_L(1-\overline{s})}.$$

Let

$$\widetilde{a}(n_\alpha) := \begin{cases} 0, & \text{if } n_\alpha \notin \mathbb{N}, \\ a(n_\alpha), & \text{if } n_\alpha \in \mathbb{N} \end{cases}$$

Theorem 1.15. (K-P) *$L(s,\alpha)$ has meromorphic continuation to all $s \in \mathbb{C}$ with simple pole at*

$$s = \frac{d+1}{2d} - \frac{i \cdot \Im\left(\sum(\mu_j - \frac{1}{2})\right)}{d}.$$

Remark. If $0 < d < 1$ then pole at $\frac{d+1}{2d} > 1$ and it contradicts the fact that $\sum \frac{|a(n)|}{n^\sigma}$ converges absolutely for $\sigma > 1$. So this gives the new proof for nonexistence for the L-function with degree $0 < d < 1$.

1.4 Degree 2 elements of $\mathcal{S}$

Automorphic forms Hecke: $\mathbb{H} = \{z = x + iy \mid x \in \mathbb{R},\ y > 0\}$

$$f : \mathbb{H} \to \mathbb{C}, \quad \text{holomorphic}$$

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ (i.e., $ad - bc = 1$ and $N|c$) where χ is a character mod N. Modular forms of weight k. Let $f\left|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right.(z) := f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(cz+d)^{-k}$. Then

$$f(z+1) = f(z)$$

so they are periodic function, so have a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}$$

since they are holomorphic. If $a(0) = 0$ then they are called cusp forms. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}.$$

$$\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}^{-1} = \Gamma_0(N)$$

so it is a normalizer. Claim

$$f_N(z) := f\Big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}(z) \text{ is modular.}$$

$$f_N(z) := f\left(\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} z\right) \text{ is modular.}$$

Because

$$\begin{aligned} f_N\left(\frac{az+b}{cz+d}\right) &= f_N\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = f\left(\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) \\ &= f\left(\underbrace{\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}^{-1}}_{\in\Gamma_0(N)} \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} z\right). \end{aligned}$$

Define $w_N f := f_N$. Then $w_N^2 f = f$. (involution) Let $\mathcal{H}_{N,k}$ be the holomorphic automorphic forms. Every $f \in \mathcal{H}_{N,k}$

$$f = f^+ + f^-$$

where $w_N f^+ = f^+$ and $w_N f^- = -f^-$. Assume $w_N f = f$.

$$\begin{aligned} \int_0^\infty f(iy) y^s \frac{dy}{y} &= \sum_{n=1}^\infty a(n) \int_0^\infty e^{-2\pi n y} y^{s+k} \frac{dy}{y} \\ &= \sum \frac{a(n)}{n^{s+k}} (2\pi)^{-s+k} \Gamma(s+k) \end{aligned}$$

$$f\left(-\frac{1}{Nz}\right) = \left(-\frac{1}{Nz}\right)^k f(z)$$

$$\begin{aligned} \int_0^\infty f(iy) y^s \frac{dy}{y} &= \left[\int_0^a + \int_a^\infty \right] \\ &= i^{-k} \int_0^a f\left(\frac{i}{Ny}\right) (Ny)^k (Ny)^{-s-k} \frac{dy}{y} + \int_a^\infty f(iy) y^{s+k} \frac{dy}{y} \\ &= i^{-k} \int_\infty^{\frac{1}{Na}} f(iy) y^{-s} \frac{dy}{y} + \int_a^\infty f(iy) y^{s+k} \frac{dy}{y} \end{aligned}$$

and take $a = \frac{1}{\sqrt{N}}$ and functional equation $s \to s-k$.

We have to use "involution" to get the functional equation.

There is no reason to assume that a modular form $f(z)$ is holomorphic.

Maass forms Maass (1940) developed a theory of non-holomorphic modular forms.

Maass forms of weight 0.

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d) f(z)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. (no holomorphic forms satisfying this condition.)

(example) Eisenstein series

$$\begin{aligned} E(z,s,\chi) :=& \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\chi(d)^{-1} y^s}{|cz+d|^{2s}} \\ -& \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\Im(\gamma z))^s \tilde{\chi}(\gamma)^{-1} \end{aligned}$$

where

$$\Gamma_\infty \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} | m \in \mathbb{Z} \right\}.$$

Then

$$E(\alpha z, s, \chi) = \sum_\gamma (\Im(\gamma \alpha z))^s \tilde{\chi}(\gamma\alpha)^{-1} \tilde{\chi}(\alpha) = \tilde{\chi}(\alpha) E(z,s,\chi).$$

But they are not cusp forms. - Is there a cusp form?

We're looking for invariant differential operators. Want a polynomial in $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ which is invariant under the transform $z \mapsto \frac{az+b}{cz+d}$.

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = -4(\Im z)^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$$

where $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Let

$$\mathcal{M}_{N,\chi,\lambda} := \left\{ f : \mathbb{H} \to \mathbb{C},\ \text{smooth} \;\middle|\; \begin{array}{c} f\left(\frac{az+b}{cz+d}\right) = \chi(d) f(z),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \\ \text{and } \Delta f = \lambda f, \ \ \text{cuspidal} \end{array} \right\}$$

Fourier expansion: $f \in \mathcal{M}_{N,\chi,\lambda}$, so $f(z+1) = f(z)$. Then $f(z)$ has a Fourier expansion.

$$f(z) = \sum_{n \neq 0} a(n,y) e^{2\pi i n x}$$

since f satisfies the differential equation $\Delta f = \lambda f$, this implies that $\Delta a(n,y)e^{2\pi i n x} = \lambda a(n,y) e^{2\pi i n x}$. i.e.,

$$\lambda a(n,y) = -y^2 \left(4\pi^2 n^2 a(n,y) + a''(n,y)\right) e^{2\pi i n x}.$$

This is a differential operator for Bessel function. The solution of the differential equation is the K-Bessel function.

$$K_\nu(y) = \frac{1}{2} \int_0^\infty e^{-\frac{y}{2}(u+u^{-1})} u^\nu \, \frac{du}{u}$$

Then

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\lambda - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}.$$

We have the L-function associated to f:

$$\begin{aligned} \Lambda_f(s) &= \int_0^\infty f(iy) y^{s-\frac{1}{2}} \, \frac{dy}{y} = \sum_{n \neq 0} a(n) \int_0^\infty \sqrt{y} K_{\lambda-\frac{1}{2}}(2\pi|n|y) y^{s-\frac{1}{2}} \, \frac{dy}{y} \\ &= \sum_{n=1}^\infty \frac{a(n)}{n^s} \Gamma\left(\frac{s+\lambda-\frac{1}{2}}{2}\right) \Gamma\left(\frac{s-\lambda+\frac{1}{2}}{2}\right) 2\pi^{-s} \end{aligned}$$

For any $f \in \mathcal{M}_{N,k,\lambda}$, then $L_f(s) \in \mathcal{S}$ with degree 2. We also have $L(s+ia_1,\chi_1)L(s+ia_2,\chi_2) \in \mathcal{S}$. This is imprimitive.

Conj If $L \in \mathcal{S}$ with degree $d = 2$ and L is primitive (cannot be factored into lower degree L-function in $\mathcal{S}$)then it must be an automorphic L-function of Hecke-Mass type (includes Eisenstein series).

Lecture 6: 2010-9-28

Conjecture. **(D. Ramakrishnan)** *If $L(s) \in \mathcal{S}$ and $L(s)$ has a pole (of order m) at $s = 1$ then*

$$\frac{L(s)}{\zeta(s)^m} \in \mathcal{S}$$

and it is holomorphic everywhere.

2 Automorphic forms

2.1 Automorphic forms

Let $n \geqslant 1$ be an integer.

- $GL(n, \mathbb{R})$
- $GL(n, \mathbb{Z})$ acts on $GL(n, \mathbb{R})$ by left matrix multiplication
- $O(n, \mathbb{R})$ = maximal compact subgroup of $GL(n, \mathbb{R})$

  $$= \left\{ h \in GL(n, \mathbb{R}) \;\middle|\; h \cdot^t h = I_n \right\}$$

  For example for $n = 2$,

  $$O(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; 0 \leqslant \theta < 2\pi \right\}$$

- $\mathbb{H}^n = GL(n, \mathbb{R})/O(n, \mathbb{R}) \cdot \mathbb{R}^\times$ = Generalized upper half plane. For $z \in \mathbb{H}^n$, then $z = xy$ where

  $$x = \begin{pmatrix} 1 & x_{1,2} & \cdots & x_{1,n} \\ & 1 & & x_{2,n} \\ & & \ddots & \vdots \\ & & & x_{n-1,n} \\ & & & 1 \end{pmatrix}, \; x_{i,j} \in \mathbb{R}$$

  $$y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}, \; y_i > 0$$

  For $n = 2$,

  $$\mathbb{H}^2 \cong \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}, \; y > 0 \right\} = \text{classical upper half-plane}$$

Invariant differential operators For $z = \begin{pmatrix} 1 & & x_{i,j} \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathbb{H}^n$,

$$D_z := \text{ polynomial } \left(\frac{\partial}{\partial x_{i,j}}, \frac{\partial}{\partial y_j} \right) \text{ such that } D_{\gamma z} = D_z, \; \forall \gamma \in GL(n, \mathbb{R})$$

Let $\mathcal{D}^n$ be the space of invariant differential operators.

For $n = 2$, $D = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

Definition 2.1. *An automorphic forms for $SL(n, \mathbb{Z})$ is a smooth function $f : \mathbb{H}^n \to \mathbb{C}$ satisfying*

(1) $f(\gamma z) = f(z), \; \forall \gamma \in SL(n, \mathbb{Z}), \; z \in \mathbb{H}^n$

(2) $Df = \lambda_D f, \; \forall D \in \mathcal{D}^n, \; \lambda_D \in \mathbb{C}$ *(This λ_D is a Harish-Chandra's character)*

(3) f has moderate growth.

For $n = 2$, $Df = \nu(1-\nu)f$ for some $\nu \in \mathbb{C}$. Let f be a cusp form for $SL(2, \mathbb{Z})$.

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{n \neq 0} \frac{a(n)}{\sqrt{|n|}} \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}$$

where

$$K_\nu(y) = \frac{1}{2} \int_0^\infty e^{-\frac{y}{2}(u + \frac{1}{u})} u^\nu \frac{du}{u}.$$

This f is called a Maass cusp form. We have associated L-function

$$L_f(s) := \sum_{n=1}^\infty \frac{a(n)}{n^s}$$

$$\begin{aligned} \int_0^\infty f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) y^s \frac{dy}{y} &= \sum_{n \neq 0} a(n) |n|^{-\frac{1}{2}} \int_0^\infty K_{\nu - \frac{1}{2}}(2\pi |n| y) y^{s + \frac{1}{2}} \frac{dy}{y} \\ &= (2\pi)^{-s - \frac{1}{2}} \sum_{n \neq 0} \frac{a(n)}{|n|^s} \underbrace{\int_0^\infty K_{\nu - \frac{1}{2}}(y) y^{s + \frac{1}{2}} \frac{dy}{y}}_{= 2^{-\frac{3}{2} + s} \Gamma\left(\frac{1 + s - \nu}{2}\right) \Gamma\left(\frac{s + \nu}{2}\right)} \end{aligned}$$

Definition 2.2.

$$T_{-1} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) := f\left(\begin{pmatrix} y & -x \\ 0 & 1 \end{pmatrix}\right)$$

$$T_{-1} f = \begin{cases} f, & \textit{even} \Rightarrow a(-n) = a(n) \ (n \neq 0) \\ -f, & \textit{odd} \Rightarrow a(-n) = -a(n) \end{cases}$$

So if f is even,

$$L_f(s) = (2\pi)^{-s - \frac{1}{2}} L_f(s)$$

If f is odd work with

$$\frac{\partial}{\partial x} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right).$$

The Functional equation

$$\Lambda_f(s) = \pi^{-s} \Gamma\left(\frac{s + \epsilon - \frac{1}{2} + \nu}{2}\right) \Gamma\left(\frac{s + \epsilon + \frac{1}{2} - \nu}{2}\right) L_f(s) = \Lambda_f(1 - s) \tag{2.1}$$

where $\epsilon = 1$ if f is odd and $\epsilon = 0$ if f is even. This $L_f \in \mathcal{S}$ with degree $d = 2$.

2.2 Converse theorem (Hecke-Maass)

Very roughly, it says that if an L-function satisfies enough functional equations then it must be the L-function of an automorphic form.

Taking inverse Mellin transform to (2.1), we get the Bessel-function expansion, since the inverse Mellin transform of Gamma functions becomes the Bessel function. Then

$$(2.1) \Rightarrow f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 1/y & 0 \\ 0 & 1 \end{pmatrix}\right) \text{ (inverse Mellin transform)}$$

Moreover, we get

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} y & x + 1 \\ 0 & 1 \end{pmatrix}\right)$$

for free, because of the expansion.

$$SL(2,\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z \mapsto \begin{pmatrix} \frac{y}{|z|^2} & -\frac{x}{|z|^2} \\ 0 & 1 \end{pmatrix}$$

To prove converse theorem, we need

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} \\ 0 & 1 \end{pmatrix}\right)$$

Theorem 2.3. (Converse Theorem for $SL(2,\mathbb{Z})$) *Assume $L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ satisfies (2.1). Then define $a(-n) = a(n)$ and set $f(z) := \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$. Then f is an even Maass form for $SL(2,\mathbb{Z})$.*

Proof. We use Mellin inversion

$$h : \mathbb{R} \to \mathbb{C},\ \widetilde{h}(s) := \int_0^{\infty} h(x)x^s \frac{dx}{x},\ h(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widetilde{h}(s)x^{-s}\, ds$$

We apply Mellin inversion to Λ_f. Define

$$\begin{aligned} F(y) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Lambda_f(s)y^{-s}\, ds = \sum_{n=1}^{\infty} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi ny) \\ &= \frac{1}{2}\sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y) \end{aligned}$$

By functional equation $F(y) = F\left(\frac{1}{y}\right)$.

Let

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) := \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}.$$

We know

$$\Delta f = \nu(1-\nu)f$$

for $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. This implies that f is real analytic, so

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_m b_m(y)x^m.$$

Again, $\Delta f = \nu(1-\nu)f$ implies that $b_m(y)$ satisfies recurrence relations.i.e.,

$$\nu(1-\nu)\sum b_m(y)x^m = -y^2\sum\left[b_m(y)m(m-1)x^{m-2} + b_m''(y)x^m\right]$$

$$\Rightarrow\ \nu(1-\nu)b_m(y) = -y^2\left[b_{m+2}(y)(m+2)(m+1) + b_m''(y)\right]$$

Lemma 2.4. *Assume $F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$ satisfies $\Delta F = \nu(1-\nu)F$ and $F\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$, $\frac{\partial}{\partial x}F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\Big|_{x=0} = 0$. Then $F \equiv 0$.*

Proof.

$$\nu(1-\nu)b_m(y) = -y^2\left[b_{m+2}(y)(m+2)(m+1) + b''_m(y)\right]$$

So by using $F\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$, we have

$$b_0(y) = 0 \Rightarrow\ b''_0(y) = 0 \Rightarrow\ b_2(y) = 0,\ b_4(y) = 0, \dots, b_{2m}(y) = 0$$

By using $\frac{\partial}{\partial x} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\Big|_{x=0} = 0$, we get

$$b_{2m+1}(y) = 0.$$

□

Choose

$$F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) - f\left(\begin{pmatrix} \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} \\ 0 & 1 \end{pmatrix}\right)$$

We know

$$F\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

by the functional equation. It is easy to show that $\frac{\partial}{\partial x} F\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\Big|_{x=0} = 0$ because $a(n) = a(-n)$. □

Lecture 7: 2010-9-30

2.3 Ramanujan Conjecture

Let $\tau(1) = 1$ and $\Delta : \mathbb{H}^2 \to \mathbb{C}$ be a modular form of weight 12 for $SL(2,\mathbb{Z})$, i.e., for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$,

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z).$$

Conjecture. **(1)** $\tau(p) \leqslant 2p^{\frac{11}{2}}$ *(First proved by Deligne)*

Conjecture. **(2)** $\tau(mn) = \tau(m)\tau(n),$ *if* $(m,n) = 1$ *(First proved by Model)*

Conjecture 1 and 2 implies that $\tau(n) \leqslant \sigma(n)n^{\frac{11}{2}}$ for any integer $n \geqslant 1$, where $\sigma(n) = \sum_{1 \leqslant d|n} 1$.

Ramanujan conjecture for Maass forms Let

$$f(z) = \sum_{n \neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$$

be a Maass form (with $\Delta f = \nu(1-\nu)f$) then

$$|a(n)| \ll n^{\epsilon}$$

Still unsolved. World record (Kim-Sarnak) $a(n) \ll n^{\frac{7}{64}}$

Let's obtain a trivial bound for $a(n)$ using an idea of Hecke.
We assume that

$$||f||_2^2 = \int\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}^2} |f(z)|^2 \ \frac{dxdy}{y^2} = 1$$

For any $\epsilon > 0$, there exists unique $\gamma \in SL(2,\mathbb{Z})$ and $z \in SL(2,\mathbb{Z})\backslash\mathbb{H}^2$, such that

$$\gamma z = x_0 + i\epsilon$$

and

$$f(\gamma z) = f(x_0 + i\epsilon) = f(z)$$

So

$$1 \gg \int_0^1 f(x+iy)e^{-2\pi inx}\ dx = a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y).$$

Choose $y = \frac{1}{n}$, then

$$\frac{|a(n)|}{\sqrt{n}}K_{\nu-\frac{1}{2}}(2\pi) \ll 1 \Rightarrow \ |a(n)| \ll n^{\frac{1}{2}}.$$

2.4 Finiteness of eigenspaces

Theorem 2.5. **(Maass)** *Fix* $\lambda = \nu(1-\nu) \in \mathbb{R}$ *and* $\nu = \frac{1}{2} + ir$. *Let*

$$\mathcal{M}_\lambda := \left\{ f : \mathbb{H}^2 \to \mathbb{C} \ \middle| \ \begin{array}{c} f(\gamma z) = f(z),\ \Delta f = \lambda f \\ f = \ \textit{Maass cusp form}\ \gamma \in \Gamma \subset SL(2,\mathbb{Z}) \end{array} \right\}$$

Then

$$\dim(\mathcal{M}_\lambda) \ll 1.$$

$$\iff \ \textit{Given}\ \lambda \exists c(\lambda) \geqslant 0 \ \textit{and}\ \dim(\mathcal{M}_\lambda) \leqslant c(\lambda)$$

Conjecture. $\dim(\mathcal{M}_\lambda) \leqslant 1$ *(there is a program by Sarnak)*

Proof. Assume that $\dim(\mathcal{M}_\lambda) = \infty$. Let $f_1,\ f_2,\ f_3 \ldots \in \mathcal{M}_\lambda$ and

$$f_i(z) = \sum_{n\neq 0} a_i(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$$

Then

$$a_2(1)f_1(z) - a_1(1)f_2(z) = \sum_{n\neq 0,1} b(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$$

Lemma 2.6. *Let* $F \in \mathcal{M}_\lambda$*, Assume that*

$$F(z) = \sum_{|n|>N} b(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi i|n|y)e^{2\pi inx}$$

Then if N *is sufficiently large then we must have* $F \equiv 0$.

Proof of Lemma. Normalize F so that

$$||F||_2^2 = \int\int_{\Gamma\backslash\mathbb{H}^n} |F(z)|^2 \ \frac{dxdy}{y^2} = 1.$$

We already proved that $|b(n)| \ll \sqrt{n}$.

Let's assume that $\Gamma = SL(2,\mathbb{Z})$. Let D be the fundamental domain and $D^* = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) D$. So

$$\begin{aligned} 1 &= \int\int_D |F(z)|^2 \frac{dxdy}{y^2} \\ &\leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y \geqslant \frac{\sqrt{3}}{2}} |F(z)|^2 \ \frac{dxdy}{y^2} \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y \geqslant \frac{\sqrt{3}}{2}} \left| \sum_{|n|>N} b(n)\sqrt{y} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x} \right|^2 \ \frac{dxdy}{y^2} \\ &= \sum_{|n|>N} \int_{\frac{\sqrt{3}}{2}}^{\infty} |b(n)|^2 \left| K_{\nu-\frac{1}{2}}(2\pi|n|y) \right|^2 \ \frac{dy}{y} \end{aligned}$$

(we actually know that $b(n) \in \mathbb{R}$) It is known that $\sqrt{y}K_{\nu-\frac{1}{2}}(y) \approx c \cdot e^{-2\pi y} \ \ll e^{-y}$, as $y \to \infty$. Because

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}(u+\frac{1}{u})} u^\nu \frac{du}{u}.$$

So

$$\begin{aligned} 1 &\ll \sum_{|n|>N} |n| \cdot \int_{\frac{\sqrt{3}}{2}}^{\infty} e^{-2\pi|n|y} \ \frac{dy}{y} \\ &\ll \sum_{|n|>N} |n| e^{-|n|} < 1 \end{aligned}$$

for sufficiently large N. So it is a contradiction. □

□

When do Maass forms exist? In Maass's paper, he constructed an example of Maass forms for some $\Gamma \subset SL(2,\mathbb{Z})$, using Hecke L-function.

The first person to prove that there are infinitely many Maass forms for $SL(2,\mathbb{Z})$ is Selberg, using the Trace formula.

Question. How many Maass forms are there with eigenvalue $\leqslant B$? i.e.,

$$\sum_{\lambda \leqslant B} \dim(\mathcal{M}_\lambda) = X(B)$$

The trace formula gives an asymptotic formula for $X(B)$.

Nonexistence of Maass forms Let $\Gamma \subset SL(2,\mathbb{R})$ and Γ is a discrete subgroup. It is possible to deform the subgroup (Phillips-Sarnak theory of spectral deformation theory).

Let F be a Maass cusp form for $\Gamma \subset SL(2,\mathbb{R})$. The eigenvalues for Δ are always real. So $\lambda = \nu(1-\nu) = -\frac{1}{4} + r^2$ where $\nu = \frac{1}{2} + ir,\ r \in \mathbb{R}$. When we deform the group Γ then we deform F, so deform λ, and that moves to the outside of $\Re(\nu) = \frac{1}{2}$, then they are not exist.

Conjecture. *For a generic group Γ there are no Maass forms.*

Conjecture. *Maass forms exists only for Arithmetic groups.*

Lecture 8: 2010-10-5

3 Selberg Trace Formula

3.1 Introduction to trace formula

Let H be a Hilbert space and $L : H \to H$ be a linear operator.

- The linear operator L is a compact linear operator if L maps bounded sets to relatively compact sets (i.e., closure is compact).
- Bounded operator L
- Unbounded operator L

Functional analysis Hilbert space has a norm on it, inner product $\langle , \rangle : H \times H \to C$, which is positive definite, we say L is self-adjoint if and only if

$$\langle Lh_1, h_2 \rangle = \langle h_1, Lh_2 \rangle$$

for any $h_1, h_2 \in H$.

We're interested in eigenfunctions and eigenvalues. If there exists $h \in H$, such that $Lh = \lambda h$ for some $\lambda \in \mathbb{C}$, then we say h is the eigenfunction (eigenvector) and λ is the eigenvalue.

Theorem 3.1. (Spectral theorem for compact operators) *For every self-adjoint compact operator L on a real or complex Hilbert space H then there exists an orthonormal basis of eigenvectors of L, say $\mu_1, \mu_2, \ldots \in H$ with $L\mu_i = \lambda_i \mu_i$, $\lambda_i \neq 0$, discrete, and every $h \in H$ can be written*

$$h = \sum_{i=1}^{\infty} \langle h, \mu_i \rangle \, \mu_i \qquad \text{(countable sum).}$$

For a bounded operators we still have a spectral theorem but there could be a continuous spectrum of λ

$$L\mu_\lambda = \lambda \mu_\lambda$$

Spectral:

$$h = \int \langle h, \mu_\lambda \rangle \underbrace{d^* \lambda}_{\substack{\text{spectral} \\ \text{measure}}}$$

The Laplace operator is an unbounded operator.

