# MATH 2050 ALGEBRAIC GEOMETRY
# PROBLEM SETS

**Problem Set 1. Due Friday September 15 in class**

1. Eisenbud-Harris exercises I-1, I-2, I-3 (graded for completion only)
2. Classify, with proof, the points of $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$, and describe the closure of each point in the Zariski topology.
3. Do the same for $\operatorname{Spec} (\mathbb{C}[x, y]/(xy))_{(x,y)}$.
4. Do the same for $\operatorname{Spec} \mathbb{Z}[x]$ (see Eisenbud-Harris II-37, II-38).
5. Liu exercise 2.1.6.

**Problem Set 2. Due Friday September 22 in class**

1. Liu exercises 2.1.2, 2.1.3, 2.1.4, 2.2.1, 2.2.2. See online errata for 2.1.4(a): replace "nilpotent" with "nil ideal"

**Problem Set 3. Due Friday September 29 in class**

1. Eisenbud-Harris exercise I-20
2. Liu exercises 2.2.4, 2.2.8, 2.2.9, and either 2.2.6 or 2.2.13.

**Problem Set 4. Due Friday October 6 in class**

1. We studied a once-punctured plane $\mathbb{A}^2_k - \{(0, 0)\}$ in class. Consider a *twice*-punctured plane. Is it isomorphic as a scheme to a once-punctured plane?
2. Verify that the disjoint union of finitely many affine schemes is an affine scheme.
3. Liu exercises 2.3.1, 2.3.14, 2.3.15.

**Problem Set 5. Due Friday October 13 in class**

1. Vakil exercises 9.2.A, 9.2.B, 9.2.F.
2. Eisenbud-Harris exercise I-46, just parts (d) through (g)
3. Liu exercises 2.3.7, 3.1.6, 3.1.8.

**Problem Set 6. Due Friday October 20 in class**

1. Eisenbud-Harris exercises II-11, II-12, and II-14. See discussion on pp. 60–61 of that book.
2. Liu exercises 2.4.1, 2.4.2 (see example 2.3.16), and 2.4.3.

**Problem Set 7. Due Friday October 27 in class**

1. Let $B = \operatorname{Spec} \mathbb{C}[t]$. Using a computer or otherwise, compute the limits of the following families of closed subschemes of $\mathbb{A}^n_B$ as $t \to 0$, in the precise sense discussed. What are the primary components of the limiting scheme? Draw pictures.
   (a) The plane curve $xy^2 = t$
   (b) Three concurrent lines becoming coplanar: the three lines through the origin and $(1, 0, 0), (0, 1, 0), (1, 1, t)$ respectively
   (c) Squashing a twisted cubic curve: the space curve whose closed points are $(ts, s^2, s^3)$ for $t, s \in \mathbb{C}$.
2. Liu exercises 2.4.4, 2.4.9, 2.4.11, 2.5.3.

**Problem Set 8. Due Friday November 3 in class**

1. Liu exercises 2.3.10, 2.3.11, 2.3.18, 2.5.7, 3.1.5.

**Problem Set 9. Due Friday November 10 in class**

1. Vakil exercise 8.2.N (see 8.2.11)
2. Let $d, n$ be positive integers, and let $N = \binom{n+d}{d} - 1$. Prove that the image of the Veronese map $v_d(\mathbb{P}^n_{\mathbb{C}}) \subset \mathbb{P}^N_{\mathbb{C}}$ does not lie on any hyperplane (i.e. vanishing locus of a linear form) of $\mathbb{P}^N$.
3. (You may replace $\mathrm{Gr}(2, 4)$ with $\mathrm{Gr}(d, n)$ as you wish.)

   Consider the Grassmannian variety $\mathrm{Gr}(2, 4)$ parametrizing 2-planes in $\mathbb{C}^4$. For each 2-element subset $I$ of $\{1, \dots, 4\}$, consider all $2 \times 4$ matrices with reduced row echelon form having leading 1s in columns $I$.
   (a) Show that the row spans of such matrices form the closed points of a *locally closed*—i.e., intersection of closed and open—subvariety $\Sigma_I$ of $\mathrm{Gr}(2, 4)$, and that each $\Sigma_I$ is isomorphic to an affine space (of what dimension?)
   (b) Argue that $\overline{\Sigma_I} = \cup \Sigma_{I'}$ for appropriately chosen $I'$. Partially order the $\Sigma_I$ according to whether the closure of one contains the other.
4. Liu exercise 3.2.6.

**Problem Set 10. Due Friday November 17 in class**

1. Practice the valuative criterion: use it to verify that $\mathbb{A}^n_k$ is proper over $k$ if and only if $n = 0$.
2. Over $\mathbb{C}$, argue that the polynomial

   $$F(x) = x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$$

   has at most 3 distinct roots iff $F$ and $F'$ have a common factor of degree 3. Set up a `Macaulay2` computation to find equations for the closed set of $\mathbb{A}^6$ consisting of those points $(b, c, d, e, f, g)$ such that $F(x)$ has at most 3 distinct roots.[1]
3. Liu exercise 3.3.12
4. Prove the statement in Liu exercise 3.3.15, using the suggested method or otherwise.

**Problem Set 11. Due Friday December 1 in class**

1. List, with justification, the points of $\operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2)$.
2. Eisenbud-Harris exercises II-6, II-7.
3. Liu exercises 3.2.9, 4.2.7.

**Problem Set 12. Due Monday December 11 in class**

1. Vakil exercise 5.4.H.
2. Liu exercises 4.2.10, 4.2.11, and 4.2.12. For 4.2.10, note "Euler's Lemma," that a homogeneous degree $d$ polynomial $F \in k[T_0, \dots, T_n]$ satisfies $\sum T_i \frac{\partial F}{\partial T_i} = d \cdot F$.

[1] If you actually want to try it, here is some sample `Macaulay2` code that may be helpful.

```
R=QQ[a..f];
M=matrix{{a,b,c},{d,e,f}};
minors(2,M)
```