3.2 Trace of an integral operator L

Choose some test function $f : \mathbb{R} \to \mathbb{C}$. Then a trace of operator L

$$\mathrm{Trace}_f(L) := \int_{\substack{\text{Spectrum} \\ \text{of } L}} f(\lambda) \, d^* \lambda.$$

For compact open

$$\mathrm{Trace}_f(L) := \sum_{i=1}^{\infty} f(\lambda_i).$$

Simple example Let $\mathbb{Z}$ = additive group of integers acting on $\mathbb{R}$ by addition.

- $\mathbb{Z}\backslash\mathbb{R} = S^1 = (0,1]$
- $H = L^2(\mathbb{Z}\backslash\mathbb{R})$ =Hilbert space.
  $\Rightarrow h \in H$ is a function $h : \mathbb{R} \to \mathbb{C}$ with $h(x+n) = h(x)$ for any $n \in \mathbb{Z}$ and $\int_0^1 |h(x)|^2 \, dx \ll 1$.
- $L = \frac{d^2}{dx^2}$ then
  - eigenfunction: $e_m(x) := e^{2\pi imx}, \ m = 0, \pm 1, \pm 2, \dots$
  - eigenvalue: $-4\pi^2 m^2$
  - Fourier theorem: for any $h \in H$,

$$h(x) = \sum_{n\in\mathbb{Z}} \langle h, e_m \rangle \, e_m(x)$$

Then the spectral theorem is the Fourier theorem. For a test function $f : \mathbb{R} \to \mathbb{C}$, we have

$$\mathrm{Trace}_f(L) = \sum_{n\in\mathbb{Z}} f(-4\pi^2 n^2) = \sum_{n\in\mathbb{Z}} f_0(n)$$

with re-normalizing f. By Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f_0(n) = \sum_{n\in\mathbb{Z}} \widehat{f}_0(n).$$

In the representation theory point of view, we have the Frobenius reciprocity law.

3.3 Selberg spectral decomposition

We are interested in working out the trace formula for

$$L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$$

where $\mathbb{H}^2 = \{x + iy \mid x \in \mathbb{R}, \ y > 0\}$.

Selberg Spectral decomposition

$$L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right) = \mathbb{C} \oplus L_{\text{cusp}} \oplus L_{\text{Eisenstein}}$$

where

- $$L_{\text{cusp}} = \left\langle \left\{ \eta_i, \ i = 1, 2, 3, \dots \;\middle|\; \begin{array}{c} \Delta\eta_i = \lambda_i\eta_i, \ \eta_i\left(\frac{az+b}{cz+d}\right) = \eta_i(z), \\ \eta_i = \text{Maass cusp form} \end{array} \right\} \right\rangle.$$

  We will prove that $\lambda_i \to \infty$ as $i \to \infty$ and form a discrete countable set by constructing a Hilbert-Schumidt operator ($\approx$ spectral theorem for compact operator).

- $$\mathrm{Vol}\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right) = \int\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}^2} \frac{dx\ dy}{y^2} = \frac{3}{\pi}.$$

- $$L_{\text{Eisenstein}} \approx \mathbb{C}E(z,s)$$

  where
  $$E(z,s) = \sum_{\substack{c,d\in\mathbb{Z},\\(c,d)=1}} \frac{y^2}{|cz+d|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \Im(\gamma z)^s \notin L^2$$
  and $\Delta E(z,s) = s(1-s)E(z,s)$.

For $H = L^2(\mathbb{R})$ and $L = \frac{d^2}{dx^2}$, $\Delta e^{i\lambda x} = -\lambda^2 e^{i\lambda x}$ for $\lambda \in \mathbb{R}$. Spectral theorem: $f \in L^2(\mathbb{R})$,
$$f(x) = \int_{\mathbb{R}} \widehat{f}(u) e^{-2\pi i u x}\, du.$$
But $e^{i\lambda x} \notin L^2(\mathbb{R})$, since $\int_{\mathbb{R}} \left|e^{i\lambda x}\right|^2 dx = \infty$.

Theorem 3.2. **(Sepctral theorem for $L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$)** *Let $f \in L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$. Then there exists η_i $(i = 0,1,2,\ldots)$ where $\eta_0 = \sqrt{\frac{\pi}{3}}$ = constant function and $\eta_i(z)$ for $i = 1,2,\ldots$ are Maass cusp forms satisfying*
$$\langle \eta_i, \eta_j\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}.$$
Here
$$\langle F, G\rangle = \iint_{SL(2,\mathbb{Z})\backslash\mathbb{H}^2} F(z)\overline{G(z)}\frac{dxdy}{y^2}.$$
Then
$$f(z) = \sum_{i=0}^{\infty} \langle f, \eta_i\rangle \eta_i(z) + \frac{1}{4\pi i}\int_{\frac12 - i\infty}^{\frac12 + i\infty} \langle f, E(*,s)\rangle E(z,s)\, ds$$
for any $z \in \mathbb{H}^2$.

Sketch of proof. **Step 1** Assume that $\langle f, 1\rangle = 0$. We show that
$$F(z) := f(z) - \frac{1}{4\pi i}\int_{\frac12 - i\infty}^{\frac12 + i\infty} \langle f, E(*;s)\rangle E(z,s)\, ds$$
is automorphic and $\int_0^1 F(x+iy)dx = 0$. Can be proved using Mellin inversion.

Step 2 Consider the space
$$L^2_{\text{cusp}} := L^2_{\text{cusp}}\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right) = \left\{ f \in \mathcal{L}^2,\ \langle f, 1\rangle = 0 \text{ and } \int_0^1 f(x+iy)\, dx = 0\right\}.$$
We will construct an integral operator mapping $L^2_{\text{cusp}} \to L^2_{\text{cusp}}$. Use spectral theorem for Hilbert-Schmidt operator. □

Lecture 9: 2010-10-7

Theorem 3.3. **(Sepctral theorem for $L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$)** *Let $f \in L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$. Assume that $\langle f, 1\rangle = 0$ and*
$$\int_{\frac12 - i\infty}^{\frac12 + i\infty} \left|\langle f, E(*,s)\rangle E(*,s)\right|\, ds \ll 1.$$

Then

$$f(z) = f_0(z) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E(z,s)\rangle\, E(z,s)\, ds$$

where $f_0 \in L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$ *and* $\int_0^1 f_0(x+iy)\, dx = 0$ *for any* $y > 0$*, i.e.,* $f_0 \in L^2_{\text{cusp}}\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$.

Remark. **(Properties of Eisenstein series)** Let $\Gamma_\infty = \{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\} \subset SL(2,\mathbb{Z})$ and

$$E(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \left(\Im(\gamma z)\right)^s.$$

Then

- $E(\gamma z, s) = E(z,s),\ \forall\gamma \in SL(2,\mathbb{Z})$
- $\Delta E(z,s) = s(1-s)E(z,s)$, where $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
- Fourier-Whittaker expansion:

  $$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

  where

  $$\phi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

  and $\phi(s)\phi(1-s) = 1$. Then $E(z,s) = \phi(s)E(z,1-s)$.
- Let $E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s)$ and it is holomorphic except for simple pole at $s = 0$ and $s = 1$.

Proof of theorem. It is enough to show

$$\int_0^1 f(x+iy)\, dx = \int_0^1 \left[\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\langle f, E(*,s)\rangle\, E(x+iy,s)\, ds\right] dx$$

because this implies that $\int_0^1 f_0(x+iy)\, dx = 0$ where

$$f_0(z) = f(z) - \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\langle f, E(*,s)\rangle\, E(x+iy,s)\, ds.$$

Mellin Transform $h : \mathbb{R} \to \mathbb{C}$, we have Mellin transform

$$\widetilde{h}(s) := \int_0^\infty h(x)x^s\,\frac{dx}{x}$$

and we have inverse Mellin transform

$$h(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\widetilde{h}(s)\, x^{-s}\, ds.$$

We compute

$$\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\langle f, E(*,s)\rangle\, E(x+iy,s)\, ds$$

by first computing

$$\begin{aligned}
\langle f, E(*,s)\rangle &= \iint\limits_{SL(2,\mathbb{Z})\backslash\mathbb{H}^2} f(z) \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\Im(\gamma z))^{\overline{s}}\, \frac{dx\,dy}{y^2} \\
&= \sum_{\gamma} \iint\limits_{\gamma\mathcal{D}} f(z) y^{\overline{s}}\, \frac{dx\,dy}{y^2} \\
&= \int_0^1 \int_0^\infty f(z) y^{\overline{s}}\, \frac{dx\,dy}{y^2}.
\end{aligned}$$

Define

$$C_f(y) := \int_0^1 f(x+iy)\, dx = \text{ constant term of } f.$$

We have shown that

$$\langle f, E(*,s)\rangle = \int_0^\infty C_f(s) y^{\overline{s}-1}\, \frac{dy}{y} = \widetilde{C_f}(\overline{s}-1).$$

Moreover,

$$\langle f, E(*,s)\rangle = \langle f, \phi(s)E(*,1-s)\rangle = \overline{\phi(s)}\widetilde{C_f}(-\overline{s}),$$

so,

$$\widetilde{C_f}(s-1) = \phi(s)\widetilde{C_f}(-s).$$

Next,

$$\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E(*,s)\rangle\, E(x+iy,s)\, ds = \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widetilde{C_f}(\overline{s}-1) E(x+iy,s)\, ds.$$

On the line $s = \frac{1}{2} + it$ for $t \in \mathbb{R}$, we have $\overline{s} - 1 = -s$. To proceed, we look at

$$\begin{aligned}
&\int_0^1 \left[\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E(*,s)\rangle\, E(x+iy,s)\, ds \right] dx \\
&= \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widetilde{C_f}(-s)\ \underbrace{\left[\int_0^1 E(x+iy,s)\, dx \right]}_{=y^s+\phi(s)y^{1-s}} ds \\
&= \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widetilde{C_f}(-s)\ \left[y^s + \phi(s)y^{1-s} \right]\, ds \\
&= \frac{1}{4\pi i}\left\{ \int_{(\frac{1}{2})} \widetilde{C_f}(-s) y^s\, ds + \int_{(\frac{1}{2})} \underbrace{\widetilde{C_f}(-s)\phi(s)}_{=\widetilde{C_f}(s-1)} y^{1-s}\, ds \right\} \\
&= C_f(y) = \int_0^1 f(x+iy)\, dx.
\end{aligned}$$

□

Lecture 10: 2010-10-12

3.4 Integral Operators

Let $\boldsymbol{X} =$ Riemanian space and $x, y, z \in \boldsymbol{X}$. Let $G =$ group acts on $\boldsymbol{X}$. Assume that there exists a function

$$k : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{C}$$

satisfying

$$k(gx, gy) = k(x, y), \qquad (\forall g \in G,\ x, y \in \boldsymbol{X}).$$

For $f \in L^2(G\backslash\boldsymbol{X})$, define

$$Kf(x) := \int_{\boldsymbol{X}} k(x, y) f(y) dy$$

then

$$K : L^2(G\backslash\boldsymbol{X}) \to^? L^2(G\backslash\boldsymbol{X}).$$

For any $g \in G$,

$$Kf(gx) = \int_{\boldsymbol{X}} k(gx, y) f(y)\ dy = \int_{\boldsymbol{X}} k(x, g^{-1}y) f(y)\ dy = \int_{\boldsymbol{X}} k(x, y) f(gy)\ dy = Kf(x).$$

We want

$$\int_{\boldsymbol{X}} \int_{\boldsymbol{X}} |k(x, y)|^2\ dxdy < \infty$$

then this implies that $K : L^2(G\backslash\boldsymbol{X}) \to L^2(G\backslash\boldsymbol{X})$ and K is Hilbert-Schmidt operator and we have a spectral theorem.

Definition 3.4. *Let $\phi : \mathbb{R} \to \mathbb{C}$, satisfying $\phi(t) \ll \frac{1}{(2+t)^{1+\epsilon}}$ for $t \geqslant 0$. We define*

$$k_\phi(z, z') = \phi\left(\frac{|z - z'|^2}{yy'}\right)$$

where $z = x + iy,\ z' = x' + iy' \in \mathbb{H}^2$.

Remark. The function

$$\frac{|z - z'|^2}{yy'}$$

is “almost" hyperbolic distance between z and z'.

Lemma 3.5. *For any $\gamma, \gamma' \in SL(2, \mathbb{R})$,*

$$k_\phi(\gamma z, \gamma' z') = k_\phi(z, z')$$

for $z, z' \in \mathbb{H}^2$.

Proof. Use $\Im\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. Then

$$\begin{aligned} k_\phi(\gamma z, \gamma' z') &= \phi\left(\frac{\left|\frac{az+b}{cz+d} - \frac{a'z'+b'}{c'z'+d'}\right|^2}{yy'|cz + d|^{-2}|c'z' + d'|^{-2}}\right) \\ &= \phi\left(\frac{(az + b)(cz + d) - (a'z' + b')(c'z' + d')}{yy'}\right) = \phi\left(\frac{|z - z'|^2}{yy'}\right). \end{aligned}$$

□

Properties of $k_\phi(z, z')$ Let $z = x + iy \in \mathbb{H}^2$ and $\Delta_z = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

(1) $k_\phi(\gamma z, \gamma' z') = k_\phi(z, z'),\ \forall \gamma, \gamma' \in SL(2, \mathbb{R})$

(2) $k_\phi(z, z') = k_\phi(z', z)$

(3) $\Delta_z k_\phi(z, z') = \Delta_{z'} k_\phi(z, z')$ because

$$\Delta_z \phi\left(\frac{|z - z'|}{yy'}\right) = \Delta_{z'} \phi\left(\frac{|z - z'|}{yy'}\right).$$

Let $\Gamma := SL(2, \mathbb{Z})$. Define

$$K_\phi(z, z') := \sum_{\gamma \in \Gamma} k_\phi(\gamma z, z') = \sum_{\gamma \in \Gamma} k_\phi(z, \gamma z').$$

Integral operators Let

$$L_0^2(\Gamma\backslash\mathbb{H}^2) := \left\{ f \in L^2(\Gamma\backslash\mathbb{H}^2), \;\middle|\; \int_0^1 f(x+iy)\, dx = 0 \right\}.$$

For $f \in L_0^2(\Gamma\backslash\mathbb{H}^2)$, define

$$I_\phi f(z) := \iint\limits_{\mathbb{H}^2} k_\phi(z,z') f(z') \frac{dx'\, dy'}{y'^2} = \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z,z') f(z')\, \frac{dx'\, dy'}{y'^2}.$$

We need to show the integral operator I_ϕ is well defined and that I_ϕ is H.S. (Hilbert-Schmidt).

Proposition 3.6. *Let $z = x+iy,\; z_1 = x_1 + iy_1 \in \mathbb{H}^2$. Then $\forall \epsilon > 0$ there exists $c_\epsilon > 0$ such that*

$$K_\phi(z,z_1) \leqslant c_\epsilon y^{-\epsilon} y_1^{1+\epsilon} \quad \textit{(quite sharp)}.$$

Remark. If this is the real growth of K_ϕ then

$$\iint\limits_{\Gamma\backslash\mathbb{H}^2} \iint\limits_{\Gamma\backslash\mathbb{H}^2} |K_\phi(z,z_1)|^2\, \frac{dx\, dy}{y^2}\, \frac{dx_1\, dy_1}{y_1^2} \approx \iint\limits_{\Gamma\backslash\mathbb{H}^2} \iint\limits_{\Gamma\backslash\mathbb{H}^2} y^{-2\epsilon} y_1^{2+2\epsilon}\, \frac{dx\, dy}{y^2}\, \frac{dx_1\, dy_1}{y_1^2} = \infty$$

so we need a modification.

Proof of Proposition.

$$K_\phi(z,z') = \sum_{\gamma\in\Gamma} \phi\left(\frac{|\gamma z - z_1|^2}{\Im(\gamma z)\Im(z_1)} \right)$$

we shall use the following

$$\phi\left(\frac{|z-z_1|^2}{yy_1}\right) = \phi\left(\frac{(x-x_1)^2 + (y-y_1)^2}{yy_1}\right) \leqslant c_\epsilon \left(\frac{1}{\left(2 + \frac{(x-x_1)^2+(y-y_1)^2}{yy_1}\right)^{1+\epsilon}} \right)$$
$$= \frac{c_\epsilon}{\left(\frac{(x-x_1)^2}{yy_1} + \frac{y}{y_1} + \frac{y_1}{y}\right)^{1+\epsilon}} \ll \frac{1}{\left(\frac{(x-x_1)^2}{yy_1} + \frac{y}{y_1}\right)^{1+\epsilon}}.$$

Let $\Gamma_\infty = \{ \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right),\; m \in \mathbb{Z} \}$. Then

$$K_\phi(z,z') = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{m\in\mathbb{Z}} \phi\left(\frac{|z - \gamma z_1 - m|}{\Im(z)\Im(\gamma z_1)} \right)$$
$$\ll \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im(z)\Im(\gamma z_1))^{1+\epsilon} \sum_{m\in\mathbb{Z}} \frac{1}{((x - \Re(\gamma z_1))^2 + m^2 + \Im(\gamma z_1)^2)^{1+\epsilon}}$$

Since

$$\sum_m \frac{1}{(Y^2+m^2)^{1+\epsilon}} \ll \int_0^\infty \frac{1}{(Y^2+u^2)^{1+\epsilon}} \ll \frac{1}{Y^{1+2\epsilon}},$$

$$\Rightarrow\; K_\phi(z,z') = \sum_{m\in\mathbb{Z}} k_\phi(z, z'+m) + \underbrace{\sum_{\substack{\Gamma\neq\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \\ \gamma\in\Gamma_\infty\backslash\Gamma}} \sum_{m\in\mathbb{Z}} k_\phi(z, \gamma z' + m)}_{=S}.$$

Then

$$|S| \ll \sum_{\substack{\Gamma \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \\ \gamma \in \Gamma_\infty \backslash \Gamma}} \sum_{m \in \mathbb{Z}} \frac{1}{(m^2 + \Im(\gamma z_1)^2)^{1+\epsilon}}$$

$$\ll \sum_{\substack{\Gamma \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \\ \gamma \in \Gamma_\infty \backslash \Gamma}} \frac{1}{\Im(\gamma z_1)^{1+2\epsilon}} \ll y_1^{1-(1+2\epsilon)} = y_1^{-2\epsilon}.$$

by using the Eisenstein series. □

Strategy We introduce two modification of K_ϕ in order to construct an integral operator with good growth at the cusps.

(1) First modification

$$K_\phi^\sharp(z, z') := K_\phi(z, z') - \sum_{m \in \mathbb{Z}} k_\phi(z, z' + m)$$

(2) Second modification

$$\widetilde{K}_\phi(z, z') := K_\phi(z, z') - \int_0^1 K_\phi(z, z' + t)\, dt$$

But they are not automorphic. We will prove that $K_\phi^\sharp$ is Hilbert-Schmidt and that $\left|\widetilde{K}_\phi - K_\phi^\sharp\right|$ is small then $\widetilde{K}_\phi$ is H.S.. Then we will show if $f \in L_0^2(\Gamma \backslash \mathbb{H}^2)$ then

$$\iint_{\Gamma \backslash \mathbb{H}^2} K_\phi(z, z')\, f(z')\, \frac{dx'\, dy'}{y'^2} = \iint_{\Gamma \backslash \mathbb{H}^2} \widetilde{K}_\phi(z, z')\, f(z')\, \frac{dx'\, dy'}{y'^2}.$$

Since f is cuspidal,

$$\begin{aligned} &\iint_{\Gamma \backslash \mathbb{H}^2} \widetilde{K}_\phi(z, z') f(z')\, \frac{dx'\, dy'}{y'^2} \\ &= \iint_{\Gamma \backslash \mathbb{H}^2} K_\phi(z, z') f(z')\, \frac{dx'\, dy'}{y'^2} - \underbrace{\int_0^1 \iint_{\Gamma \backslash \mathbb{H}^2} K_\phi(z, z') f(z')\, \frac{dx'\, dy'}{y'^2}\, dt}_{=0} \\ &= \iint_{\Gamma \backslash \mathbb{H}^2} K_\phi(z, z') f(z')\, \frac{dx'\, dy'}{y'^2}. \end{aligned}$$

Trace formula integrate on the diagonal

$$\underbrace{\iint_{\Gamma \backslash \mathbb{H}^2} \widetilde{K}_\phi(z, z)\, \frac{dx\, dy}{y^2}}_{=\text{ Spectral side}} = \underbrace{\sum_{\text{group } \Gamma}}_{=\text{ Arithmetic side}}$$

since $K_\phi(z, z')$ is automorphic in both variables

$$\iff K_\phi(\gamma z, z') = K_\phi(z, \gamma z') = K_\phi(z, z') \quad (\gamma \in \Gamma).$$

We know that $K_\phi(z, z') \ll y^{-\epsilon} y'^{1+\epsilon}$ so it is L^2 in the variable z so it has a Selberg spectral expansion in z. Fix z', we have

$$K(z, z') = \sum_{\substack{\eta_i, \\ \text{Maass forms}}} \langle K(*, z'), \eta_i \rangle\, \eta_i(z) + \frac{1}{4\pi i} \int_{\Re(s) = \frac{1}{2}} \langle K(*, z'), E(*, s) \rangle\, E(z, s)\, ds.$$

Lecture 11: 2010-10-14

Review

Let $|\phi(t)| \ll \frac{1}{|2+t|^{1+\epsilon}}$ for $t \geqslant 0$. Let $\Gamma = SL(2,\mathbb{Z})$. Then

$$k_\phi(z,z') = \phi\left(\frac{|z-z'|^2}{yy'}\right)$$

and

$$K_\phi(z,z') = \sum_{\gamma\in\Gamma} k_\phi(\gamma z, z').$$

Integral operator

$$I_\phi : L_0^2\left(\Gamma\backslash\mathbb{H}^2\right) \to L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$$

For $f \in L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$, we have

$$\begin{aligned} I_\phi f(z) &= \int_0^\infty \int_{-\infty}^\infty k_\phi(z,z') f(z') \frac{dx'\,dy'}{y'^2} \\ &= \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z,z')\ f(z')\ \frac{dx'\,dy'}{y'^2}, \end{aligned}$$

but this integral doesn't have the good growth property. Let

$$K_\phi^\sharp(z,z') = K_\phi(z,z') - \sum_{\gamma\in\Gamma_\infty} k_\phi(\gamma z, z')$$

where $\Gamma_\infty = \left\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \in SL(2,\mathbb{Z})\right\}$ and

$$\widetilde{K}_\phi(z,z') = K_\phi(z,z') - \int_0^1 K_\phi(z+t,z')\ dt.$$

Theorem 3.7. *Fix $\epsilon > 0$, $x, x_1 \in \mathbb{R}$ and $y, y_1 \to \infty$, we have*

- $$|K_\phi(z,z_1)| \ll y^{-\epsilon} y_1^{1+\epsilon}$$
- $$\left|K_\phi^\sharp(z,z_1)\right| \ll (yy_1)^{-\epsilon}$$
- $$\left|K_\phi^\sharp(z,z_1) - \widetilde{K}_\phi(z,z_1)\right| \ll (yy_1)^{-\epsilon} + \int_0^\infty \left|\phi'(t)\right|\ dt$$

Remark. Basically, first two are done in the last times by using

$$K_\phi(z,z_1) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{m\in\mathbb{Z}} k_\phi(z, \gamma z_1 + m).$$

We will prove the last one.

Proof.

$$\begin{aligned}
K^{\sharp}_{\phi}(z,z_1) - \widetilde{K}_{\phi}(z,z_1) &= \int_0^1 k_{\phi}(z,z_1+t)\,dt - \sum_{m\in\mathbb{Z}} k_{\phi}(z,z_1+m)\\
&= \int_0^1 K^{\sharp}_{\phi}(z,z_1+t)\,dt + \int_0^1 \sum_{m\in\mathbb{Z}} k_{\phi}(z,z_1+m+t)\,dt - \sum_{m\in\mathbb{Z}} k_{\phi}(z,z_1+m)\\
&= \int_0^1 K^{\sharp}_{\phi}(z,z_1+t)\,dt + \int_{-\infty}^{\infty} k_{\phi}(z,z_1+t)\,d(t-[t])\\
&\ll (yy_1)^{-\epsilon} + \int_{-\infty}^{\infty} k_{\phi}(z,z_1+t)\,d(t-[t])
\end{aligned}$$

Integration by parts and using $|t-[t]| \leqslant 1$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} k_{\phi}(z,z_1+t)\,d(t-[t]) &= \int_{-\infty}^{\infty} \phi\left(\frac{(x-x_1+t)^2+(y-y_1)^2}{yy_1}\right)\,d(t-[t])\\
&= \int_{-\infty}^{\infty} (t-[t])\,d\phi\left(\frac{(x-x_1+t)^2+(y-y_1)^2}{yy_1}\right)\\
&\ll \int_0^{\infty} \left|\phi'(r)\right|\,dr
\end{aligned}$$

□

Definition 3.8. **(Correct definition of $I_{\phi} : L_0^2\left(\Gamma\backslash\mathbb{H}^2\right) \to L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$)** *For $f \in L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$, define*

$$I_{\phi}f(z) = \iint\limits_{\Gamma\backslash\mathbb{H}^2} \widetilde{K}_{\phi}(z,z')f(z')\,\frac{dx'\,dy'}{y'^2}.$$

Remark. This is Hilbert-Schmidt

$$\iint\limits_{\Gamma\backslash\mathbb{H}^2}\iint\limits_{\Gamma\backslash\mathbb{H}^2} \left|\widetilde{K}_{\phi}(z,z_1)\right|^2\,\frac{dx\,dy}{y^2}\,\frac{dx_1\,dy_1}{y_1^2} \ll \iint\iint \left((yy_1)^{-2\epsilon}+c_{\phi}^2\right) \ll 1$$

where $c_{\phi} = \int_0^{\infty} |\phi'(t)|\,dt$.

Properties of I_{ϕ}

(1) $I_{\phi} : L_0\left(\Gamma\backslash\mathbb{H}^2\right)^2 \to L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$

(2) $\Delta I_{\phi} = I_{\phi}\Delta$, i.e., I_{ϕ} and Δ are commute where $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$.
(∵)

$$\begin{aligned}
\Delta I_{\phi}f(z) &= \Delta \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_{\phi}(z,z')f(z')\,\frac{dx'\,dy'}{y'^2}\\
&= \Delta_z \iint\iint\limits_{\mathbb{H}^2} \phi\left(\frac{|z-z'|}{yy'}\right) f(z')\,\frac{dx'\,dy'}{y'^2}\\
&= \iint\limits_{\mathbb{H}^2} \left(\Delta_{z'}\phi\left(\frac{|z-z'|^2}{yy'}\right)\right)\cdot f(z')\,\frac{dx'\,dy'}{y'^2}, \qquad \text{since self-adjoint}\\
&= \iint\limits_{\mathbb{H}^2} \phi\left(\frac{|z-z'|}{yy'}\right)\left(\Delta_{z'}f(z')\right)\,\frac{dx'\,dy'}{y'^2}\\
&= I_{\phi}\Delta f(z)
\end{aligned}$$

3.5 Selberg Transform

Fix $\lambda \in \mathbb{C}$. Assume $\Delta f = \lambda f$ is a one-dimensional space. Then,

$$\begin{aligned} I_\phi(\Delta f) &= I_\phi \lambda f(z) = \lambda I_\phi f(z) \\ &= \Delta(I_\phi f) \\ &\Rightarrow \Delta(I_\phi f) = \lambda \cdot I_\phi f \\ &\Rightarrow I_\phi f = \mu \cdot f, \quad \text{for some } \mu \in \mathbb{C}. \end{aligned}$$

Warning! We don't know it is a one-dimensional space!

Theorem 3.9. (Selberg) *Assume $f \in L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$ is an eigenfunction of the Laplacian (i.e., $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$). Then there exists unique $h_\phi(\lambda) \in \mathbb{C}$ such that*

$$I_\phi f(z) = h_\phi(\lambda) f(z).$$

To prove this theorem, recall radial symmetry.

Definition 3.10. *A function $f : \mathbb{H}^2 \to \mathbb{C}$ is said to be radially symmetric around $w \in \mathbb{H}^2$ if we set $z - w = x + iy = re^{i\theta}$ for $0 \leqslant r,\ 0 \leqslant \theta < 2\pi$, polar coordinates, and $f(z) =$ function of r only*

$$\Longleftrightarrow \quad \frac{\partial}{\partial\theta} f(re^{i\theta}) = 0$$

Lemma 3.11. *Fix $\lambda \in \mathbb{C}$. Up to a scalar factor there is only one regular solution to $\Delta f = \lambda f$ where f is radially symmetric around 0.*

Proof. Assume that $\Delta f = \lambda f$ and $f(r) = r^c(1 + a_1 r + a_2 r^2 + \cdots)$ write $\Delta f = \lambda f$ in polar coordinate assuming $\frac{\partial}{\partial\theta} f = 0$,

$$-\frac{1}{4}(1 - r^2)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) = \lambda f \tag{3.1}$$

Then (3.1) has exactly 2 solutions, $g(r)$ and $(\log r)g(r)$ and since it is regular, $f(r) = c \cdot g(r)$, for some constant $c \in \mathbb{C}$. ☐

Proof of Selberg's theorem on existence and uniqueness of $h_\phi(\lambda)$. (i.e., $I_\phi f = h_\phi(\lambda) \cdot f$.) Assume $I_\phi f = h_\phi(\lambda) \cdot f$

$$\Longleftrightarrow \quad \iint\limits_{\mathbb{H}^2} \phi\left(\frac{|z - z_1|}{yy'}\right) f(z_1)\, \frac{dx_1\, dy_1}{y_1^2} = h_\phi(\lambda) \cdot f(z)$$

Make a fractional linear transformation

$$z_1 \mapsto \frac{az_1 + b}{cz_1 + d}$$

mapping $\mathbb{H}^2 \to U$, the unit disc and $z \mapsto 0$, set $f(z_1) = f^*(w)$ for $w \in U$,

$$\Longleftrightarrow \quad \iint\limits_{U} \phi\left(\frac{4|w|^2}{1 - |w|^2}\right)\ f^*(w)\ d^*w = h_\phi(\lambda) \cdot f^\sharp(0)$$

where

$$f^\sharp(w) := \int_0^{2\pi} f^\sharp(|w|e^{i\theta})\ d\theta.$$

Then $f^\sharp$ is radially symmetric. After integrating,

$$\iff \iint\limits_U \phi\left(\frac{4|w|^2}{1-|w|^2}\right) f^\sharp(w)\, d^*w = h_\phi(\lambda)\cdot f^\sharp(0)$$

this has a unique solution. $\square$

Theorem 3.12. (Selberg's Transform) *Assume* $f \in L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$ *and* $\Delta f = \lambda f$ *with* $\lambda = \frac{1}{4}+r^2$. *Then*

$$I_\phi f = h_\phi(\lambda)\cdot f$$

where

$$h_\phi(\lambda) = \frac{1}{\sqrt{2}}\int_0^\infty t^{ir}\Phi\left(t-2+t^{-1}\right)\frac{dt}{t}$$

and

$$\Phi(x) := \sqrt{2}\int_x^\infty \phi(u)\frac{du}{\sqrt{u-x}}$$

is the Abel transform.

Lecture 12: 2010-10-19

Abel Transform

Definition 3.13. (Abel transform) *Let* $f:\mathbb{R}\to\mathbb{C}$ *be a function with a good property. Then the Abel transform of* f *is defined by*

$$F(x) = \int_{-\infty}^{\infty} f\left(x+\frac{\xi^2}{2}\right) d\xi.$$

Proposition 3.14. (Inverse Abel Transform)

$$f(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} F'\left(x+\frac{\eta^2}{2}\right) d\eta$$

Proof.

$$F'(x) = \int_{-\infty}^{\infty} f'\left(x+\frac{\xi^2}{2}\right) d\xi$$

$$\begin{aligned}\Rightarrow\ -\frac{1}{2\pi}\int_{\mathbb{R}} F'\left(x+\frac{\eta^2}{2}\right) d\eta &= -\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f'\left(x+\frac{\xi^2+\eta^2}{2}\right) d\xi\, d\eta \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty f'\left(x+\frac{r^2}{2}\right) r\, dr\, d\theta \\ &= -\int_0^\infty f'(x+u)\, du = f(x)\end{aligned}$$

$\square$

Example of Abel transform. $f(x) = e^{-\alpha x}$, $F(x) = \sqrt{\frac{2\pi}{\alpha}}e^{-\alpha x} \Rightarrow$ $e^{-2\pi x}$ is its own Abel transform.

Selberg Transform Last time we proved that if $f \in L_0^2\left(\Gamma\backslash\mathbb{H}^2\right)$ and $\Delta f = \lambda f$ (usually we use $\lambda = s(1-s)$) then there exists a unique $h_\phi(\lambda) \in \mathbb{C}$ such that

$$I_\phi f = h_\phi(\lambda) f$$

where

$$I_\phi f(z) = \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z, z_1) f(z_1)\, \frac{dx_1 dy_1}{y_1^2}$$

because K_ϕ has the rotation symmetry, i.e., it only depends on the radius, not the angle, and two operators commute. Here

$$K_\phi(z, z_1) = \sum_{\gamma\in\Gamma} \phi\left(\frac{|\gamma z - z_1|^2}{\Im(\gamma z)}\right).$$

Theorem 3.15. (Selberg Transform)

$$h_\phi\left(s(1-s)\right) = \frac{1}{\sqrt{2}} \int_0^\infty y_1^{s-\frac{1}{2}} \Phi\left(y_1 - 2 + \frac{1}{y_1}\right) \frac{dy_1}{y_1}$$

where Φ is the Abel transform of ϕ, i.e.,

$$\Phi(x) = \int_{-\infty}^{\infty} \phi\left(x + \frac{\xi^2}{2}\right)\, d\xi.$$

Proof. We compute $h_\phi(s(1-s))$ by using $f(z) = y^s$ which satisfies $\Delta y^s = s(1-s)y^s$. In this case,

$$I_\phi y^s = \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z, z_1) y_1^s\, \frac{dx_1\, dy_1}{y_1^2}.$$

We know

$$|K_\phi(z, z_1)| \ll y_1^\epsilon y^{1+\epsilon}$$

$$\left| \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z, z_1)\, y_1^s\, \frac{dx_1\, dy_1}{y_1^2} \right| \ll \int_0^1 \int_{\frac{\sqrt{3}}{2}} y_1^\epsilon y^{1+\epsilon} y_1^{\Re(s)}\, \frac{dx_1\, dy_1}{y_1^2}$$

If we choose $\Re(s) < 1 - 2\epsilon$ then the integral defining the transform $I_\phi y^s$ converges absolutely. So everything is well-defined. Since $\Delta y^s = s(1-s)y^s$, this implies

$$I_\phi y^s = h_\phi(s(1-s)) y^s \tag{3.2}$$

and $h_\phi(s(1-s))$ is unique. We can explicity copute $h_\phi(s(1-s))$ as follows.

$$(3.2) \Rightarrow\ h_\phi(s(1-s)) y^s = \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z, z_1) y_1^s\, \frac{dx_1\, dy_1}{y_1^2}$$

$$\begin{aligned}
\Rightarrow\ h_\phi\left(s(1-s)\right) &= y^{-s}\iint\limits_{\mathbb{H}^2}\phi\left(\frac{|z-z_1|^2}{yy_1}\right)y_1^s\,\frac{dx_1\,dy_1}{y_1^2}\\
&= y^{-s}\int_0^\infty\int_{-\infty}^\infty\phi\left(\frac{(x-x_1)^2+(y-y_1)^2}{yy_1}\right)y_1^{s-1}\,dx_1\,\frac{dy_1}{y_1}\\
&\qquad\qquad\qquad\text{and make a transform } x_1\mapsto x+x_1\\
&= y^{-s}\int_0^\infty\int_{-\infty}^\infty\phi\left(\frac{x_1^2+(y-y_1)^2}{yy_1}\right)y_1^{s-1}\,dx_1\,\frac{dy_1}{y_1}\\
&= \int_0^\infty\int_{-\infty}^\infty\phi\left(\frac{x_1^2}{yy_1}+\frac{y^2-2yy_1+y_1^2}{yy_1}\right)\cdot\left(\frac{y_1}{y}\right)^s y_1^{-1}\,dx_1\,\frac{dy_1}{y_1}\\
&\qquad\qquad\qquad\text{and make a transform } x_1\mapsto\sqrt{yy_1}x_1\\
&= \int_0^\infty\int_{-\infty}^\infty\phi\left(x_1^2+\frac{y^2-2yy_1+y_1^2}{yy_1}\right)\left(\frac{y_1}{y}\right)\sqrt{yy_1}y_1^{-1}\,\frac{dx_1\,dy_1}{y_1}
\end{aligned}$$

So,

$$\begin{aligned}
h_\phi(s(1-s)) &= \int_0^\infty\int_{-\infty}^\infty\phi\left(x_1^2+\frac{y}{y_1}-2+\frac{y_1}{y}\right)\,dx_1\,\left(\frac{y_1}{y}\right)^{s-\frac12}\,\frac{dy_1}{y_1}\\
&= \int_0^\infty\frac{1}{\sqrt2}\left[\int_{-\infty}^\infty\phi\left(\frac{\xi^2}{2}+\frac{1}{y_1}-2+y_1\right)\,d\xi\right]\,y_1^{s-\frac12}\,\frac{dy_1}{y_1}\\
&= \frac{1}{\sqrt2}\int_0^\infty\Phi\left(\frac{1}{y_1}-2+y_1\right)\,y_1^{s-\frac12}\,\frac{dy_1}{y_1}.
\end{aligned}$$

□

3.6 Crude Trace Formula (Spectral side)

$$K_\phi(z,z_1)=\sum_{\gamma\in\Gamma}\phi\left(\frac{|\gamma z-z_1|^2}{yy_1}\right)$$

$$|K_\phi(z,z_1)|\ll y_1^{-\epsilon}y^{1+\epsilon}.$$

Then $K_\phi(z,z_1)\in L^2$ in z_1. So it has the Selberg spectral expansion in z_1.

$$K_\phi(z,z_1)=\sum_{j=0}^\infty\langle K_\phi(z,*),\eta_j\rangle\,\eta_j(z_1)+\frac{1}{4\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\langle K_\phi(z,*),E(*,s)\rangle\;E(z_1,s)\;ds$$

Here $\Delta\eta_j=\lambda_j\eta_j$ is an orthnormal basis of Maass cuspforms for $j=0,1,2,\dots,$ and $\eta_0=\sqrt{\frac{3}{\pi}}$.

$$\langle K_\phi(z,*),\eta_j\rangle=\iint\limits_{\Gamma\backslash\mathbb{H}^2}K_\phi(z,z_1)\overline{\eta_j(z_1)}\,\frac{dx_1\,dy_1}{y_1^2}=h_\phi(\lambda_j)\overline{\eta_j(z)}$$

Similarly

$$\langle K_\phi(z,*),E(*,s)\rangle=h_\phi(s(1-s))\overline{E(z,s)}$$

So we get the spectral side of the trace formula.

$$K_\phi(z,z_1)=\sum_{j=0}^\infty h_\phi(\lambda_j)\eta_j(z_1)\overline{\eta_j(z)}+\frac{1}{4\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}h_\phi(s(1-s))E(z_1,s)\overline{E(z,s)}\;ds$$

Let $s=\frac12+ir$ and define the modified kernel.

Definition 3.16. (Modified Kernel)

$$\widehat{K}_\phi(z,z_1) := K_\phi(z,z_1) - \frac{1}{4\pi i}\int_{-\infty}^{\infty} h_\phi\left(\frac{1}{4}+r^2\right)\overline{E(z,\frac{1}{2}+ir)}E(z_1,\frac{1}{2}+ir)\;dr.$$

Now, we can integrate over the diagonal, i.e., $z = z_1$, we have

$$\iint\limits_{\Gamma\backslash\mathbb{H}^2} \widehat{K}_\phi\,(z,z)\;\frac{dx\;dy}{y^2} = \sum_{j=0}^{\infty} h_\phi(\lambda_j)\;\iint\limits_{\Gamma\backslash\mathbb{H}^2} \eta_j(z)\overline{\eta_j(z)}\;\frac{dx\;dy}{y^2} = \sum_{j=0}^{\infty} h_\phi(\lambda_j).$$

Proposition 3.17. (Crude Trace formula)

$$\iint\limits_{\Gamma\backslash\mathbb{H}^2} \widehat{K}_\phi\,(z,z)\;\frac{dx\;dy}{y^2} = \sum_{j=0}^{\infty} h_\phi(\lambda_j) = \textit{ Trace}$$

Remark. We proved the crude trace formula for any smooth function $\phi : \mathbb{R} \to \mathbb{C}$, where $\phi(t) \ll \frac{1}{(2+t)^{1+\epsilon}}$, $t \geqslant 0$. In proving the crude trace formula, we have used the spectral theorem for Hilbert-Schmidt integral operators. This tells there exists an orthonormal basis of cusp forms

$$\left\{\eta_j \;\middle|\; \begin{array}{c} j = 0,1,2,\ldots \\ \Delta\eta_j = \lambda_j\eta_j \end{array}\right\}$$

where $I_\phi\eta_j = h_\phi(\lambda_j)\eta_j$. By the spectral theorem,

$$h_\phi(\lambda_j) \to 0$$

as $\lambda_j \to \infty$. At this point we don't know that λ_j for $j = 1,2,3\ldots$ are discrete.

Theorem 3.18. *Let $\{\eta_j\}_{j=1,2,\ldots}$ be an orthonormal basis of Maass cusp forms satisfying $\Delta\eta_j = \lambda_j\eta_j$. Then $\lambda_j \geqslant 0$ and discrete (there is no limit point of λ_j).*

Remark. When we get the full trace formula we will prove there are infinitely many η_j and that $\lambda_j \sim \frac{j}{12}(??)$ as $j \to \infty$.

Lemma 3.19. *Let $r \geqslant 0$. Then there exists smooth compactly supported $\phi : [0,\infty) \to \mathbb{C}$ satisfying $h_\phi\left(\frac{1}{4}+r^2\right) > 0$.*

Proof.

$$\begin{aligned} h_\phi\left(\frac{1}{4}+r^2\right) &= \frac{1}{\sqrt{2}}\int_0^\infty y^{ir}\Phi\left(y-2+\frac{1}{y}\right)\;\frac{dy}{y} \\ &= \frac{1}{\sqrt{2}}\int_1^\infty (y^{ir}+y^{-ir})\Phi\left(y-2+\frac{1}{y}\right)\;\frac{dy}{y} \\ &= \sqrt{2}\int_1^\infty \cos(\log y\cdot r)\Phi\left(y-2+\frac{1}{y}\right)\;\frac{dy}{y} \end{aligned}$$

where $\Phi(x) = \int_{-\infty}^{\infty}\phi(x+\frac{\xi^2}{2})\;d\xi$. Choose Φ to have supported near 0. Then $\Phi(y-2+\frac{1}{y})$ has support near 1. Then for $1 \leqslant y \leqslant 1+\epsilon$ with ϵ sufficiently small we have $\cos((\log y)r) > \frac{3}{4}$. □

Assume infinitely many λ_j near some $r > 0$. Then

$$\begin{aligned} \text{Trace} &= \iint\limits_{\Gamma\backslash\mathbb{H}^2} \widehat{K}_\phi(z,z)\frac{dx\;dy}{y^2} \\ &= \sum_j h_\phi(\lambda_j) = \infty \end{aligned}$$

if λ_j cluster to some r. So it should be discrete.

Lecture 13: 2010-10-21

3.7 Geometric side of the Trace formula

Orbital Integral A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ is of three types:

- Parabolic $|a+d| = 2$
- Hyperbolic $|a+d| > 2$
- Elliptic $|a+d| < 2$

and

$$SL(2,\mathbb{Z}) = \bigcup_{\substack{\text{conj. classes} \\ \sigma}} [\sigma]$$

where

$$[\sigma] = \left\{ \alpha\sigma\alpha^{-1} \,|\alpha \in SL(2,\mathbb{Z}) \right\}.$$

Then

$$SL(2,\mathbb{Z}) = \left([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)] \cup \left[\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right] \right) \bigcup \left(\underset{\sigma \text{ hyperbolic}}{\cup} [\sigma] \right) \bigcup \left(\underset{\sigma \text{ parabolic}}{\cup} [\sigma] \right) \bigcup \left(\underset{\sigma \text{ elliptic}}{\cup} [\sigma] \right).$$

Trace formula Let $\Gamma = SL(2,\mathbb{Z})$.

$$\begin{aligned}
\iint\limits_{\Gamma\backslash\mathbb{H}^2} \widehat{K}(z,z)\frac{dx\,dy}{y^2} &= \iint\limits_{\Gamma\backslash\mathbb{H}^2} \sum_{\gamma\in\Gamma} \widehat{k}_\phi\left(\gamma z, z\right) \; \frac{dx\,dy}{y^2} \\
&= \underbrace{2 \iint\limits_{\Gamma\backslash\mathbb{H}^2} \widehat{k}_\phi(z,z) \; \frac{dx\,dy}{y^2}}_{= \text{Identity class}} + \iint\limits_{\Gamma\backslash\mathbb{H}^2} \sum_{\substack{[\sigma] \\ \text{hyperbolic}}} \sum_{\alpha\in\Gamma_\sigma\backslash\Gamma} \widehat{k}_\phi(\alpha\sigma\alpha^{-1}z, z) \; \frac{dx\,dy}{y^2} \\
&+ \iint\limits_{\Gamma\backslash\mathbb{H}^2} \sum_{\substack{[\sigma] \\ \text{elliptic}}} \sum_{\alpha\in\Gamma_\sigma\backslash\Gamma} \widehat{k}_\phi(\alpha\sigma\alpha^{-1}z, z) \; \frac{dx\,dy}{y^2} \\
&+ \iint\limits_{\Gamma\backslash\mathbb{H}^2} \sum_{\substack{[\sigma] \\ \text{paraboli}}} \sum_{\alpha\in\Gamma_\sigma\backslash\Gamma} \widehat{k}_\phi(\alpha\sigma\alpha^{-1}z, z) \; \frac{dx\,dy}{y^2}
\end{aligned}$$

where $\Gamma_\sigma = \{\gamma \in \Gamma \mid \gamma\sigma = \sigma\gamma\}$. So

$$\begin{aligned}
\text{Trace} &= \iint\limits_{\Gamma\backslash\mathbb{H}^2} \widehat{K}(z,z) \; \frac{dx\,dy}{y^2} = \sum_{j=0}^{\infty} h(r_j) \qquad \text{(Spectral side)} \\
&= \underbrace{C(I)}_{\substack{= \text{identity} \\ \text{conj. class}}} + C(Ell) + C(Hyp) + C(Par) \qquad \text{(Geometric side).}
\end{aligned}$$

Definition 3.20. *The integral*

$$C\left([\sigma]\right) := \iint\limits_{\Gamma\backslash\mathbb{H}^2} \sum_{\alpha\in\Gamma_\sigma\backslash\Gamma} \widehat{k}_\phi\left(\alpha\sigma\alpha^{-1}z, z\right) \; \frac{dx\,dy}{y^2}$$

is called an orbital integral for the conjugacy class $[\sigma]$.

We can easily compute

$$C([\sigma]) = \iint\limits_{\Gamma\backslash\mathbb{H}^2} \sum_{\alpha\in\Gamma_\sigma\backslash\Gamma} \widehat{k}_\phi(\sigma\alpha^{-1}z, \alpha^{-1}z)\, \frac{dx\, dy}{y^2}$$
$$= \sum_{\alpha\in\Gamma_\sigma\backslash\Gamma} \iint\limits_{\alpha(\Gamma\backslash\mathbb{H}^2)} \widehat{k}_\phi(\sigma z, z)\, \frac{dx\, dy}{y^2} = \iint\limits_{\Gamma_\sigma\backslash\mathbb{H}^2} \widehat{k}_\phi(\sigma z, z)\, \frac{dx\, dy}{y^2}$$

Definition 3.21. (Alternative definition of the orbital integral)

$$C([\sigma]) = \iint\limits_{\Gamma_\sigma\backslash\mathbb{H}^2} \widehat{k}_\phi(\sigma z, z)\, \frac{dx\, dy}{y^2}.$$

Compact hyperbolic group $\Gamma \subset SL(2,\mathbb{R})$ We assume that

- $\Gamma \subset SL(2,\mathbb{R})$ discrete, finite index subgroup such that

$$\Gamma = [\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)] \cup \left(\cup_{\sigma \text{ hyper}}[\sigma]\right)$$

- Vol $\left(\Gamma\backslash\mathbb{H}^2\right) < \infty$.

The Kernel function

$$K_\phi(z, z_1) = \sum_{\gamma\in\Gamma} k_\phi(\gamma z, z_1) = \sum_{\gamma\in\Gamma} \phi\left(\frac{|\gamma z - z_1|}{\Im(\gamma z)\Im(z_1)}\right)$$

and this is the trace class $\iff$ $\iiiint |K_\phi(z, z_1)|^2\ d^*z\ d^*z_1 < \infty$. Then there is no Eisenstein series since we don't have parabolic elements.

Review:Selberg Transform. Assume $f \in L^2\left(\Gamma\backslash\mathbb{H}^2\right)$ such that $\Delta f = \left(\frac{1}{4} + r^2\right) f$ for some $r \in \mathbb{R}$. Then $I_\phi f = h_\phi(r) f$ and

$$I_\phi f(z) = \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z, z_1) f(z_1)\, \frac{dx\, dy}{y^2}.$$

- Abel Transform:

$$\Phi(w) = \int_{-\infty}^{\infty} \phi(x^2 + w)\, dx = \int_w^\infty \frac{\phi(t)}{\sqrt{t-w}}\, dt$$

- Inverse transform

$$\phi(t) = -\frac{1}{\pi}\int_t^\infty \frac{\Phi'(w)}{\sqrt{w-t}}\, dw$$

- Fourier transform

$$g(u) = \Phi(e^u + e^{-u} - 2)$$
$$h(r) = \int_0^\infty g(u)\, e^{iru}\, du, \qquad g(u) = \frac{1}{2\pi}\int_0^\infty h(r)\cos(ru)\, dr$$

for an orthonormal basis of Maass forms $\eta_j \in L^2\left(\Gamma\backslash\mathbb{H}^2\right)$, $\Delta\eta_j = \left(\frac{1}{4} + r^2\right)\eta_j$.

$$\text{Trace} = \sum_{j=0}^\infty h(r_j) = \iint\limits_{\Gamma\backslash\mathbb{H}^2} K_\phi(z, z)\, \frac{dx\, dy}{y^2}$$
$$= C(\text{Identity}) + \sum_{\substack{[\sigma],\\ \text{hyperbolic}}} \iint\limits_{\Gamma_\sigma\backslash\mathbb{H}^2} k_\phi(\sigma z, z)\, \frac{dx\, dy}{y^2}$$

Computation of the Identity conjugacy class

$$C(Id) = \iint\limits_{\Gamma\backslash\mathbb{H}^2} \phi\left(\frac{|z-z|^2}{y^2}\right)\,\frac{dx\,dy}{y^2} = \phi(0)\mathrm{Vol}\left(\Gamma\backslash\mathbb{H}^2\right)$$

Want to express $\phi(0)$ in terms of h

$$\begin{aligned}\phi(0) &= -\frac{1}{\pi}\int_0^\infty \frac{\Phi'(w)}{\sqrt{w}}\,dw = -\frac{1}{\pi}\int_0^\infty \frac{g'(u)}{e^{\frac{u}{2}}-e^{-\frac{u}{2}}}\,du\\ &= \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty rh(r)\frac{\sin(ru)}{e^{\frac{u}{2}}-e^{-\frac{u}{2}}}\,du\,dr\end{aligned}$$

Computation of the hyperbolic oribital integrals Recall: $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is hyperbolic $\iff$ one of the following three conditions holds.

(i) $|a+d| > 2$;

(ii) there exist two real fixed points $w_1, w_2 \in \mathbb{R}$;

(iii) $\exists \rho\in\mathbb{R},\ |\rho|>1$ and $m\in SL(2,\mathbb{R})$ such that $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = m\begin{pmatrix} \rho & 0\\ 0 & \rho^{-1}\end{pmatrix}m^{-1}$.

Definition 3.22.

$$N\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right) := \rho^2$$

Lemma 3.23. *Assume* $\sigma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$ *is hyperbolic with* $N(\sigma)=\rho^2$*. Then for* $\alpha\in m^{-1}\Gamma m$*, we have*

$$\alpha\begin{pmatrix} \rho & 0\\ 0 & \rho^{-1}\end{pmatrix}\alpha^{-1} = \begin{pmatrix} \rho & 0\\ 0 & \rho^{-1}\end{pmatrix} \iff \alpha = \begin{pmatrix} \rho^\ell & 0\\ 0 & \rho^{-\ell}\end{pmatrix} \text{ for } \ell\in\mathbb{Z}.$$

$$\Gamma_\sigma = \{\alpha\in\Gamma \mid \alpha\sigma = \sigma\alpha\}$$

$$\Gamma_{\left(\begin{smallmatrix} \rho & 0\\ 0 & \rho^{-1}\end{smallmatrix}\right)} = \left\{\begin{pmatrix} \rho^\ell & 0\\ 0 & \rho^{-\ell}\end{pmatrix} \mid \ell\in\mathbb{Z}\right\}$$

and

$$\Gamma_{\left(\begin{smallmatrix} \rho^k & 0\\ 0 & \rho^{-k}\end{smallmatrix}\right)} = \Gamma_{\left(\begin{smallmatrix} \rho & 0\\ 0 & \rho^{-1}\end{smallmatrix}\right)}.$$

Lemma 3.24. *A fundamental domain for* $\Gamma_{\left(\begin{smallmatrix} \rho & 0\\ 0 & \rho^{-1}\end{smallmatrix}\right)}\backslash\mathbb{H}^2$ *is*

$$\mathcal{D}_\rho = \left\{z = x+iy \mid -\infty\leqslant x\leqslant\infty,\ 1\leqslant y\leqslant\rho^2\right\}.$$

Proof. $\begin{pmatrix} \rho & 0\\ 0 & \rho^{-1}\end{pmatrix}z = \rho^2 z$ and $\bigcup_{\ell\in\mathbb{Z}}\rho^{2\ell}\mathcal{D}_\rho = \mathbb{H}^2$. □

Remark. For hyperbolic $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ with $N\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right) = \rho^2$, then $\ln\left(\rho^2\right)$ = hyperbolic length of this geodesic. (because $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)i = m\begin{pmatrix} \rho & 0\\ 0 & \rho^{-1}\end{pmatrix}m^{-1}i = \rho^2 i = e^r i$ if $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = k_1\begin{pmatrix} e^{-r/2} & \\ & e^{r/2}\end{pmatrix}k_2$ for $k_1,k_2\in SO(2,\mathbb{R})$)

Assume that $\sigma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = m\begin{pmatrix} \rho^\ell & 0\\ 0 & \rho^{-\ell}\end{pmatrix}m^{-1}\in\Gamma$, with $m\in SL(2,\mathbb{R})$ and $[\sigma]$ is a primitive class, i.e., every other conjugacy class $[\sigma']$ (for $\sigma'\in\Gamma$) with the same fixed points modulo Γ is a unique power of σ, i.e., $\sigma' = \sigma^\ell$ for some integer $1 \leqslant \ell$. Then

$$\begin{aligned}
C(Hyp) &= \sum_{\substack{[\sigma],\\ \text{hyperbolic}}} \iint_{\Gamma_\sigma \backslash \mathbb{H}^2} k_\phi(\sigma z, z)\, \frac{dx\, dy}{y^2} \\
&= \sum_{\ell=1}^{\infty} \iint_{\Gamma_{\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right)} \backslash \mathbb{H}^2} k_\phi\left(\left(\begin{smallmatrix} \rho^\ell & 0 \\ 0 & \rho^{-\ell} \end{smallmatrix}\right) z, z \right)\, \frac{dx\, dy}{y^2} \\
&= \sum_{\ell=1}^{\infty} \iint_{\Gamma_{\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right)} \backslash \mathbb{H}^2} k_\phi\left(\rho^{2\ell} z, z \right)\, \frac{dx\, dy}{y^2}.
\end{aligned}$$

For each $\ell \geqslant 1$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{1}^{\rho^2} \phi\left(\frac{(\rho^{2\ell}-1)(x^2+y^2)}{\rho^{2\ell} y^2} \right)\, \frac{dx\, dy}{y^2} \\
&\qquad = \int_{-\infty}^{\infty} \int_{1}^{\rho^2} \phi\left(\frac{(\rho^{2\ell}-1)^2}{\rho^{2\ell}} (1+\xi^2) \right)\, d\xi\, \frac{dy}{y} \\
&\qquad = 2 \log \rho \int_{0}^{\infty} \phi\left(\frac{(\rho^{2\ell}-1)^2}{\rho^{2\ell}} (1+\xi^2) \right)\, d\xi
\end{aligned}$$

and let

$$u = \frac{(\rho^{2\ell}-1)^2}{\rho^{2\ell}} (1+\xi^2), \text{ and } \xi = \sqrt{\frac{\rho^{2\ell}}{(\rho^{2\ell}-1)^2} u - 1}$$

then

$$\begin{aligned}
&= 2 \log \rho \frac{\rho^\ell}{(\rho^{2\ell}-1)} \int_{\frac{(\rho^{2\ell}-1)^2}{\rho^{2\ell}}} \frac{\phi(u)}{\sqrt{u - \frac{(\rho^{2\ell}-1)^2}{\rho^{2\ell}}}}\, du \\
&= \frac{\log \rho}{\rho^{\frac{\ell}{2}} - \rho^{-\frac{\ell}{2}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-ir \log(\rho^{2\ell})}\, dr \\
&= \frac{\log \rho}{\rho^{\ell/2} - \rho^{-\ell/2}} g(2\ell \log \rho)
\end{aligned}$$

So for each primitive class $[\sigma]$, with $N(\sigma) = \rho^2$, we have

$$\sum_{\ell=1}^{\infty} \frac{\log \rho}{\rho^{\ell/2} - \rho^{-\ell/2}} \cdot g(2\ell \log \rho).$$

Trace formula for compact hyperbolic groups

$$\begin{aligned}
\sum_{j=0}^{\infty} h(r_j) &= \mathrm{Vol}\left(\Gamma \backslash \mathbb{H}^2 \right) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} r \cdot \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, h(r)\, dr \\
&\quad + \sum_{[\wp]} \sum_{\ell=1}^{\infty} \frac{\log(N(\wp))}{N(\wp)^{\ell/2} - N(\wp)^{-\ell/2}} \cdot g\left(\ell \log N(\wp) \right)
\end{aligned}$$

where $\wp = m \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) m^{-1} \in \Gamma$, $m \in SL(2, \mathbb{R})$ with $|\rho| > 1$, i.e., $N(\wp) = \rho^2$, and it is minimal.

Selberg Zeta function

$$Z(s) = \prod_{\substack{[\wp] \text{ primitive hyperbolic} \\ \text{conjugacy classes}}} \prod_{\ell=0}^{\infty} \left(1 - \frac{1}{N(\wp)^{s+\ell}}\right), \qquad (\text{ if } \Re(s) > 1).$$

Theorem 3.25. *The Selberg Zeta function $Z(s)$ has meromorphic continuation to all $s \in \mathbb{C}$ and $Z(s)$ satisfies RH. i.e., $Z(s) = 0$ with $\Re(s) = \frac{1}{2} \iff s = 1/2 + ir$.*

Question. Can we generalize the Selberg zeta function for higher rank groups?

Lecture 14: 2010-10-26

3.8 Selberg Zeta function

Explicit formula relating prime numbers and zeros of Rieman zeta function

$$\begin{array}{c} \text{sum over primes} \\ \text{with test function} \end{array} = \begin{array}{c} \text{sum over zeros of } \zeta(s) \\ \text{wth transform of the test function} \end{array}$$

(often called “Weil’s explicit formula")

$$-(\log(\zeta(s))' = \frac{-\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}}$$

where

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \quad (\text{ for } k = 1, 2, \ldots); \\ 0, & \text{otherwise.} \end{cases}$$

1. Left hand side: sum over primes with a test function: Weil’s explicit formula Choose a test function (assuming enough decay) $h : \mathbb{R} \to \mathbb{C}$ where

$$\widetilde{h}(s) = \int_0^{\infty} h(u) u^s \, \frac{du}{u}$$

is holomorphic. Then,

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) \widetilde{h}(s) \, ds = \sum_p \log p \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\widetilde{h}(s) e^{-(\log p)ks}}{p^{k/2}} \, ds$$
$$= \sum_p \log p \sum_{k=1}^{\infty} \frac{h(k \log p)}{p^{k/2}}$$

$$\Rightarrow \sum_p \log p \sum_{k=1}^{\infty} \frac{h(k \log p)}{p^{k/2}} = \begin{array}{c} \text{sum over zeros of } \zeta(s) \\ \text{wth transform of the test function} \end{array}$$

2. Right hand side: sum over zeros of $\zeta(s)$ with transform of the test function For a test function $h : \mathbb{R} \to \mathbb{C}$ (with enough decay), we have,

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) \widetilde{h}(s) \, ds = \widetilde{h}\left(\frac{1}{2}\right) - \sum_{\substack{j, \\ \zeta(\frac{1}{2}+ir_j)=0}} \widetilde{h}(ir_j) \tag{3.3}$$
$$+ \frac{1}{2\pi i} \int_{-2-i\infty}^{-2+i\infty} -\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) \widetilde{h}(s) \, ds. \tag{3.4}$$

Note: We don't assume that r_j's are real!

By functional equation, $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s)$, so $\Lambda\left(\frac{1}{2}+s\right) = \Lambda\left(\frac{1}{2}-s\right)$, i.e.,

$$\pi^{-\frac{\frac{1}{2}+s}{2}}\Gamma\left(\frac{\frac{1}{2}+s}{2}\right)\zeta\left(\frac{1}{2}+s\right) = \pi^{\frac{-\frac{1}{2}+s}{2}}\Gamma\left(\frac{\frac{1}{2}-s}{2}\right)\zeta\left(\frac{1}{2}-s\right).$$

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2}+s\right) = \frac{\zeta'}{\zeta}\left(\frac{1}{2}-s\right) + 1 - \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{\frac{1}{2}+s}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{\frac{1}{2}-s}{2}\right)\right)$$

use this relation and (3.3), with assuming that $\widetilde{h}(s) = -\widetilde{h}(-s)$, we have

$$\frac{2}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}\left(\frac{1}{2}+s\right)\ \widetilde{h}(s)\ ds = \widetilde{h}\left(\frac{1}{2}\right) - \sum_{\substack{j,\\ \zeta(\frac{1}{2}+ir_j)=0}} \widetilde{h}(ir_j)$$
$$+ \frac{1}{2\pi i}\int_{-2-i\infty}^{-2+i\infty}\left(-1+\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}+\frac{\Gamma'}{\Gamma}\right)\right)\widetilde{h}(s)\ ds.$$

Therefore, we get

$$2\cdot\sum_p \log p\sum_{k=1}^{\infty}\frac{h(k\log p)}{p^{k/2}}$$
$$= \widetilde{h}\left(\frac{1}{2}\right) - \sum_{\substack{j,\\ \zeta(\frac{1}{2}+ir_j)=0}} \widetilde{h}(ir_j)\frac{1}{2\pi i} + \int_{-2-i\infty}^{-2+i\infty}\left(-1+\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}+\frac{\Gamma'}{\Gamma}\right)\right)\widetilde{h}(s)\ ds.$$

Selberg zeta function Last time: Selberg Trace formula for cocompact case.

$$\sum_{\substack{[\wp],\text{ hyperbolic}\\ \text{conjugacy classes}}}\sum_{k=1}^{\infty}\frac{\log(N(\wp))}{N(\wp)^{\frac{k}{2}} - N(\wp)^{-\frac{k}{2}}}g\left(k\log N(\wp)\right) \tag{3.5}$$
$$= \sum_{j=0}^{\infty} h(r_j) - \mathrm{Vol}\left(\Gamma\backslash\mathbb{H}^2\right)\frac{1}{4\pi}\int_{-\infty}^{\infty} r\cdot\frac{e^{-\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\ h(r)\ dr$$

Selberg asked Can we construct an Euler product

$$Z(s) = \prod_{\wp}\prod_{k=0}^{\infty}\left(1-\frac{1}{N(\wp)^{s+k}}\right)$$

So that when we do Weil's explicit formula for $\frac{Z'}{Z}\left(\frac{1}{2}+s\right)$ we get (3.5).

Lemma 3.26.

$$\frac{Z'}{Z}\left(\frac{1}{2}+s\right) = \sum_p\sum_{k=1}^{\infty}\frac{\log N(\wp)}{N(\wp)^{k(s+1/2)} - N(\wp)^{-k(-s+1/2)}}$$

Proof.

$$\frac{Z'}{Z}(s) = (\log Z(s))' = \frac{d}{ds}\sum_{\wp}\sum_{k=0}^{\infty}\log\left(1-N(\wp)^{-s-k}\right)$$
$$= \sum_{\wp}\sum_{k=0}^{\infty}\log(N(\wp))\frac{N(\wp)^{-s-k}}{1-N(\wp)^{-s-k}}$$
$$= \sum_{\wp}\sum_{k=0}^{\infty}\log(N(\wp))N(\wp)^{-s-k}\sum_{m\geqslant 0}N(\wp)^{-ms-mk}$$
$$= \sum_{\wp}\sum_{m\geqslant 1}\frac{\log N(\wp)}{N(\wp)^{ms} - N(\wp)^{m(s-1)}}$$

□

Question. Convergence?

Lemma 3.27.

$$\sum_{N(\wp)\leqslant X} 1 \ll X$$

Proof. elementary □

So, the Euler product for $Z(s)$ converges absolutely for $\Re(s) > 1$. (The summation in Lemma 3.26 is also absolutely convergent for $\Re(s) > 1$.)

By choosing suitable g and h in (3.5), we can show

$$\sum_{r_j\leqslant X} 1 = \mathcal{O}(X).$$

Following Hejhal's idea *(Hejhal, "Selberg trace formula for $PSL(2,\mathbb{R})$", Springer Lecture Notes)* Fix $\alpha, \beta \in \mathbb{C}$ for $\frac{1}{2} < \Re(\alpha) < \Re(\beta)$. Choose

$$h(r) = \frac{1}{r^2+\alpha^2} - \frac{1}{r^2+\beta^2} = \frac{\beta^2-\alpha^2}{(r^2+\alpha^2)(r^2+\beta^2)},$$

$$g(u) = \frac{1}{2\alpha}e^{-\alpha|u|} - \frac{1}{2\beta}e^{-\beta|u|},$$

and

$$|h(r)| = O\left(|r|^{-4}\right) \qquad \text{(as } |r| \to \infty\text{)}.$$

We rewrite the trace formula using these g, h, so

$$\frac{1}{2\alpha}\sum_{\wp}\sum_{k=1}^{\infty}\frac{\log N(\wp)}{N(\wp)^{k/2}-N(\wp)^{-k/2}}\cdot\frac{1}{N(\wp)^{k\alpha}} - \frac{1}{2\beta}\sum_{\wp}\sum_{k=1}^{\infty}\frac{\log N(\wp)}{N(\wp)^{k/2}-N(\wp)^{-k/2}}\cdot\frac{1}{N(\wp)^{k\beta}} \tag{3.6}$$
$$= \sum_{j=0}^{\infty}\left(\frac{1}{r^2+\alpha^2} - \frac{1}{r_j^2+\beta^2}\right) - \frac{\mathrm{Vol}\left(\Gamma\backslash\mathbb{H}^2\right)}{4\pi}\int_{-\infty}^{\infty} r\left(\frac{1}{r^2+\alpha^2} - \frac{1}{r^2+\beta^2}\right)\tanh(\pi r)\,dr$$

Next, choose $\alpha = s - \frac{1}{2}$ and $\beta = \beta - \frac{1}{2}$, by lemma, we get

$$\begin{aligned} &\frac{1}{2s-1}\frac{Z'}{Z}\left(s-\frac{1}{2}\right) - \frac{1}{2\beta-1}\frac{Z'}{Z}\left(\beta-\frac{1}{2}\right) \\ &= \underbrace{\sum_{j=0}^{\infty}\left(\frac{1}{r_j^2+\left(s-\frac{1}{2}\right)^2} - \frac{1}{r_j^2+\left(\beta-\frac{1}{2}\right)^2}\right) - \frac{\mathrm{Vol}}{4\pi}\int\frac{1}{r^2+\left(s-\frac{1}{2}\right)^2}}_{\text{invariant } s\mapsto 1-s \,\Rightarrow\, \text{FE}} \end{aligned} \tag{3.7}$$

Right side of (3.7) has simple poles when $r_j^2 + \left(s-\frac{1}{2}\right)^2 = 0$ i.e., when $s = \frac{1}{2} \pm ir_j$. We can prove using (3.7) that $\frac{Z'}{Z}(s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with poles at $s = \frac{1}{2} \pm ir_j$. Can also get growth of $\frac{Z'}{Z}(s)$ as $|s| \to \infty$. Then

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{Z'}{Z}\left(\frac{1}{2}+s\right)H(s)\,ds = \sum_{\wp}(\ldots) = \sum r_j(\ldots).$$

Theorem 3.28. *Selberg zeta function*

$$Z(s) = \prod_{\wp} \prod_{k=0}^{\infty} \left(1 - N(\wp)^{-s-k}\right)$$

is an entire function of $s \in \mathbb{C}$ *which satisfies*

(1) $Z(s)$ *has trivial zeros at* $s = \ell = -1, -2, -3, \ldots$ *with multiplicity* $(2g-1)(2\ell+1)$*;*

(2) $s = 0$ *is a zero of multiplicity* $2g-1$ *and* $s = 1$ *is a zero of multiplicity* 1*;*

(3) the non-trivial zeros are located at $s = \frac{1}{2} \pm ir_j$ *where* $\frac{1}{4} + r_j^2 =$ *eigenvalue of* Δ*;*

(4) (Functional equation)

$$Z(s) = Z(1-s) \exp\left[\mathrm{Vol}\left(\Gamma\backslash\mathbb{H}^2\right) \int_0^{s-\frac{1}{2}} r \tanh(\pi r)\, dr\right].$$

Remark. $Z(s)$ grows like $e^{|s|^{2+\epsilon}}$, (then $DG(s)Z(s) = DG(1-s)Z(1-s)$ where $DG(s)$ is a double gamma function.) so it is a function of Hadamard order 2. All L-functions in Langlands Program has order 1.

Generalization of Selberg's zeta function

- Ihara zeta function is a p-adic version of Selberg zeta function.
- Ruelle zeta function - zeta function defined by dynamical systems. Selberg zeta function is a special case.

Lecture 15: 2010-10-28

3.9 Trace formula for $\Gamma\backslash\mathbb{H}^2$ (for $\Gamma = PSL(2,\mathbb{Z})$)

For $\phi\left(\frac{|z-z|^2}{y^2}\right) = \phi(0)$, we have

$$\sum_{j=0}^{\infty} h_\phi(r_j) = \mathrm{Vol}\left(\Gamma\backslash\mathbb{H}^2\right)\phi(0) + \sum_{\substack{[\gamma] \\ \text{hyperbolic}}} \iint_{\Gamma_\sigma\backslash\mathbb{H}^2} k_\phi(z, \gamma z)\, \frac{dx\, dy}{y^2}$$

$$+ \sum_{\substack{[\gamma], \\ \text{elliptic}}} \iint_{\Gamma_\gamma\backslash\mathbb{H}^2} k_\phi(z, \gamma z)\, \frac{dx\, dy}{y^2} + C(Par).$$

Here

$$C(Par) = \sum_{\substack{[\gamma] \text{ parabolic} \\ \text{conjugacy classes}}} \underbrace{\iint_{\Gamma_\gamma\backslash\mathbb{H}^2} k_\phi(z, \gamma z)\, \frac{dx\, dy}{y^2}}_{=\text{(secretely)}\infty} - \underbrace{\iint_{\Gamma\backslash\mathbb{H}^2} \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle K, E\rangle\, E(z,s)\, \frac{dx\, dy}{y^2}}_{\text{(secretely)}=\infty}$$

but it is not well defined. To remedy this, Selberg cuts the fundamental domain and takes a limit.

Definition 3.29. *Let* $[\gamma] =$ *parabolic conjugacy class. Let*

$$\Gamma_\gamma = \{\sigma \in \Gamma \mid \sigma\gamma = \gamma\sigma\}.$$

We define

- $$D_\gamma^Y = \left\{ z \in D_\gamma = \Gamma_\gamma \backslash \mathbb{H}^2 \mid \Im(z) \leqslant Y \right\};$$
- $$D^Y = \left\{ z \in D = \Gamma \backslash \mathbb{H}^2 \mid \Im(z) \leqslant Y \right\}.$$

So,

$$C(Par) = \lim_{Y \to \infty} \left[\sum_{\substack{[\gamma] \text{ parabolic} \\ \text{conjugacy classes}}} \iint\limits_{D_\gamma^Y} k_\phi(z, \gamma z) \, \frac{dx\, dy}{y^2} - \iint\limits_{D_\gamma^Y} \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle K, E \rangle \, E(z, s) \, \frac{dx\, dy}{y^2} \right].$$

Lemma 3.30. *The parabolic conjugacy classes* $[\gamma]$ *are all of the form* $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ *with* $m \in \mathbb{Z}$, $m \neq 0$.

Proof. For $m \neq n$, we have

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & am+b \\ c & cm+d \end{smallmatrix}\right) \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \begin{pmatrix} 1 & a^2 m \\ -c^2 m & 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \neq \sigma \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \sigma^{-1}.$$

□

For $m \in \mathbb{Z},\ m \neq 0$, we have

$$\Gamma_{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)} = \Gamma_\infty = \left\{ \left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right),\ \ell \in \mathbb{Z} \right\}$$

$$\Rightarrow D_{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)} = \Gamma_\infty \backslash \mathbb{H}^2$$

and

$$D^Y_{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)} = \left\{ x + iy,\ 0 \leqslant x < 1,\ y \leqslant Y \right\}.$$

Therefore,

$$C(Par) = \lim_{Y \to \infty} \left[\sum_{\substack{m \in \mathbb{Z},\ m \neq 0, \\ \gamma = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)}} \int_0^Y \int_0^1 k_\phi\left(z, \gamma z\right) \, \frac{dx\, dy}{y^2} - \iint\limits_{D^Y} \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle K, E \rangle \, E(z, s) \, \frac{dx\, dy}{y^2} \right].$$

First, we compute

$$\iint\limits_{D^Y} \frac{1}{4\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \langle K, E \rangle \, E(z, s) \, \frac{dx\, dy}{y^2}.$$

Proposition 3.31. **(Maass-Selberg relation)** *Let* $s = \sigma + ir$. *Then,*

$$\iint\limits_{D^Y} E(z, s) \, E(z, \overline{s}) \, \frac{dx\, dy}{y^2} = \frac{Y^{2\sigma - 1} - |M(s)|^2 Y^{1 - 2\sigma}}{2\sigma - 1} + \frac{M(\overline{s}) Y^{2ir} - M(s) Y^{-2ir}}{2ir}$$

where

$$\int_0^1 E(x + iy, s) \, dx = y^s + M(s) y^{1-s}, \qquad M(s) M(1-s) = 1$$

and

$$M(s) = \frac{\sqrt{\pi}\,\Gamma(s - 1/2)\zeta(2s - 1)}{\Gamma(s)\zeta(2s)}.$$

Remark. Computed by Green's theorem. Usually use two variables s and w:

$$\int_{D^Y} E(z,s)E(z,w)\ \frac{dx\,dy}{y^2}.$$

Since $\langle K, E\rangle = h_\phi(r)E(z,\overline{s})$, we have

$$\begin{aligned}\Rightarrow \iint\limits_{D^Y} \frac{1}{4\pi i}\int_{(1/2)} \langle K,E\rangle\, E(z,s)\ ds\ \frac{dx\,dy}{y^2}\\ = \iint\limits_{D^Y} \frac{1}{4\pi i}\int_{(1/2)} h_\phi(r)E(z,s)E(z,\overline{s})\ ds\ \frac{dx\,dy}{y^2}.\end{aligned}$$

We can write

$$\begin{aligned} I_Y &= \lim_{\sigma\to\frac{1}{2}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{r^{2\sigma-1} - |M(s)|^2\,Y^{1-2\sigma}}{2\sigma-1}\ dr\\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{M(\overline{s})Y^{2ir} - M(s)Y^{-2ir}}{2ir}\ dr\\ &= I_Y^1 + I_Y^2.\end{aligned}$$

Compute I_Y^1 and I_Y^2.

(1)

$$\begin{aligned} I_Y^1 &= \lim_{\sigma\to\frac{1}{2}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{r^{2\sigma-1}-Y^{1-2\sigma}}{2\sigma-1}\ dr + \lim_{\sigma\to\frac{1}{2}}\frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{(1-M(s))^2}{2\sigma-1}\ dr\\ &= \log Y\cdot g(0) + \lim_{\sigma\to\frac{1}{2}}\frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{(1-M(s))^2}{2\sigma-1}\ dr\end{aligned}$$

Tayor series around $\sigma = 1/2$

$$\begin{aligned} 1 - M(\sigma + ir_M(\sigma - ir) &= \left[1 - M(\frac{1}{2}+ir)M(\frac{1}{2}-ir)\right]\\ &+ \left[M'(\frac{1}{2}+ir)M(\frac{1}{2}-ir) + M(\frac{1}{2}+ir)M'(\frac{1}{2}-ir)\right]\left(\sigma-\frac{1}{2}\right)\\ &+ \text{ higher powers of } (\sigma - \frac{1}{2})\end{aligned}$$

since $M(s)M(1-s) = 1$,

$$1 - |M(\sigma+ir)|^2 = \left[\frac{M'}{M}\left(\frac{1}{2}+ir\right)\frac{M'}{M}\left(\frac{1}{2}-ir\right)\right](\sigma-\frac{1}{2}) + \text{ higher power}$$

$$I_Y^1 = (\log Y)g(0) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\left[\frac{M'}{M}(\frac{1}{2}+ir) + \frac{M'}{M}(\frac{1}{2}-ir)\right]\ dr$$

(2)

$$\begin{aligned} I_Y^2 &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{\left[M(\frac{1}{2}-ir)Y^{2ir} - M(\frac{1}{2}+ir)Y^{-2ir}\right]}{2ir}\ dr\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(r)\frac{\Re\left(M(\frac{1}{2}-ir\right)\sin(2r\log Y)}{r}\ dr\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\phi(\frac{r}{\log Y})\frac{\Re\left(M(\frac{1}{2}-i\frac{r}{\log Y}\right)\sin(2r)}{r}\ dr\\ &= \frac{1}{4\pi}h(0)M(\frac{1}{2}) + \mathcal{O}(\frac{1}{\log Y})\end{aligned}$$

Next we consider

$$\sum_{\substack{m\in\mathbb{Z},\ m\neq 0,\\ \gamma=\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)}} \int_0^Y \int_0^1 k_\phi\left(z,\gamma z\right)\ \frac{dx\ dy}{y^2}.$$

Proposition 3.32.

$$\begin{aligned} P &:= \sum_{m\in\mathbb{Z},\ m\neq 0} \int_0^Y \int_0^1 k_\phi(z, z+m)\ \frac{dx\ dy}{y^2} \\ &= g(0)\log Y + \frac{h(0)}{4} - g(0)\log 2 - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\ dr + \mathcal{O}(\frac{1}{\log Y}) \end{aligned}$$

Proof.

$$\begin{aligned} P &= \sum_{m\neq 0}\int_0^Y\int_0^1 \phi\left(\frac{|z-(z+m)|^2}{y^2}\right)\ \frac{dx\ dy}{y^2} = \sum_{m\neq 0}\int_0^Y \phi\left(\frac{m^2}{y^2}\right)\ \frac{dy}{y^2} \\ &= \sum_{m\neq 0}\frac{1}{|m|}\int_{\frac{|m|}{Y}}^{\infty}\phi(y^2)\ dy = 2\int_0^\infty\left(\sum_{0<m\leqslant yY}\frac{1}{m}\right)\phi(y^2)\ dy \end{aligned}$$

Lemma 3.33.

$$\sum_{m\leqslant X}\frac{1}{m} = \log X + c + o(1)$$

$$\begin{aligned} P &= 2\log Y\int_0^\infty \phi(y^2)\ dy + 2\int_0^\infty \phi(y^2)\log y\ dy + c\int_0^\infty \phi(y^2)\ dy + o(1) \\ &= g(0)\left[\log Y + c\right] + \frac{1}{2}\int_0^\infty \log y\ \phi(y)\frac{dy}{\sqrt{y}}o(1) \end{aligned}$$

$$\begin{aligned} &\int_0^\infty \log y\ \phi(y)\frac{dy}{\sqrt{y}}\ \text{(Inverse Abel Transform)} \\ &\qquad = \frac{-1}{\pi}\int_0^\infty\int_y^\infty \frac{\log y}{\sqrt{y(w-y)}}\Phi'(w)\ dw\ dy \\ &\qquad = -\frac{1}{\pi}\int_0^\infty\int_0^1 \frac{\log wy}{\sqrt{y(1-y)}}\Phi'(w)\ dy\ dw \\ &\qquad - \frac{1}{\pi}\int_0^\infty\left(\pi\log w - 2\pi\log 2\right)\Phi'(w)\ dw \\ &\qquad = -\int_0^\infty \log(1-e^{-u})\ d\Phi(u) + \frac{h(0)}{4} - \log 2g(0) \end{aligned}$$

□

3.10 Selberg Zeta function for $\Gamma = PSL(2,\mathbb{Z})$

$$Z(s) = \prod_{\substack{[P],\text{ hyperbolic}\\ \text{conjugacy class}}}\ \prod_{k=0}^{\infty}\left(1-N(P)^{-s-k}\right)$$

Consider a hyperbolic conjugacy class in Γ: Let $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$ with the trace $T = a+d$ and $|T|>2$. Then

$$\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \sigma\left(\begin{smallmatrix}\rho & 0\\ 0 & \rho^{-1}\end{smallmatrix}\right)\sigma^{-1}$$

for some $\rho > 1$ and $N(\gamma) = \rho^2$. Clearly $\left[\left(\begin{smallmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{smallmatrix}\right)\right]$ is again a hyperbolic conjugacy class.

$$Z(s) = \prod_{\substack{[P],\text{primitive} \\ \text{hyperbolic conjugacy class}}} \prod_{\ell=1}^{\infty} \prod_{k=0}^{\infty} \left(1 - N(P)^{\ell(-s-k)}\right)$$

If $\left(\begin{smallmatrix} a & b \\ d & d \end{smallmatrix}\right) = \sigma \left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{smallmatrix}\right) \sigma^{-1} \Rightarrow T = \rho + \rho^{-1}$, so $\rho^2 - \rho T + 1 = 0$. Then

$$\rho = \frac{T \pm \sqrt{T^2 - r}}{2} = \text{ unit in } \mathbb{Q}\left(\sqrt{T^2 - 4}\right).$$

So we can rewrite,

$$Z(s) = \prod_{\substack{\text{fundamental discriminant } D \\ \text{of real quadratic fields}}} \prod_{k=0}^{\infty} \left(1 - \epsilon_D^{-s-k}\right)^{h(D)}$$

where ϵ_D = fundamental unit in $\mathbb{Q}(\sqrt{D})$ and $h(D)$ = class number of $\mathbb{Q}(\sqrt{D})$.

Peter Sarnak's thesis:

$$\sum_{\epsilon_D \leqslant X} h(D) \sim X \quad \text{as } X \to \infty$$

Lecture 16: 2010-11-4

4 Selberg Trace formula (Representation theory)

4.1 Selberg Trace formula for a finite group

Fix G = finite group, V = finite dimensional vector space over $\mathbb{C}$ and $\pi : G \to GL(V)$ be a representation of G. Equivalently, we could consider the group algebra

$$\mathbb{C}[G] := \{\phi : G \to \mathbb{C}\}$$

with convolution

$$(\phi_1 * \phi_2)(g) := \sum_{g_1 \cdot g_2 = g} \phi_1(g_1)\phi_2(g_2)$$

and consider finite dimensional $\mathbb{C}[G]$-modules.

Example

(1) π_{triv} with $V = \mathbb{C}$

(2) π_{reg} with $V = \mathbb{C}[G]$ with actions by left translation.

Notation For $g \in G$, $v \in V$,

$$\pi(g).v$$

denotes the action.

Example For $h, g \in G$ and $\phi \in \mathbb{C}[G]$,

$$\pi_{\text{reg}}(h).\phi(g) = \phi(hg).$$

(Selberg Trace formula → break into conjugacy classes) For the Selberg Trace formula, we consider the action of G on itself by conjugation. Let

$$\mathrm{Conj}[G] := \text{ set of conjugacy classes of } G.$$

If $g \in G$ then

$$[g] = \{\sigma g \sigma^{-1} \mid \sigma \in G\} \in \mathrm{Conj}[G].$$

Definition 4.1. (Class functions)

$$\mathrm{Class}[G] = \left\{ \phi : G \to \mathbb{C} \;\middle|\; \begin{array}{c} \phi(\sigma g \sigma^{-1}) = \phi(g), \\ \forall \sigma, g \in G \end{array} \right\}$$

and $\phi \in \mathrm{Class}[G]$ *are called class functions.*

Definition 4.2. (Character of π)

$$\chi_\pi(g) := \mathrm{Trace}(\pi(g)) \qquad (\textit{for } g \in G)$$

Proposition 4.3. *(1)* $\chi_\pi \in \mathrm{Class}[G]$*;*

(2) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$*;*

(3) $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \cdot \chi_{\pi_2}$*;*

(4) $\chi_{\mathrm{triv}}(g) = 1, \qquad (\forall g \in G)$*;*

(5) $\chi_{\mathrm{reg}}(g) = \begin{cases} |G|, & g = Id, \\ 0, & \textit{otherwise.} \end{cases}$

Definition 4.4. (Inner product on $\mathrm{Class}[G]$**)** *Let* $\phi_1, \phi_2 \in \mathrm{Class}[G]$. *We define*

$$\langle \phi_1, \phi_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \phi_1(g) \overline{\phi_2(g)}.$$

Proposition 4.5. *The character of irreducible representations of G form an orthonormal basis for* $\mathrm{Class}[G]$.

Remark. This proposition is analogue of the spectral theory.

We need an analogue of the geometric side of the Selberg Trace formula for our situation of $G =$ finite group. There is another basis for $\mathrm{Class}[G]$, called the geometric basis given by using characteristic functions for conjugacy classes.

Definition 4.6. *Fix* $h \in G$*, define* $1_{[h]} : \mathrm{Class}[G] \to \mathbb{C}$ *as*

$$1_{[h]}(g) := \begin{cases} 1, & \textit{if } g \in [h], \\ 0, & \textit{otherwise.} \end{cases}$$

Fix $h_1,\ h_2 \in G$, consider

$$\begin{aligned} \langle 1_{[h_1]}, 1_{[h_2]} \rangle &= \frac{1}{|G|} \sum_{g \in G} 1_{[h_1]}(g) \cdot 1_{[h_2]}(g) \\ &= \begin{cases} \frac{|[h_1]|}{|G|}, & [h_1] = [h_2] \\ 0, & [h_1] \neq [h_2] \end{cases} . \end{aligned}$$

They are orthogonal basis.

Given (π, V) representation $\pi : G \to GL(V)$, we may expand χ_π in two different base using

$$V = \oplus_{i\in I} V_i^{m(\pi_i)}, \qquad (V_i \text{ irreducible}).$$

Then

$$\underbrace{\sum_{i\in I} m(\pi_i)\chi_{\pi_i}}_{\text{spectral side}} = \underbrace{\sum_{[g]\in \mathrm{Calss}[G]} \mathrm{Trace}(\pi(g)) \cdot 1_{[g]}}_{\text{geometric side}}$$

To get the analogue of the Selberg Trace formula, we consider $\Gamma \subset G$ and move to the situation of $\Gamma\backslash G$. Need to consider induced representations.

Induced representations Fix $\pi : \Gamma \to GL(W)$ where W = finite dimensional vector space over $\mathbb{C}$. Want to define

$$\pi_{\mathrm{Ind}} = \mathrm{Ind}_\Gamma^G(\pi).$$

$$\pi_{\mathrm{Ind}} : G \to GL(V)$$

where

$$V = \left\{ f : G \to W \;\middle|\; \begin{matrix} f(\gamma g) = \pi(\gamma).f(g), \\ \forall \gamma \in \Gamma,\ g \in G \end{matrix} \right\}$$

$$\pi_{\mathrm{Ind}}(h).f(g) := f(gh), \qquad (\forall h, g \in G,\ f \in V).$$

Let's look at the simplest possible case, i.e., let's induce π_{triv} (i.e., $\pi_{\mathrm{triv}}(\gamma) = 1$ for all $\gamma \in \Gamma$). The group algebra $\mathbb{C}[G]$ acts on V by a matrix with an entry for every pair of cosets $x, y \in \Gamma\backslash G$. Let $\phi : G \to \mathbb{C}$ (i.e., $\phi \in \mathbb{C}[G]$) then ϕ acts on V as follows.

Definition 4.7.

$$K_\phi(x, y) = \sum_{\gamma\in\Gamma} \phi(x^{-1}\gamma y), \qquad (x, y \in \Gamma\backslash G).$$

Then $K_\phi(x, y)$ determines a matrix

$$\begin{pmatrix} & & \\ & K_\phi(x,y) & \\ & & \end{pmatrix}_{(x,y\in\Gamma\backslash G)}$$

and let this matrix acts on V.

Theorem 4.8.

$$\chi_{\mathrm{Ind}}(\phi) = \sum_{\gamma\in\mathrm{Conj}(\Gamma)} \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma) \cdot \sum_{x\in G_\gamma\backslash G} \phi(x^{-1}\gamma x)$$

and

$$\sum_{x\in G_\gamma\backslash G} \phi(x^{-1}\gamma x) := \int_{[\gamma]\in G} \phi \qquad \textit{(orbital integral).}$$

We can also express this using characteristic functions of conjugacy classes. Therefore

$$\textit{STF for finite group} : \underbrace{\sum_{\gamma\in\mathrm{Conj}(\Gamma)} \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma) \cdot 1_{[\gamma]}(\phi)}_{\textit{geometric side}} = \underbrace{\sum_{i\in I} m_i\chi_i(\phi)}_{\textit{spectral side}}$$

(Frobenius Reciprocity)

Remark.

$$\mathrm{Trace} = \int K_\phi(x, x)$$

4.2 Selberg Trace formula for an infinite group (elementary example)

Let $G = \mathbb{R}$. Then the irreducible unitary representations are just characters, $e^{i\lambda x}$ for $\lambda \in \mathbb{R}$.

- Spectral side → Fourier transform.

For STF, we consider the case of a subgroup $\Gamma \subset \mathbb{R}$ and $\Gamma\backslash\mathbb{R}$.

Simplest case : $\Gamma = \mathbb{Z}$ and consider $\mathbb{Z}\backslash\mathbb{R}$.

- Irreducible unitary representations: $e_n(x) := e^{2\pi i n x}$ for $n \in \mathbb{Z}$.

For $\phi : \mathbb{R} \to \mathbb{C}$, define

$$K_\phi(x, y) := \sum_{n\in\mathbb{Z}} \phi(x + n - y), \qquad (x, y \in \mathbb{R}).$$

Spectral expansion : Fix y.

$$K_\phi(x, y) = \sum_{n\in\mathbb{Z}} \langle K_\phi(*, y), e_n\rangle\, e_n(x)$$

where

$$\langle \phi_1, \phi_2\rangle := \int_{\mathbb{Z}\backslash\mathbb{R}} \phi_1(x)\overline{\phi_2(x)}\, dx.$$

Then

$$\begin{aligned}\langle K_\phi(*, y), e_n\rangle &= \int_0^1 K_\phi(x, y)e^{-2\pi i n x}\, dx\\ &= \int_0^1 \sum_{m\in\mathbb{Z}} \phi(x + m - y)e^{-2\pi i n x}\, dx\\ &= \int_{\mathbb{R}} \phi(x - y)e^{-2\pi i n x}\, dx = e^{-2\pi i n y}\widehat{\phi}(n)\end{aligned}$$

where $\widehat{\phi}(n) = \int_{\mathbb{R}} \phi(x)e^{-2\pi i n x}\, dx$.

Selberg Trace formula :

$$K_\phi(x, x) = \sum_{n\in\mathbb{Z}} \phi(n) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(n)$$

i.e., *Poisson Summation formula*.

Lecture 17: 2010-11-9

4.3 Preliminaries

Notations

- G = algebraic group defined over field F
- $F \subset R$ = ring
- $G(R)$ = R-points of G

Base Change Let $K \supset F$. We consider G_K as an algebraic group over K, called the base change.

Remark. The only difference between G and G_K is that for $G_K(R)$ we may only consider rings R that contain K.

Torus

Definition 4.9. (Algebraic Torus) *A torus T over a field F is defined to be an algebraic group over F such that the base change*

$$T_{\overline{F}} \cong GL(1)^k, \qquad (\textit{for some integer } k = 1, 2, 3, \ldots).$$

Recall $GL(1) = \mathbb{G}_m$ = multiplicative group in algebraic geometry, with defining equation $ab = 1$.

Definition 4.10. (Split Torus) *A torus T over F is split if*

$$T \cong GL(1)^k \qquad (\textit{for some integer } k = 1, 2, 3, \ldots).$$

Definition 4.11. (Anisotropic Torus) *A torus T is anisotropic if*

$$\mathrm{Hom}(T, GL(1)) = \langle (0) \rangle, \qquad \textit{i.e., it is trivial.}$$

Otherwise, it is called isotropic.

Example

(i) Split torus:

$$GL(1) = \{ab = 1 \mid a, b \in F\}$$

is a split torus.

(ii) Anisotropic

$$\begin{aligned} SO(2) &= \left\{g \in SL(2) \;\middle|\; g^{-1} = {}^t g\right\} \\ &\cong S^1 = \left\{x, y \in F \;\middle|\; x^2 + y^2 = 1\right\}. \end{aligned}$$

So, over $\mathbb{R}$, the torus $SO(2, \mathbb{R}) = \left\{\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)\right\}$ is anisotropic.

Remark. Over $\mathbb{C}$,

$$GL(1) \cong SO(2)$$

with $a = x + iy$ and $b = x - iy$.

A good way to think about tori T (an algebraic group over F) having 2 pieces of structure

(1) Base change $T_{\overline{F}}$.

(2) Galois descent needed to recover polynomial equations defining T as an algebraic group.

Reductive Group

Definition 4.12. (Reductive group) *$G \subset GL(n)$ and $G = {}^tG$.*

Definition 4.13. (Split reductive group) *A reductive group G is split if it contains a maximal torus (not a proper subgroup of another torus) which is split.*

Example (Split reductive groups)

- (A_n)
$$SL(n+1) = \{g \in GL(n+1) \mid \det(g) = 1\}$$
- (B_n) Odd orthogonal groups
$$SO(2n+1) = \left\{g \in GL(2n+1) \;\middle|\; {}^tgQ_{2n+1}g = Q_{2n+1},\ \det g = 1\right\}$$
where
$$Q_{2n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & Q_{2n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
- (C_n) Sympletic group $Sp(2n)$
- (D_n) Even special orthogonal groups $SO(2n)$

Each of these split with its maximal torus = diagonal elements.

4.4 Jacquet-Langlands Correspondence

This is the first algebraic application of the trace formula. They work with a quaternion algebra.

Example of a quaternion algebra Fix positive integers q and r (coprime) and square-free. Define $D[q,r]$ to be the quaternion algebra over $\mathbb{Q}$ with defining relations $J_1,\ J_2,\ J_3 \in D[q,r]$ such that

- $J_1^2 = q, \quad J_2^2 = r, \quad J_3^2 = -qr;$
- $J_1J_2 + J_2J_1 = 0, \quad J_1J_3 + J_3J_1 = 0, \quad J_2J_3 + J_3J_2 = 0;$
- $J_1J_2 = J_3, \quad J_2J_3 = -rJ_1, \quad J_3J_1 = -qJ_2.$

Then
$$D[q,r] := \{x_0 + x_1J_1 + x_2J_2 + x_3J_3 \mid x_0,\ x_1,\ x_2,\ x_3 \in \mathbb{Q}\}.$$
For $x = x_0 + x_1J_1 + x_2J_2 + x_3J_3 \in D[q,r]$, define
$$\overline{x} = x_0 - x_1J_1 - x_2J_2 - x_3J_3,$$
$$N(x) = x \cdot \overline{x}$$
and
$$Tr(x) = x + \overline{x}.$$
Then
$$D[q,r] \hookrightarrow M_2(\mathbb{R}) = \begin{array}{c}2 \times 2 \text{ matrices with} \\ \text{coefficients in } \mathbb{R}.\end{array}$$
with
$$x = x_0 + x_1J_1 + x_2J_2 + x_3J_3 \hookrightarrow \begin{pmatrix} x_0 + x_1\sqrt{q} & \sqrt{r}(x_2 + x_3\sqrt{q}) \\ \sqrt{r}(x_2 - x_3\sqrt{q}) & x_0 - x_1\sqrt{q} \end{pmatrix} \qquad (\text{for } N(x) = 1).$$

Definition 4.14. *A subring $\mathcal{O} \subset D[q,r]$ is called an order provided $1 \in \mathcal{O}$ and $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank 4. Then*

$$\operatorname{disc}(\mathcal{O}) := |\det(Tr[\xi_j, \xi_k])|$$

where $\{\xi_j\}$ is the $\mathbb{Z}$-basis for $\mathcal{O}$.

Definition 4.15. *Let $\mathcal{O}$ be an order in $D[q,r]$. We define*

$$\Gamma_{\mathcal{O}} \subset SL(2,\mathbb{R})$$

to be the set

$$\{\operatorname{mat}(x) \ | \ x \in \mathcal{O}, \ N(x) = 1\}.$$

Theorem 4.16.

$$\operatorname{Vol}\left(\Gamma_{\mathcal{O}} \backslash \mathbb{H}^2\right) < \infty.$$

Naive Jacquet-Langlands correspondence To each Maass form ϕ in $L^2\left(\Gamma_{\mathcal{O}} \backslash \mathbb{H}^2\right)$ with $\Delta\phi = \lambda\phi$ there exists $N \in \mathbb{Z}^+$ and a Maass form Φ for $L^2\left(\Gamma_0(N) \backslash \mathbb{H}^2\right)$ with $\Delta\Phi = \lambda\Phi$. (Reference: D. Hejhal, “A classical approach to a well-known spectral correspondence on quaternion groups")

Fix $F =$ number field and $\mathbb{A}_F =$ adeles over F,

- $H =$ division algebra of degree p^2 over F, for a prime p
- $G =$multiplicative group of H
- $Z =$ center of $G(\mathbb{A}_F)$
- $\overline{G} = \operatorname{center}(G) \backslash G$

Let

$$\omega : F^\times \backslash \mathbb{A}_F^\times \to \text{ roots of unity}$$

be a unitary Hecke character. Let

$$\begin{aligned} V &= L^2_\omega\left(\overline{G}(F) \backslash \overline{G}(\mathbb{A}_F)\right) \\ &= \left\{ f : G(\mathbb{A}_F) \to \mathbb{C} \ \middle| \ \begin{array}{c} \int_{\overline{G}(F)\backslash\overline{G}(\mathbb{A}_F)} |f(g)|^2 \ dg < \infty \\ f(\gamma z g) = \omega(z) f(g), \ \forall z \in Z, \ \gamma \in G(F), \ g \in G(\mathbb{A}_F) \end{array} \right\}. \end{aligned}$$

Consider the automorphic representation (ρ, V) with

$$\rho : G(\mathbb{A}_F) \to GL(V),$$

an action by right translation, i.e.,

$$\rho(h).f(g) = f(gh), \qquad (\forall f \in V, \ g, h \in G(\mathbb{A}_F)).$$

We want to construct a new representation (ρ_ϕ, V) coming from a function

$$\phi : G(\mathbb{A}_F) \to^{\text{smooth}} \mathbb{C}$$

satisfying

(i) $\phi(zg) = \omega(z)^{-1}\phi(g), \qquad \forall g \in G(\mathbb{A}_F), \ z \in Z;$

(ii) ϕ is compactly supported mod Z.

Definition 4.17.

$$\rho_\phi := \int_{G(\mathbb{A}_F)} \phi(y)\ \rho(y)\ dy$$

and ρ_ϕ acts on $f \in V$.

For $f \in V$ and $x \in G(\mathbb{A}_F)$, compute the action,

$$\begin{aligned}\rho_\phi.f(x) &= \int_{G(\mathbb{A}_F)} \phi(y)\ (\rho(y).f(x))\ \ dy \\ &= \int_{G(\mathbb{A}_F)} \phi(y)f(xy)\ dy = \int_{G(\mathbb{A}_F)} \phi(x^{-1}y)f(y)\ dy.\end{aligned}$$

(We assume that dy is a Haar measure.) Then

$$\rho_\phi.f(x) = \int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in G(F)} \phi(x^{-1}\gamma y)f(y)\ dy.$$

Definition 4.18.

$$K_\phi(x,y) := \sum_{\gamma \in G(F)} \phi(x^{-1}\gamma y).$$

Proposition 4.19.

$$\rho_\phi = \int_{G(\mathbb{A}_F)} \phi(y)\rho(y)\ dy = \int_{G(F)\backslash G(\mathbb{A}_F)} K_\phi(*,y)\ dy.$$

Since

$$\text{Vol}\,(G(F)\backslash G(\mathbb{A}_F)) < \infty$$

$\Rightarrow$ kernel $K(x,y)$ is of Hilbert-Schmidt type so of trace class!

$$\text{Tr}(\rho_\phi) = \int_{\overline{G}(F)\backslash \overline{G}(\mathbb{A}_F)} K(x,x)\ dx$$

Reference: Gelbart-Jacquet, “Forms of $GL(2)$ from the analytic point of view" (1979).

Lecture 18: 2010-11-11

- $G =$ reductive group
- $\mathbb{A} =$ adele ring over F
- $\omega : F^\times\backslash\mathbb{A}^\times \to$ root of unity (unitary Hecke character)
- $V = L^2_\omega(G(F)\backslash G(\mathbb{A})) \ni f(\gamma z g) = \omega(z)f(g)$ for any $g \in \mathbb{A}$, $z \in Z(G(\mathbb{A}))$ and $\gamma \in G(F)$

Representation $\pi : G(\mathbb{A}) \to GL(V)$ right action, i.e., $\pi(h).f(g) = f(gh)$, automorphic representation (π, V). Given $\phi(zg) = \omega^{-1}(z)\phi(g)$ for all $z \in Z(G(\mathbb{A}))$ and $g \in G(\mathbb{A})$, the function ϕ compactly supported (mod $Z(G(\mathbb{A}))$.

We may construct an operator on V

$$\pi(\phi) := \int_{G(\mathbb{A})} \phi(y)\pi(y)\ dy.$$

The operator $\pi(\phi)$ acts on $f \in V$ as follows:

$$\begin{aligned}\pi(\phi).f(x) &:= \int_{G(\mathbb{A})} \phi(y) f(xy)\ dy \\ &= \int_{G(F)\backslash G(\mathbb{A})} \underbrace{\left(\sum_{\gamma \in G(F)} \phi(x^{-1}\gamma y) \right)}_{= K_\phi(x,y)\text{ , Selberg kernel}} f(y)\ dy\end{aligned}$$

If $\iint |K_\phi(x,y)|^2\ dx\ dy < \infty$, i.e., Hilbert Schmidt, then

$$\operatorname{Tr}(\pi(\phi)) = \sum_i \underbrace{m(\pi_i)}_{\text{multiplicity}}\ \operatorname{Tr}(\pi_i(\phi)) \qquad \text{(Spectral side)}$$

where π_i is an irreducible representation with multiplicity $m(\pi_i)$.

If K_ϕ is not Hilbert-Schmidt then we can modify the Kernel to $K_\phi^\circ(x,y)$, where

$$K_\phi^\circ(x,y) = K_\phi(x,y) - \int \text{Eisenstein series}$$

For higher rank this is quite complicated and solved by Arthur.

Geometric side

$$\begin{aligned}\text{Trace} &= \int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma \in G(F)} \phi\left(x^{-1}\gamma x\right)\ dx \\ &= \int_{G(F)\backslash G(\mathbb{A})} \sum_{\substack{[\gamma] \text{ conjugacy} \\ \text{classes in } G(F)}} \sum_{\sigma \in G_\gamma(F)\backslash G(F)} \phi\left(x^{-1}\sigma^{-1}\gamma\sigma x\right)\ dx\end{aligned}$$

where $G_\gamma(F) = \{\sigma \mid \sigma\gamma = \gamma\sigma\}$. Then

$$= \sum_{[\gamma]} \int_{G_\gamma(F)\backslash G(\mathbb{A})} \phi(x^{-1}\gamma x)\ dx \qquad \text{(global orbital integral)}$$

Assume $\phi(g) = \prod_v \phi_v(g_v)$ for $g_v \in F_v$, then

$$\begin{aligned}&= \sum_{[\gamma]} \operatorname{Vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\right) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \phi(x^{-1}\gamma x)\ dx \\ &= \sum_{[\gamma]} \operatorname{Vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\right) \prod_v \int_{G_\gamma(F_v)\backslash G(F_v)} \phi_v(x_v^{-1}\gamma x_v)\ dx_v\end{aligned}$$

Then

$$\int_{G_\gamma(F_v)\backslash G(F_v)} \phi_v(x_v^{-1}\gamma x_v)\ dx_v$$

is called a local orbital integral.

Idea of Jacquet-Langlands Compare local orbital integral on different groups G and try to find " matchings" $\Rightarrow$ there exists a transfer between automorphic representations on both groups. (Langlands transfer and functoriality)

Definition 4.20. *G = algebraic group over F. Fix $K \supset F$. Let $G_\gamma = \{\sigma \mid \sigma\gamma = \gamma\sigma\}$. We define,*

(1) γ is **regular** *if G_γ is commutative*

(2) γ is **semisimple** *if G_γ is connected and reductive.*

(3) γ is **regular semisimple** *if it is both regular and semisimple $\iff$ G_γ = torus.*

(4) γ is **anisotropic** *if it is regular semisimple and G_γ = anisotropic torus.*

Elliptic, hyperbolic and parabolic?

Definition 4.21.

- *γ is* **ellipitic** *if γ is regular semisimple and its characteristic polynomial,*

$$P_\gamma(x) = x^2 - \mathrm{Tr}(\gamma)x + 1$$

is irreducible over F $\iff$ elliptic means that γ does not belong to any parabolic subgroups.

- *γ is* **hyperbolic** *if the roots of its characteristic polynomial are distinct and lie in F.*

We want to compare local orbital integrals in 2 groups, G and G'.

- $G = GL(2)$
- G' = unit group of a quaternion algebra D over a field F

Definition 4.22. (Essentially square integrable) *An irreducible admissible automorphic representation of $GL(2, F_v)$ is essentially square integrable if it is a twist of a representation which is not principal series (i.e., supercuspidal or Steinberg).*

Theorem 4.23. (Jacquet-Langlands correspondence) *There is a natural bijection*

$$\left\{ \begin{matrix} \text{Automorphic representations} \\ \text{of } G' \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Automorphic representations of } G \text{ which are essentially square} \\ \text{integrable at every place where } D \text{ ramifies} \end{matrix} \right\}$$

Further if π' is an automorphic representation of G' and π is an automorphic representation of G (corresponding) then π_v is completely determined by π'_v where π' ramifies at v.

Remark. In 2009, A. Snowden produced a thesis proving Jacquet-Langlands correspondence without the trace formula just using Fourier theory.

Quaternion algebras Fix $a, b \in F^\times$. A quaternion algebra D over F is generated by 2 elements i, j satisfying

$$i^2 = a, \quad j^2 = b, \quad ij = -ji.$$

Let

$$\begin{aligned} D &= \{x_0 + x_1 i + x_2 j + x_3 ij \mid x_0,\ x_1,\ x_2,\ x_3 \in F\} \\ &:= \left(\frac{a, b}{F}\right). \end{aligned}$$

Example

$$\left(\frac{-1, -1}{R}\right) = \text{Hamilton quaternions.}$$

Proposition 4.24. *Let v be a place of F and F_v its local field. Then there exists a unique quaternion algebra over F_v which is a division ring. Let $K \supset F$ then $D_K = D \otimes_{\overline{F}} K$ is again a quaternion algebra over K.*

Definition 4.25. *We say K splits D if $D_K \hookrightarrow M_2(K)$, where $M_2(K)$ is a 2×2 matrix group with coefficients in K.*

Definition 4.26. **(unramified vs. ramified)** *D is unramified at v if F_v splits D. Otherwise it is ramified.*

Lecture 19: 2010-11-16

We have two groups G' and G.

$$\begin{tikzcd} G' & & G \\ \text{quaternions} \arrow[u] & & GL(2) \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \text{automorphic representations of } G' \arrow[r, leftrightarrow, "\text{Jacquet-Langlands}", "\text{correspondence}"'] & \text{automorphic representations of } G \end{tikzcd}$$

co-compact finite dimensional representations	finite volume bu not co-compact infinite dimensional representations
NO parabolic conjugacy classes	there ARE parabolic conjugacy classes

To prove: choose test functions, the trace formulas match.

Work over $\mathbb{Q}$ Let's look at the contributions of the parabolic conjugacy classes for $G = GL(2)$ (over $\mathbb{Q}$). Set up the following notations.

- $\mathbb{A}$ = adeles over $\mathbb{Q}$
- $\mathbb{A}^1 = \{g \in \mathbb{A} \mid |g|_{\mathbb{A}} = 1\}$
- $G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) \mid |\det g|_{\mathbb{A}} = 1\}$
- T = diag. torus
- $B = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$ = Borel, $N = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$ = unipotent radical of B
- $\text{Vol}\left(G(\mathbb{Q})\backslash G(\mathbb{A})^1\right) < \infty$
- $L^2\left(G(\mathbb{Q})\backslash G(\mathbb{A})^1\right) = \left\{ f \in G(\mathbb{Q})\backslash G(\mathbb{A})^1 \to \mathbb{C} \;\middle|\; \int |f(g)|^2 \, dg < \infty \right\}$
- Kernel:

$$K_\phi(x, y) := \sum_{\gamma \in G(\mathbb{Q})} \phi(x^{-1}\gamma y)$$

  but not of trace class. We need to modify the kernel by subtracting an integral of Eisenstein series.

- Iwasawa decomposition
$$G(\mathbb{A}) = N(\mathbb{A}) \cdot T(\mathbb{A}) \cdot K$$
and $g = ntk$ for any $g \in G(\mathbb{A})$.

Langlands Eisenstein series

Definition 4.27. *For $g \in G(\mathbb{A}) = GL(2, \mathbb{A})$, define*
$$H(g) := H(t) := H\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \frac{1}{2}\log\left(\left|\frac{a}{b}\right|_{\mathbb{A}}\right)$$
where $g = ntk \in G(\mathbb{A})$.

Choose
$$\phi : N(\mathbb{A})T(\mathbb{Q})\mathbb{A}_\infty \backslash G(\mathbb{A})^1 \to \mathbb{C}$$
where
$$\mathbb{A}_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$
satisfying
$$||\phi||^2 = \int_K \int_{T(\mathbb{Q})\backslash T(\mathbb{A})} |\phi(tk)|^2 \; dt \; dk < \infty$$
and for an arbitrary complex number s, define
$$\phi_s(x) := \phi(x) \cdot \exp\left((s+1)H(x)\right) \qquad (\text{ for } x \in G(\mathbb{A})^1).$$
For $y \in G(\mathbb{A})$, define a right action:
$$R_s(y) \cdot \phi(x) := \exp\left((s+1)H(x)\right) \cdot \phi_s(xy).$$

Definition 4.28. (Langlnads Eisensetein series)
$$E(x, \phi, s) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_s(\gamma x) \qquad (\textit{for } x \in G(\mathbb{A})).$$

- *Intertwining operator:*
$$M(s)\phi(x) := \exp\left((s+1)H(x)\right) \int_{N(\mathbb{A})} \phi_s(wux) \; du$$
where $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Remark. Langlands proves the following properties:

- $E(xy, \phi, s) = E(x, R_s(y)\phi, s)$
- $M(s)R_s(y) = R_{-s}(y)M(-s)$
- $E(x, M(s)\phi, -s) = E(x, \phi, s)$ (Functional equation)
- $M(-s)M(s) = 1$

Proposition 4.29. (Langlnads) *The map*
$$F \mapsto \frac{1}{2}\int_{i\mathbb{R}} E(x, F, s) \; ds$$
is an isometry of Hilbert spaces of satisfying
$$F(-s) = M(s)F(s) \quad \textit{onto} \quad L^2_{\text{cont}}.$$

Definition 4.30. *For $T \in \mathbb{R}$ (T will be our cut-off), let*

$$\xi_T := \text{ characteristic function of } \left\{ g \in G(\mathbb{A})^1 \mid |H(g)| > T \right\}$$

i.e.,

$$\xi_T(g) = \begin{cases} 1, & \text{if } H(g) > T; \\ 0, & \text{otherwise.} \end{cases}$$

Hilbert-Schmidt :

$$K^T_{\text{modified}}(x,y) =^? K(x,y) - \frac{1}{2}\int_{i\mathbb{R}} K(x,y)E(x, \underbrace{R_s\phi}_{\leftarrow \xi_T}, s)\, ds$$

(This will be Hilbert-Schmidt)

Trace formula for $G = GL(2)$ For $T \in \mathbb{R}$ we have the modified kernel:

$$K^T_\phi(x,y) = K_\phi(x,y) - \sum_{\delta \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \xi_T(\delta x) \cdot \int_{N(\mathbb{A})} \sum_{\gamma \in T(\mathbb{Q})} \phi(x^{-1}\gamma n y)\, dn$$

(Hilbert-Schmidt). Then we have the trace formula:

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} K^T_\phi(x,x)\, dx.$$

Jacquet-Langlands correspondence

$$\text{units in quaternion algebra} = G' \longleftrightarrow G = GL(2)$$

Let v be a place of $\mathbb{A}$.

- v unramified for $G' \hookrightarrow GL(2, \mathbb{A})$
- v ramified for G'

Let $S = \{$ ramified places$\}$. We want to choose test functions

$$\phi = \prod_v \phi_v, \qquad \phi' = \prod_v \phi'_v,$$

so that the trace formula for G and G' match.

$$\lim_{T\to 0} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} K^T_\phi(x,x)\, dx = \int_{G(\mathbb{Q})\backslash G'(\mathbb{A})} K_{\phi'}(x,x)\, dx$$

(i) If v is unramified for G' then it can be shown that if we choose $\phi_v = \phi'_v$ then the local orbital integrals match.

(ii) If v is ramified, choose $\phi_v = \phi'_v = 1$.

Then it only remains to eliminate the contribution of the parabolic conjugacy classes on G. (Bottom of p. 244 in Gelbart-Jcquet "Forms of $GL(2)$ from the analytic point of view".)

$$\int \phi\left(g^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} g\right) dg = \lim_{a\to 1} \left|1 - a^{-1}\right| \int \phi\left(g^{-1}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) dg$$

($\to$ this is "basically" what appears), i.e., the nilpotent orbital integral is a limit of hyperbolic orbital integrals
$\Rightarrow$ with previous choices, the contribution of the parabolic conjugacy classes is 0.

Lecture 20: 2010-11-18

5 Kuznetsov Trace formula

Kuznetsov Trace formula: use double coset
-geometry side is quite different

↓

Jacquet's Relative trace formula

Reference : "On the estimation of Fourier coefficients of modular forms" by A. Selberg, PSPM 1965

5.1 Kloosterman sums for $SL(2,\mathbb{Z})$

For $c \geqslant 1$,

$$S(m,n;c) = \sum_{\substack{a=1,(a,c)=1,\\ a\overline{a}\equiv 1 \pmod{c}}}^{c} e^{2\pi i\left(\frac{am+\overline{a}n}{c}\right)}.$$

Selberg introduced the Kloosterman zeta function

$$Z(m,n;s) = \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c^{2s}}.$$

Theorem 5.1. **(Selberg)** *The Kloosterman zeta function $Z(m,n;s)$ converges absolutely and uniformaly for $\Re(s) > \frac{3}{4}$ and it has a meromorphic continuation to all $s \in \mathbb{C}$ with simple poles at*

$$s = \frac{1}{2} + ir_j$$

where $\frac{1}{4} + r_j^2 = \lambda_j$ is an eigenvalue of the Laplacian Δ, i.e., there exists a Maass form $\eta_j(z) \in L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$ and $\Delta\eta_j = \lambda_j\eta_j$.

Remark. With a modification, the Kloosterman zeta function has poles only at λ_j, and has a functional equation.

Proposition 5.2. **(Properties of Kloosterman sums)**

(1) $S(m,n;c) = S(n,m;c)$

(2) $S(m,n;cc') = S(m\overline{c},n\overline{c};c')\cdot S(m\overline{c'},n\overline{c'};c)$ *provided* $(c,c') = 1$ *and* $c\overline{c} \equiv 1 \pmod{c'}$, $c'\overline{c'} \equiv 1 \pmod{c}$. *(Kloosterman zeta function doesn't have an Euler product, but it has a multiplicative relation in it.)*

(3) $S(m,n;c) = S(mn,1;c)$, *if* $(m,n,c) = 1$.

(4) $S(m,n;c) = \sum_{r|(m,n,c)} rS\left(\frac{mn}{r^2},1;\frac{c}{r}\right)$, *(Selberg, 1938, Helsinki).*

(5) $S(m,n;c) = S(-m,-n;c)$, *so* $S(m,n,;c) \in \mathbb{R}$.

Proof.

$$S(m,n;c) = \sum_{\substack{a=1,(a,c)=1,\\ a\overline{a}\equiv 1 \mod c}}^{c} e^{2\pi i\left(\frac{am+\overline{a}n}{c}\right)}$$

(1) $a \mapsto \overline{a}$

(2)

$$\begin{aligned}
&S(m\bar{c}, n\bar{c}; c') \cdot S(m\overline{c'}, n\overline{c'}; c) \\
&= \left(\sum_{a'=1}^{c'} e^{2\pi i \frac{a' m\bar{c} + \overline{a'} n\bar{c}}{2}}\right) \cdot \left(\sum_{a=1}^{c} e^{2\pi i \frac{am\overline{c'} + an\overline{c'}}{2}}\right) \\
&= \sum_{a'} \sum_{a} e^{2\pi i \frac{\left[m(c\bar{c}a' + c'\overline{c'}a) + n(c\bar{c}\overline{a'} + c'\overline{c'}\bar{a})\right]}{cc'}}
\end{aligned}$$

(3) It is enough to prove when $c = p^{\ell}$. Either $(m,p) = 1$ or $(n,p) = 1$. If $(m,p) = 1$ then let $a \mapsto na$ and $(n,p) = 1$ then let $a \mapsto ma$.

□

Growth of Kloosterman sums

Theorem 5.3. (A. Weil)

$$|S(m,n;p)| \leqslant 2\sqrt{p}, \qquad (p \nmid nm).$$

Remark. This theorem is a deep result, using RH, for curve over finite fields.

Theorem 5.4. (Salie)

$$S(n,n;p^{\alpha}) = \begin{cases} 2p^{\frac{\alpha}{2}} \cos\left(\frac{4\pi n}{p^{\alpha}}\right), & \alpha \textit{ even} \\ 2\left(\frac{n}{p}\right) p^{\frac{\alpha}{2}} \cos\left(\frac{4\pi n}{p^{\alpha}}\right), & \alpha \textit{ odd},\ p \equiv 1 \mod 4 \\ -2\left(\frac{n}{p}\right) p^{\frac{\alpha}{2}} \sin\left(\frac{4\pi n}{p^{\alpha}}\right), & \alpha \textit{ odd},\ p \equiv 3 \mod 4 \end{cases}$$

If $p \nmid mn$,

$$S(m,n;p^{\alpha}) = 0$$

unless $m \equiv nu^2 \mod p^{\alpha}$ *for some* $u \in \mathbb{Z}/p^{\alpha}\mathbb{Z}$ *and in this case*

$$S(m,n;p^{\alpha}) = S(nu,nu;p^{\alpha}).$$

5.2 Poincaré series for $\Gamma = SL(2,\mathbb{Z})$

Definition 5.5. (Poincaré Series) *Fix* $n \in \mathbb{Z}$. *Let* $z = x + iy \in \mathbb{H}^2$, $s \in \mathbb{C}$ *and*

$$\begin{aligned}
P_n(z,s) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s e^{2\pi i n \gamma(z)} \\
&= \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} e^{2\pi i n \frac{az+b}{cz+d}}, \qquad \left(\textit{for } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash SL(2,\mathbb{Z})\right)
\end{aligned}$$

where $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \subset SL(2,\mathbb{Z})$.

Remark. (i) When $n = 0$, $P_0(z,s) = E(z,s)$.

(ii) Kloosterman sums will appear in the Fourier expansion of $P_n(z,s)$. Assume that $\Re(s) > 1$. Let's look at the mth coefficient

$$\begin{aligned}
&\int_0^1 P_n(z,s) e^{-2\pi i m x}\, dx \\
&= \int_0^1 \sum_{c,d} \frac{y^s}{[(cx+d)^2 + c^2y^2]^s} e^{2\pi i n \frac{az+b}{cz+d} - 2\pi i m x}\, dx
\end{aligned}$$

Let $x \mapsto x - \frac{d}{c}$. Then

$$= \underbrace{\delta_{m,n} y^s e^{-2\pi n y}}_{\text{for } c=0} + \sum_{\substack{(c,d)=1,\\ c\neq 0}} \int_{-\frac{d}{c}}^{\frac{-d}{c}+1} \frac{y^s}{c^{2s}(x^2+y^2)^s} e^{2\pi i n \frac{az+b-\frac{ad}{c}}{cz} - 2\pi i m(x-\frac{d}{c})}\, dx$$

Look at the second term. Let $d = \ell c + r$, we have

$$y^s \sum_{c\neq 0} \frac{1}{c^{2s}} \sum_{\ell\in\mathbb{Z}} \sum_{\substack{r=1,(r,c)=1,\\ ar\equiv 1 \pmod c}}^{c} \int_{-\ell-\frac{r}{c}}^{1-\ell-\frac{r}{c}} \frac{e^{2\pi i n\left(\frac{a}{c}-\frac{1}{c^2 z}\right)}}{(x^2+y^2)^s} e^{-2\pi i m(x-\ell-\frac{r}{c})}\, dx$$

since $e^{2\pi i m\ell} = 1$, we have ($ar \equiv 1 \pmod c$)

$$= y^s \sum_{c\neq 0} \frac{1}{c^{2s}} \sum_{\substack{r=1,(r,c)=1,\\ ar\equiv 1 \pmod c}}^{c} \int_{-\infty}^{\infty} \frac{e^{\frac{-2\pi i n}{c^2 z}} e^{\frac{2\pi i n a}{c}} e^{2\pi i m\frac{r}{c}}}{(x^2+y^2)^s} e^{-2\pi i m x}\, dx$$

$$= y^s \sum_{c\neq 0} \frac{S(m,n;c)}{c^{2s}} \int_{-\infty}^{\infty} \frac{e^{\frac{-2\pi i n}{c^2 z}} e^{-2\pi i m x}}{(x^2+y^2)^s}\, dx$$

To proceed further, use Taylor expansion.

$$e^{-\frac{2\pi i n \bar z}{c^2|z|^2}} = \sum_{\ell=0}^{\infty} (-1)^\ell \left(\frac{2\pi i n \bar z}{c^2|z|^2}\right)^\ell$$

So

$$\underbrace{\int_0^1 P_n(z,s) e^{-2\pi i m x}\, dx}_{\text{Spectral side}} = y^s \sum_{\ell=0}^{\infty} (-1)^\ell \underbrace{Z(m,n;s+\ell)}_{\text{geometric side}} \int_{-\infty}^{\infty} \frac{(2\pi i n(x-iy))^\ell}{(x^2+y^2)^{s+\ell}} e^{-2\pi i n x}\, dx$$

$$+ \delta_{m,n} y^s e^{-2\pi n y}.$$

Lecture 21: 2010-11-23

Review of Poincaré series for $SL(2,\mathbb{Z})$

$$P_n(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s\, e^{2\pi i n\gamma(z)}$$

(when $n = 0$, it is an Eisenstein series) We worked out the Fourier expansion

$$\int_0^1 P_n(x+iy,s) e^{-2\pi i m x}\, dx = \delta_{m,n} y^s e^{-2\pi n y} + \sum_{\ell=0}^{\infty} Z_{m,n}(s+\ell) H_\ell(s+\ell, ny)$$

where $\delta_{m,n} = \begin{cases} 1, & m = n; \\ 0, & m \neq n \end{cases}$,

$$Z_{m,n}(s) = \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c^{2s}}$$

and

$$H_0(s,ns) = \frac{2\pi^s n^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi n y)}{\Gamma(s)}.$$

This Fourier expansion is the geometric side.

Spectral side

Proposition 5.6. *If* $n \neq 0$,

$$P_n(z,s) \in L^2\left(\Gamma\backslash\mathbb{H}^2\right) \quad \textit{provided } \Re(s) > 1.$$

Idea of Proof. For Eisenstein series,

$$E(z,s) = P_0(z,s) = y^s + \phi(s)y^{1-s} + \sum \underbrace{\text{other stuff}}_{\substack{\text{rapid decay as}\\ y\to\infty}}$$

No matter how we choose s one of y^s or y^{1-s} is not L^2. But for $n > 0$,

$$P_n(z,s) = \underbrace{y^s e^{2\pi i n x} e^{-2\pi n y}}_{\substack{L^2\text{, because } e^{-2\pi n y}\\ \text{decays rapidly}}} + \sum_{\substack{\gamma\neq\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\\ \gamma\in\Gamma_\infty\backslash\Gamma}} \underbrace{\Im(\gamma z)^s\, e^{2\pi i n\gamma(z)}}_{\ll y^{1-\Re(s)}}$$

$\square$

Recall the Selberg Spectral decomposition, if $F \in L^2$ then

$$F(z) = \sum_{j=0}^{\infty} \langle F, \eta_j\rangle\, \eta_j(z) + \frac{1}{4\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle F, E(*,w)\rangle\, E(z,w)\, dw$$

where η_j is a normalized Maass form for $j = 1, 2, \ldots$ and $\eta_0 = \text{Vol}\left(\Gamma\backslash\mathbb{H}^2\right)^{-\frac{1}{2}}$. So, we have

$$P_n(z) = \sum_{j=0}^{\infty} \langle P_n(*,s), \eta_j\rangle\, \eta_j(z) + \frac{1}{4\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle P_n(*,s), E(*,w)\rangle\, E(z,w)\, dw.$$

Let's compute (for $\Re(s) > 1$ and $n > 0$)

$$\begin{aligned}
\langle P_n(*,s), \eta_j\rangle &= \iint_{\Gamma\backslash\mathbb{H}^2} P_n(z,s)\overline{\eta_j(z)}\, \frac{dx\, dy}{y^2} \\
&= \iint_{\Gamma_\infty\backslash\mathbb{H}^2} y^s e^{2\pi i n z}\overline{\eta_j(z)}\, \frac{dx\, dy}{y^2} \\
&= \int_{x=0}^{1}\int_{y=0}^{\infty} y^s e^{2\pi i n x} e^{-2\pi n y} \sum_{m\neq 0} a_j(m)\sqrt{y}K_{ir}(2\pi|m|y)\, e^{-2\pi i m x}\, \frac{dx\, dy}{y^2} \\
&= a_j(n)\int_0^{\infty} K_{ir_j}(2\pi n y) y^{s-\frac{1}{2}} e^{-2\pi n y}\, \frac{dy}{y} \\
&= a_j(n)\frac{\sqrt{\pi}}{(4\pi n)^{s-\frac{1}{2}}}\frac{\Gamma(s-\frac{1}{2}+ir_j)\Gamma(s-\frac{1}{2}-ir_j)}{\Gamma(s)}
\end{aligned}$$

Then

$$\begin{aligned}
P_n(z) = {} & \frac{\sqrt{\pi}}{(4\pi n)^{s-\frac{1}{2}}\Gamma(s)}\sum_{j=1}^{\infty} a_j(n)\Gamma\left(s-\frac{1}{2}+ir_j\right)\Gamma\left(s-\frac{1}{2}-ir_j\right)\eta_j(z) \\
& + \frac{1}{4\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle P_n(*,s), E(*,w)\rangle\, E(z,w)\, dw
\end{aligned}$$

We actually need to show that "$\sum_{j=1}^{\infty} \cdots$" is convergent. Since η_j is normalized, we have $a_j(n) = a_j(1)\rho_j(n)$ and $\rho_j(n)$ doesn't depend on r_j. But $a_j(1)$ depends on r_j since

$$\iint\limits_{\Gamma\backslash\mathbb{H}^2} |\eta_j(z)|^2 \, \frac{dx\,dy}{y^2} = 1 \geqslant \int_{\frac{\sqrt{3}}{2}}^{\infty} \left|K_{ir_j}(2\pi y)\right|^2 \, \frac{dy}{y^2} \cdot a_j(1)$$

and because $K_{ir_j}(y)$ is rapidly decay as $y \to \infty$, the coefficient $a_j(1)$ blows up. But Γ factor in the summation is decay. (See Hoffstein-R?)

Let's compute

$$\langle P_n(*,s), E(*,w)\rangle = \int_{x=0}^{1}\int_{y=0}^{\infty} y^s e^{2\pi nz}\,\overline{E(z,w)}\,\frac{dx\,dy}{y^2}.$$

Since

$$E\left(z, \frac{1}{2}+ir\right) = y^{\frac{1}{2}+ir} + \phi\left(\frac{1}{2}+ir\right) y^{\frac{1}{2}-ir} + \sum_{m\neq 0} \frac{2\pi^{\frac{1}{2}+ir}\,|m|^{ir}\sigma_{-2ir}(m)}{\Gamma\left(\frac{1}{2}+ir\right)\zeta(1+2ir)} \sqrt{y}K_{ir}(2\pi|m|y)\, e^{2\pi imx},$$

we have

$$\langle P_n(*,s), E(*,w)\rangle = \frac{2\pi^{\frac{1}{2}+ir} n^{ir}\sigma_{-2ir}(n)}{\Gamma\left(\frac{1}{2}+ir\right)\zeta(1+2ir)} \cdot \int_0^{\infty} y^s e^{-2\pi ny}\sqrt{y}K_{ir}(2\pi ny)\,\frac{dy}{y^2}.$$

5.3 Kuznetsov Trace formula (preliminary coarse version)

For a positive integer m, we have

$$\begin{aligned}
&\int_0^1 P_n(x+iy,s)e^{-2\pi imx}\,dx \\
&= \delta_{m,n}y^s e^{-2\pi ny} + \sum_{\ell=0}^{\infty}\sum_{c=1}^{\infty} \frac{S(m,n;c)}{c^{2s+2\ell}} H_\ell(s+\ell, ny) \\
&= \frac{\sqrt{\pi}}{(4\pi n)^{s-\frac{1}{2}}\Gamma(s)} \sum_{j=1}^{\infty} \Gamma\left(s-\frac{1}{2}+ir_j\right)\Gamma\left(s-\frac{1}{2}+ir_j\right) a_j(m)\overline{a_j(n)}\sqrt{y}K_{ir_j}(2\pi my) \\
&+ \frac{1}{2}\int_{-\infty}^{\infty} \frac{(\pi m)^{-ir}}{\Gamma\left(\frac{1}{2}-ir\right)\zeta(1-2ir)}\sigma_{2ir}(m)\overline{\sigma_{2ir}(n)}\frac{\Gamma\left(s-\frac{1}{2}+ir\right)\Gamma\left(s-\frac{1}{2}-ir\right)}{(4\pi n)^{s-\frac{1}{2}}\Gamma(s)}\sqrt{y}K_{ir}(2\pi my)\,\frac{dy}{y}.
\end{aligned}$$

The spectral side has meromorphic continuation to all $s \in \mathbb{C}$. There will be poles at $s = \frac{1}{2} + ir_j - k$ (for $k = 0, 1, 2, \ldots$) and

$$\text{Residue} = \frac{\sqrt{\pi}\Gamma(2ir_j)}{(4\pi n)^{ir_j}\Gamma(\frac{1}{2}+ir_j)} a_j(n)\overline{a_j(n)}\sqrt{y}K_{ir_j}(2\pi my).$$

Theorem 5.7. **(Selberg)** *Let $m > 0$ and $n \neq 0$ be integers. Then the Kloosterman zeta function*

$$Z_{m,n}(s) = \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c^{2s}}$$

has meromorphic continuation to all $s \in \mathbb{C}$ with simple poles at $s = \frac{1}{2} \pm ir_j - k$ for $k = 0, 1, 2, \ldots,$ and $j = 1, 2, \ldots$. Moreover,

$$\operatorname*{Res}_{s=\frac{1}{2}+ir_j} Z_{m,n}(s) = \frac{1}{2}a_j(m)a_j(n)(m|n|)^{-ir_j}\frac{\Gamma(2ir_j)}{(2\pi)^{2ir_j}}.$$

Theorem 5.8. (Goldfeld-Sarnak) *Let $m > 0$ and $n \neq 0$ be integers. Then $Z_{m,n}(s)$ is holomorphic if $\Re(s) > \frac{1}{2}$ and satisfies the growth condition*

$$|Z_{m,n}(s)| = O\left(\frac{|mn||s|^{\frac{1}{2}}}{\Re(s) - \frac{1}{2}}\right), \qquad (\Re(s) > \frac{1}{2}).$$

Proof. The proof is based on the following lemma.

Lemma 5.9.

$$\iint\limits_{\Gamma\backslash\mathbb{H}^2} |P_m(z,s)|^2 \ \frac{dx\,dy}{y^2} = O\left(|m|\frac{|s|}{|\Im(s)|\cdot(\Re(s) - \frac{1}{2})}\right)$$

If we take the inner product of two Poincaré series, then we get the Kloosterman zeta function.

Let $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. Consider

$$(\Delta - s(1-s))\cdot y^s e^{2\pi imx}e^{-2\pi my} = 4\pi msy^{s+1}e^{2\pi imz}.$$

Define the resolvent operator

$$R_\lambda := (\Delta - \lambda)^{-1}$$

$$\Rightarrow\ P_m(z,s) = 4\pi ms \cdot R_{s(1-s)}P_m(z, s+1)$$

(i.e., Poincaré series is the shifted eigenfunction of the Resolvent operator) and using the theorem of analysis, we have

$$||R_\lambda|| \leqslant \frac{1}{\text{dist}(\lambda,\ \text{spectrum of }\Delta)}$$

$$\Rightarrow\ ||P_m(*,s)|| \leqslant |4\pi ms|\,\frac{||P_m(z,s+1)||}{\text{dist}(s(1-s),\mathbb{R})}$$

$$\text{dist}(s(1-s),\mathbb{R}) \geqslant \Im(s(1-s))$$

If $s = \sigma + it$, then

$$\Im(s(1-s)) = \Im((\sigma+it)(1-\sigma-it)) = t(2\sigma - 1).$$

and plug them to the bound, we get the lemma. □

Lecture 22: 2010-11-30

Last time we found an identity

$$\text{sums of shifted Kloosterman zetas} = \text{eigenfunction of } \Delta \text{ spectral data}$$

- LHS: Kloosterman zeta function

  $$Z_{m,n}(s) = \sum_{c=1}^{\infty}\frac{S(m,n;c)}{c^{2s}}$$

  for

  $$S(m,n;c) = \sum_{\substack{a=1,\\(a,c)=1}}^{c} e^{2\pi i\frac{am+\bar{a}n}{c}}$$

  where $a\bar{a} \equiv 1 \pmod{c}$.

- Kuznetsov inserted an arbitrary test function into Selberg's identity.
- on the geometric side, we will explain how to compute with double cosets.

Bruhat Decomposition

$$GL(n) = B_n W_n B_n, \qquad \text{for } B_n = \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}$$

and $W_n =$ Weyl group of $n \times n$ matrices which have exactly one “1" in each row and column otherwise zero.

Proof of Bruhat Decomposition. Let

$$g = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \cdots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \in GL(n, F)$$

then there exists $b_1 \in B_n$ such that

$$gb_1 = \begin{pmatrix} & * & \\ 0\dots0 & 1 & 0\dots0 \end{pmatrix}$$

where 1 lies in the position $n\ell$ (cleaning the last row). There exists $b_2 \in B_n$ such that

$$b_2 g b_1 = \begin{pmatrix} * & & * \\ \dots & 0 & \dots \\ 0\dots0 & 1 & 0\dots0 \end{pmatrix}.$$

In the finite number of similar procedure, we get the Bruhat decomposition. ☐

Bruhat decomposition for $GL(2)$: Since $W_2 = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\}$, every $\gamma \in GL(2, \mathbb{R})$ is in one of two possible forms.

$$\begin{aligned} \gamma &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} & \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (c = 0) \\ \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} & \leftarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (c \neq 0) \end{cases} \end{aligned}$$

5.4 Kuznetsov Trace formula

Let

$$P_n(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} p\left(2\pi n \Im(\gamma z)\right) \sqrt{\Im(\gamma z)} e^{2\pi i n \Re(\gamma z)}$$

$$Q_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} q\left(2\pi m \Im(\gamma z)\right) \sqrt{\Im(\gamma z)} e^{2\pi i m \Re(\gamma z)}$$

where $p, q : \mathbb{H} \to \mathbb{C}$, smooth, rapid decay. To get the Kuznetsov trace formula, we compute

$$\langle P_n, Q_m \rangle$$

in two ways.

Geometric side Let $\Gamma = SL(2,\mathbb{Z})$

$$\begin{aligned}\langle P_n, Q_m\rangle &= \iint\limits_{\Gamma\backslash\mathbb{H}^2} P_n(z) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{q(2\pi m\Im(\gamma z))\sqrt{Im(\gamma z)}e^{-2\pi i m\Re(\gamma z)}}\ \frac{dx\ dy}{y^2}\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\ \iint\limits_{\gamma^{-1}\cdot\Gamma\backslash\mathbb{H}^2} P_n(z)\overline{q}(2\pi m y)\sqrt{y}e^{-2\pi i m x}\ \frac{dx\ dy}{y^2}\\ &= \int_0^1\int_0^\infty P_n(z)\overline{q(2\pi m y)}\sqrt{y}e^{-2\pi i m x}\ \frac{dx\ dy}{y^2}\end{aligned}$$

We now compute,

$$\begin{aligned}\int_0^1 P_n(z)e^{-2\pi m x}\ dx &= \int_0^1 \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} p(2\pi n\Im(\gamma z))\cdot\sqrt{\Im(\gamma z)}e^{2\pi i n\Re(\gamma z)}e^{-2\pi i m x}\ dx\\ &= \underbrace{\int_0^1 p(2\pi n y)\sqrt{y}e^{2\pi i(n-m)x}\ dx}_{=\delta_{m,n}p(2\pi n y)\sqrt{y}}\\ &\quad + \sum_{\substack{c,b,\\ (c,b)=1}} \int_0^1 p\left(2\pi n\Im\left(\left(\begin{smallmatrix} c^{-1} & 0\\ 0 & c\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)\begin{pmatrix} 1 & \overline{b}/c\\ 0 & 1\end{pmatrix} z\right)\right)\sqrt{(\ )}e^{-2\pi i n\Re(\ )}e^{-2\pi i m x}\ dx\end{aligned}$$

where $\gamma = \begin{pmatrix} b & \frac{b\overline{b}-1}{c}\\ c & \overline{b}\end{pmatrix}$, for $b\overline{b}\equiv 1 \pmod{c}$. Then

$$\begin{aligned}&\sum_{c,b}\int_0^1 p\left(\frac{2\pi n}{c^2}\Im\left(\frac{-1}{z+\overline{b}/c}\right)\right)\sqrt{\frac{1}{c^2}\cdot\Im\left(\frac{-1}{z+\overline{b}/c}\right)}\cdot e^{-2\pi i n\frac{b}{c}}e^{-2\pi i\frac{n}{c^2}\Re\left(-\frac{1}{z+\overline{b}/c}\right)}e^{-2\pi i m x}\ dx\\ &= \sum_{c,b}\int_{-\overline{b}/c}^{1-\overline{b}/c} p\left(\Im\frac{-2\pi n}{c^2 z}\right)\sqrt{\Im\left(-\frac{1}{c^2 z}\right)}\underbrace{e^{\frac{-2\pi i n b}{c}}e^{\frac{-2\pi i m\overline{b}}{c}}}_{\text{Kloosterman sum}}e^{\frac{-2\pi i n}{c^2}\Re(-\frac{1}{z})}e^{-2\pi i m x}\ dx.\end{aligned}$$

Finally we have,

$$\begin{aligned}&\int_0^1 P_n(z)e^{-2\pi i m x}\ dx\\ &= \sum_{c\neq 0} S(m,n;c)\cdot\int_{-\infty}^{\infty}\sqrt{\frac{1}{c^2 y(x^2+1)}}p\left(\frac{2\pi|n|}{c^2 y(x^2+1)}\right)e^{-2\pi i\frac{nx}{c^2 y(x^2+1)}}e^{-2\pi i m x y}\ dx.\end{aligned}$$

So,

$$\begin{aligned}\langle P_n, Q_m\rangle = \sum_{c\neq 0} S(m,n;c)\int_{x=-\infty}^{\infty}\int_0^\infty q(2\pi m y)\sqrt{y}\sqrt{\frac{1}{c^2 y(x^2+1)}}\\ \cdot p\left(\frac{2\pi|n|}{c^2 y(x^2+1)}\right)e^{-2\pi i\frac{nx}{c^2 y(x^2+1)}}e^{-2\pi i m x y}\ dx\ \frac{dy}{y^2}\end{aligned}$$

Spectral side

$$P_n(z) = \sum_{j=0}^{\infty}\langle P_n, \eta_j\rangle\,\eta_j(z) + \frac{1}{4\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+\infty}\langle P_n, E(*,s)\rangle\,E(z,s)\ ds$$

and

$$\begin{aligned}\langle P_n,\eta_j\rangle &= \iint\limits_{\Gamma\backslash\mathbb{H}^2} P_n(z)\overline{\eta_j(z)}\ \frac{dx\ dy}{y^2} = \int_0^1\int_0^\infty p(2\pi ny)\sqrt{y}e^{2\pi inx}\overline{\eta_j(z)}\ \frac{dx\ dy}{y^2}\\ &= \sqrt{2\pi}\cdot\overline{A_j(n)}\underbrace{\int_0^\infty p(y)K_{ir_j}(y)\ \frac{dy}{y^{\frac{3}{2}}}}_{\substack{\text{Kontorobich-Lebedev transform}\\ :=p^\sharp(r_j)}}\end{aligned}$$

So

$$P_n(z) = \sqrt{2\pi}\sum_{j=0}^{\infty}\overline{A_j(n)}p^\sharp(r_j)\eta_j(z) + \text{Similar Eisenstein series.}$$

Therefore,

$$\langle P_n, Q_m\rangle = 2\pi\sum_j A_j(n)\overline{A_j(m)}p^\sharp(r_j)\overline{q^\sharp(r_j)} +\ \text{Cont.}$$

As an application, Kuznetsov proved

$$\sum_{c=1}^{X}\frac{S(m,n;c)}{c} \ll \begin{cases} X^{\frac{1}{6}+\epsilon}, \\ X^\epsilon,\ \text{conjectured by Selberg}\end{cases}.$$

Lebedev Inversion

$$\begin{aligned}\psi^\sharp(t) &= \int_0^\infty \psi(y)K_{it}(y)\ \frac{dy}{y}\\ \psi(y) &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{\psi^\sharp(t)K_{it}(y)}{\Gamma(it)\Gamma(-it)}\ dt.\end{aligned}$$

The problem of $A_j(n)$ Assume that

$$\begin{aligned}||\eta||_2^2 &= \iint\limits_{\Gamma\backslash\mathbb{H}^2}|\eta_j(z)|^2\ \frac{dx\ dy}{y^2} = 1\\ &= \frac{1}{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^2)}\iint\limits_{\Gamma\backslash\mathbb{H}^2}|\eta(z)|^2\ \operatorname*{Res}_{s=1}E(z,s)\ \frac{dx\ dy}{y^2} = 1\\ &= \operatorname*{Res}_{s=1}\frac{1}{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^2)}\int_0^1\int_0^\infty|\eta(z)|^2\ y^s\ \frac{dx\ dy}{y^2}\\ &= \operatorname*{Res}_{s=1}\frac{1}{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^2)}\int_0^1\int_0^\infty\left|\sum_{n\neq 0}A(n)K_{ir}(2\pi|n|y)\ e^{2\pi inx}\right|^2\ y^s\ \frac{dx\ dy}{y^2}\\ &= \frac{1}{\mathrm{Vol}}\operatorname*{Res}_{s=1}\sum\int_0^\infty\frac{A(n)^2}{n^s}\frac{\sqrt{\pi}\Gamma\left(\frac{s-2ir}{2}\right)\Gamma\left(\frac{s-2ir}{2}\right)}{4\Gamma\left(\frac{s}{2}\right)}\\ &= \operatorname*{Res}_{s=1}\frac{A(1)^2}{\mathrm{Vol}}L(s,\eta\times\eta)\frac{\sqrt{\pi}\Gamma\left(\frac{s-2ir}{2}\right)\Gamma\left(\frac{s+2ir}{2}\right)}{4\Gamma\left(\frac{s}{2}\right)}\end{aligned}$$

where $A(n) = A_r(1)\cdot a(n)$.

$$|A_j(1)|^2 = \frac{\mathrm{Vol}\cdot 8}{\Gamma\left(\frac{1}{2}-ir\right)\Gamma\left(\frac{1}{2}+ir\right)L(1,\mathrm{Sym}^2\eta)}.$$

Lecture 23: 2010-12-7

Let $\mathcal{D}$ = discrete spectrum, $\mathcal{E}$ = Eisenstein series, $\mathcal{M}$ = main term and $\mathcal{K}$ = Kloosterman sums Then

$$\underbrace{\mathcal{D}+\mathcal{E}}_{\text{spectral side}} = \underbrace{\mathcal{M}+\mathcal{K}}_{\text{geometric side}} .$$

Notations

- $\Delta\eta_j = \left(\frac{1}{4}+r_j^2\right)\eta_j$
- $\eta_j(z) = \sum_{n\neq 0}\mu_j(n)\sqrt{y}K_{ir_j}(2\pi|n|y)e^{2\pi inx}$: Maass forms
- $p, q : \mathbb{H}^2 \to \mathbb{C}$ test functions
- Lebedev-Bessel transform:

$$\psi(y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\psi^\sharp(t)K_{it}(y)}{\Gamma(it)\Gamma(-it)}\,dt, \qquad \psi^\sharp(t) = \int_0^\infty \psi(y)K_{it}(y)\,\frac{dy}{y}$$

Then we have

$$\mathcal{D} = \frac{16\pi\sqrt{nm}}{\mathrm{Vol}\,(\Gamma\backslash\mathbb{H}^2)}\sum_j \frac{\overline{\mu_j}(n)\mu_j(m)p^\sharp(t_j)q^\sharp(t_j)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)L\left(1,\mathrm{Sym}^2(\eta_j)\right)},$$

$$\mathcal{E} = \frac{1}{4\pi}\int_{\mathbb{R}} \underbrace{\mathfrak{E}\left(n,\frac{1}{2}+it\right)}_{n\text{th coefficient}}\mathfrak{E}\left(m,\frac{1}{2}+it\right)\,p^\sharp(t)q^\sharp(t)\,dt,$$

$$\mathfrak{M} = \int_0^\infty p(ny)q(my)\,\frac{dy}{y},$$

and

$$\mathcal{K} = \int_0^\infty y^{\frac{1}{2}}q(2\pi my)\sum_{c\neq 0}\frac{S(m,n;c)}{c} \times \int_0^\infty (u^2+1)^{-\frac{1}{2}}p\left(\frac{2\pi|n|}{c^2y(u^2+1)}\right)\exp\left(\frac{-2\pi iny}{c^2y(u^2+1)}\right)\,e^{-2\pi imny}\,du\,\frac{dy}{y}.$$

5.5 Application in the direction to the Analytic number theory

Theorem 5.10. (Goldfeld-Kontorovich) *Let* $\Gamma = SL(2,\mathbb{Z})$. *Then*

$$\frac{16\pi}{\mathrm{Vol}\,(\Gamma\backslash\mathbb{H}^2)}\sum_{|t_j|<T}\frac{1}{L\left(1,\mathrm{Sym}^2\eta_j\right)} = \underbrace{\frac{1}{\pi}\int_{-T}^{T}\frac{\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right)}{\Gamma(it)\Gamma(-it)}\,dt}_{\approx T^2} + O(T)$$

$$\Rightarrow \sum_{|t_j|\leqslant T} 1 \sim cT^2.$$

Proof. (1) Discrete side $\mathcal{D}$: Choose $m = n = 1$. Let

$$1_{[-T,T]}(t) = \begin{cases} 1, & -T \leqslant t \leqslant T, \\ 0, & \text{otherwise} \end{cases}$$

Choose

$$q^\sharp(t) = 1_{[-T,T]}(t)\, \frac{\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right)}{p^\sharp(t)}$$

$$\Rightarrow\ \mathcal{D} = \frac{16\pi}{\mathrm{Vol}\left(\Gamma\backslash\mathbb{H}^2\right)} \sum_{-T\leqslant t_j\leqslant T} \frac{1}{L\left(1, \mathrm{Sym}^2(\eta_j)\right)}$$

(2) Main term $\mathcal{M}$:

$$\begin{aligned} \mathcal{M} &= \int_0^\infty p(y)q(y)\,\frac{dy}{y} \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{p^\sharp(t)q^\sharp(t)}{\Gamma(it)\Gamma(-it)}\,dt \\ &= \frac{1}{\pi}\int_{-T}^{T} \frac{\Gamma\left(\frac{1}{2}-it\right)\Gamma\left(\frac{1}{2}+it\right)}{\Gamma(it)\Gamma(-it)}\,dt \sim \frac{T^2}{\pi} \end{aligned}$$

Now choose p intelligently to make everything else small.

(3) Eisenstein series $\mathcal{E}$:

$$\begin{aligned} \mathcal{E} &= \frac{1}{4\pi}\int_{\mathbb{R}} \frac{p^\sharp(t)q^\sharp(t)}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2\,|\zeta(1+2it)|^2}\,dt \\ &= \frac{1}{4}\int_{-T}^{T} \frac{1}{|\zeta(1+2it)|^2}\,dt = \mathcal{O}(T) \end{aligned}$$

by using the prime number theorem. ($\zeta(1) \neq 0$)

(4) Kloosterman sum $\mathcal{K}$:

$$\mathcal{K} = \text{complicated} \ll T.$$

We need to choose $p(y)$ in a special way:

$$p(y) = yK_0(y).$$

□

6 Jacquet's relative trace formula

Notations

- F = number field
- $\mathbb{A}$ = adele ring of F
- G = reductive group over F
- H = abelian subgroup of $G \times G$

We define an action of H on G as follows: for $(h,h') \in H$ and $g \in G$,

$$(h,h').g := h^{-1}gh'.$$

We define a stabilizer

$$H_g = \left\{(h,h') \in H \;\middle|\; h^{-1}gh' = g\right\}$$

and we also have orbits

$$[\delta] := \left\{h^{-1}\delta h' \;\middle|\; (h,h') \in H(F)\right\}.$$

Define the Selberg kernel function

$$K_\phi(x,y) = \sum_{\gamma \in G(F)} \phi(x^{-1}\gamma y)$$

where $\phi : G \to \mathbb{C}$ is a smooth function and $K_\phi : G \times G \to \mathbb{C}$.

In the relative trace formula, we choose a character

$$\chi : H(F)\backslash H(\mathbb{A}) \to \mathbb{C}^\times$$

and we compute

$$\int_{H(F)\backslash H(\mathbb{A})} K_\phi(x,y)\ \chi(x,y)\ d^\times((x,y)).$$

Kuznetsov Trace formula: Zagier's idea Let $z = x+iy,\ z' = x'+iy'$, then

$$K_\phi(z,z') = \sum_{\gamma\in\Gamma} \phi\left(\frac{|\gamma z - z|^2}{\Im(\gamma z)\Im(z')}\right).$$

Take a double integral,

$$\underbrace{\iint_{\Gamma\backslash\mathbb{H}^2}\iint_{\Gamma\backslash\mathbb{H}^2} K_\phi(z,z')e^{-2\pi i m x}e^{-2\pi i n x'}\ \frac{dx\ dy}{y^2}\ \frac{dx'\ dy'}{y'^2}}_{=\langle P_n,P_m\rangle}.$$

Jacquet wants to generalize this idea:

$$\begin{array}{ccc} H(F)\backslash H(\mathbb{A}) & \leftrightarrow & \Gamma\backslash\mathbb{H}^2 \times \Gamma\backslash\mathbb{H}^2 \\ \chi & \leftrightarrow & e^{-2\pi i m x}e^{-2\pi i n x'} \end{array}.$$

Geometric side

$$\begin{aligned} &\int_{H(F)\backslash H(\mathbb{A})} \sum_{\gamma\in G(F)} \phi(x^{-1}\gamma y)\ \chi(x,y)\ d^\times((x,y)) \\ &= \int_{H(F)\backslash H(\mathbb{A})} \sum_{[\delta]}\sum_{\gamma\in[\delta]} \phi(x^{-1}\gamma y)\chi(x,y)\ d^\times((x,y)) \\ &= \sum_{[\delta]} \int_{H_\delta(F)\backslash H(\mathbb{A})} \phi(x^{-1}\delta y)\chi(x,y)\ d^\times((x,y)) \end{aligned}$$

for $[\delta] = \left\{h^{-1}\delta h' \mid (h,h') \in H\right\}$ is a double coset. The way group acting naturally leads us to double cosets.

Definition 6.1. (Orbital Integral)

$$I_\delta(\phi) := \int_{H_\delta(F)\backslash H(\mathbb{A})} \phi(x^{-1}\delta y)\ \chi(x,y)\ d^\times((x,y))$$

Remark. Problem :

$$I_\delta(\phi) \neq 0 \quad ?$$

Definition 6.2. (Relevant orbital integral) *An orbital integral I_δ is relevant if χ is trivial on $H_\delta(\mathbb{A})$.*

Proposition 6.3. *If I_δ is not relevant (i.e., χ is not trivial on $H_\delta(\mathbb{A})$), then $I_\delta \equiv 0$.*

Proof.

$$I_\delta(\phi) = \int\limits_{H_\delta(F)\backslash H(\mathbb{A})} \phi(x^{-1}\delta y)\chi((u,v)\cdot(x,y))d^\times((x,y)) = \chi(u,v)I_\delta(\phi)$$

If I_δ is not relevant, then there exists $(u,v) \in H_\delta(\mathbb{A})$ such that $\chi(u,v) \neq 1$. ☐

Lecture 24: 2010-12-9

6.1 Relative Trace Formula

Let

- F = number field
- $\mathbb{A}_F$ = adele ring of F
- G = reductive group
- $H \subset G \times G$, subgroup

Kernel function

$$K_\phi(x,y) = \sum_{\gamma\in G(F)} \phi(x^{-1}\gamma y), \qquad (\text{ for } x,y \in G(\mathbb{A}))$$

and

$$G(F) = \cup\,[\delta]$$

where

$$[\delta] = \left\{h^{-1}\delta h' \;\middle|\; (h,h') \in H(F)\right\}.$$

Let

$$H_\delta = \left\{(h,h') \in H \;\middle|\; h^{-1}\delta h' = \delta\right\}.$$

Then

$$\int_{H(F)\backslash H(\mathbb{A})} K_\phi(x,y)\chi(x,y)d^\times((x,y))$$

and this is computed in two ways; spectral and geometric, leading to an identity called relative trace formula.

Example : Kuznetsov Trace formula Let $N = \{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}$ and $F = \mathbb{Q}$. We want to put $H = N \times N$ (this leads to the Kuznetsov Trace formula). We need characters on $H(\mathbb{A})$, i.e., characters on $N(\mathbb{A})$ (basically in the Tate's thesis).

Definition 6.4.

$$e_v(x_v) := \begin{cases} e^{2\pi i x_\infty}, & v = \infty \\ e^{-2\pi i \{x_p\}}, & v = p \end{cases}$$

where $x_v \in \mathbb{Q}_v$. *Then we define a character on* $\mathbb{A}$ *as*

$$e(x) = \prod_{v \leqslant \infty} e_v(x_v), \qquad (\textit{for } x \in \mathbb{A}).$$

Let $m_1,\ m_2 \in \mathbb{Q}$ and $n_1, n_2 \in N$. Let $\theta_{m_1}(n_1)$ and $\theta_{m_2}(n_2)$ be characters of N. For $n = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \in N(\mathbb{A})$ define

$$\theta_{m_1}(n) := e(m_1 t).$$

Then we have

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} K_\phi(n_1, n_2)\overline{\theta_{m_1}(n_1)}\theta_{m_2}(n_2)\ dn_1\ dn_2. \tag{6.1}$$

Fourier Theorem Let $F : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ and assume that $F(x + \alpha) = F(x)$ for all $\alpha \in \mathbb{Q}$ and $x \in \mathbb{A}_{\mathbb{Q}}$. Then

$$F(x) = \sum_{\alpha \in \mathbb{Q}} \widehat{F}(\alpha) \cdot e(\alpha x)$$

where

$$\widehat{F}(\alpha) = \int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} F(x) e(-\alpha x)\ dx$$

So, in (6.1), we expect to get Fourier coefficients.

Let $\{u_j\}_{j=1,2,3,\ldots,}$ denote adelic cusp forms. So

$$u_j : GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

There is a one-to-one correspondence between adelic cusp forms for $GL(2,\mathbb{A}_{\mathbb{Q}})$ and classical Maass forms given as follows:

$$GL(2,\mathbb{A}_{\mathbb{Q}}) \ni g = \gamma\left(\begin{pmatrix} y_\infty & x_\infty\\ 0 & 1\end{pmatrix}, \begin{pmatrix} r_\infty & 0\\ 0 & r_\infty\end{pmatrix}, I_2, \ldots, I_2, \ldots\right) k$$

where $I_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right),\ k \in K,\ r_\infty \neq 0$ and $x_\infty + iy_\infty \in SL(2,\mathbb{Z})\backslash\mathbb{H}^2$.

If $\eta_j(z) = \sum_{m \neq 0} A_j(m)\sqrt{y} K_{ir_j}(2pi|m|y) e^{2\pi i m x}$, for $z = x_\infty + iy_\infty$, then

$$\underbrace{u_j(g)}_{\substack{\text{adelic}\\ \text{automorphic form}}} = \underbrace{\eta_j(x_\infty + iy_\infty)}_{\substack{\text{classical}\\ \text{Maass forms}}}$$

Spectral side

$$\begin{aligned} K_\phi(x,y) &= \sum_{\gamma \in G(F)} \phi(x^{-1}\gamma y) = \text{ Selberg Kernel function} \\ &= \sum_j h_\phi(\lambda_j) u_j(x) \overline{u_j(y)} + \text{cont} \end{aligned}$$

where $h_\phi(\lambda_j)$ is a Spectral transform of ϕ. Then

$$\begin{aligned} (6.1) &= \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_j h_\phi(\lambda_j) u_j(n_1) \overline{u_j(n_2)\theta_{m_1}(n_1)} \theta_{m_2}(n_2)\, dn_1\, dn_2 + \text{ cont} \\ &= \sum_j h_\phi(\lambda_j) \underbrace{\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} u_j(n_1)\overline{\theta_{m_1}(n_1)}\, dn_1}_{\substack{m_1\text{th Fourier} \\ \text{coefficient of } u_j}} \cdot \underbrace{\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \overline{u_j(n_2)}\theta_{m_2}(n_2)\, dn_2}_{\substack{m_2\text{th Fourier} \\ \text{coefficient of } u_j}} \\ &= \sum_j h_\phi(\lambda_j) \cdot \widehat{u}_j(m_1)\overline{\widehat{u}_j(m_2)} + \text{ cont} \qquad \text{(Spectral side)}. \end{aligned}$$

Geometric side Let $[\delta] = \left\{ n_1 \delta n_2^{-1} \;\middle|\; n_1, n_2 \in N(\mathbb{Q}) \right\}$. Define orbital integrals

$$I_\delta(\phi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \phi(n_1^{-1}\delta n_2)\, \overline{\theta_{m_1}(n_1)}\theta_{m_2}(n_2)\, dn_1\, dn_2$$

where $H_\delta = \left\{ (n_1, n_2) \in N \times N \;\middle|\; n_1^{-1}\delta n_2 = \delta \right\}$. Then

$$\text{(Spectral side)} = \sum I_\delta(\phi).$$

We need to determine which I_δ are relevant. The orbits $[\delta]$ are in one-to-one correspondence with double cosets

$$N(\mathbb{Q})\backslash GL(2,\mathbb{Q})/N(\mathbb{Q})$$

By Bruhat decomposition

$$GL(2,\mathbb{Q}) = \{N(\mathbb{Q}) \cdot A(\mathbb{Q})\} \bigcup \left\{ N(\mathbb{Q})A(\mathbb{Q}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N(\mathbb{Q}) \right\}$$

and

$$\mathbb{Q}^\times \cdot N(\mathbb{Q})\backslash GL(2,\mathbb{Q})/N(\mathbb{Q}) = \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \gamma \\ 1 & 0 \end{pmatrix}, \quad \gamma \in \mathbb{Q}^\times \right\}.$$

Case 1 $\delta = \left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right)$ If $[\delta]$ is relevant then character is trivial on $H_\delta(\mathbb{A})$. If $\left(\begin{smallmatrix} 1 & t_1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & t_2 \\ 0 & 1 \end{smallmatrix}\right) \in H_\delta(\mathbb{A})$, and $[\delta]$ is relevant, then $\left(\begin{smallmatrix} 1 & -t_1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t_2 \\ 0 & 1 \end{smallmatrix}\right) = z\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $z = \left(\begin{smallmatrix} z_0 & 0 \\ 0 & z_0 \end{smallmatrix}\right)$ with $z_0 \in \mathbb{Q}^\times$.

Case 2 $\delta = \left(\begin{smallmatrix} 0 & \gamma \\ 1 & 0 \end{smallmatrix}\right)$ then $m_1\gamma = m_2$ every thing is relevant.

6.2 Applications

Jacquet's program for the R.T.F We have G and G', two reductive groups. Then we have $H \subset G \times G$ and $H' \subset G' \times G'$. Let χ be a character on H and χ' be a character on H'.

$$\int_{H(F)\backslash H(\mathbb{A})} K_\phi(x,y)\chi(x,y)\, d^\times((x,y)) =^? \int_{H'(F)\backslash H'(\mathbb{A})} K_{\phi'}(x,y)\chi'(x,y)\, d^\times((x,y))$$

Jacquet Let π be a cuspidal representation for some reductive group G and $L(s,\pi)$ be an L-functions with a functional equation $s \mapsto 1-s$.

Theorem 6.5.

$$L\left(\frac{1}{2},\pi\right) \geqslant 0$$

has now been proved for many cases.

Proof. Relative trace formula

$$L\left(\frac{1}{2},\pi\right) = \int_{H(F)\backslash H(\mathbb{A})} (\ ) \int_{H(F)\backslash H(\mathbb{A})} \overline{(\ )} \geqslant 0.$$

☐

Recently (Whitehouse)

- $F =$ number field
- E/F quadratic extension
- π cuspidal automorphic representation of $GL(2,\mathbb{A}_F)$
- π_E base change to E
- $\Omega : \mathbb{A}_F^\times E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}$
- $L(s,\pi_E \times \Omega)$ and $s \mapsto 1-s$

Periodic integral. Let $D =$ quaternion algebra over F.

(i) π transfer to π^D on $D^\times/F$

(ii) $E \hookrightarrow D$

$$P_D(\phi) = \int_{\mathbb{A}_F^\times E^\times \backslash \mathbb{A}_E^\times} \phi(t)\Omega^{-1}(t)\ dt$$

Theorem 6.6. (Waldspurger, Jacquet)

$$P_D(\phi) = 0 \iff L\left(\frac{1}{2},\pi_E \times \Omega\right) = 0$$

Using R.T.F, one can show

$$\frac{\left|P_D\left(\phi_\pi\right)\right|^2}{(\phi_\pi,\phi_\pi)} = \underbrace{c(E,\pi,\Omega)}_{\text{can be computed}} \cdot L\left(\frac{1}{2},\pi_E \otimes \Omega\right)$$

Texed by Min Lee

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