1. Let $R$ be an integral domain, and let $f, g \in R$. Prove that $(f) = (g)$ if and only if $f = ga$ for some unit $a$.

(Updated: extra credit) Also, give an example of a commutative ring $R$ with identity $1 \neq 0$ such that the above does not hold.

2. Let $K$ be a field, and consider the ring $K[[x]]$ of formal power series.
   
   (a) Prove that $K[[x]]$ is an integral domain.
   
   (b) Prove that the ideals of $K[[x]]$ are 0 or are of the form $(x^n)$ for some integer $n \geq 0$.
   
   (c) Which of the ideals of $K[[x]]$ are principal? maximal? prime? Prove your answers.

3. Let $m, n \geq 1$ be integers. Express, with proof, the ideals in $\mathbb{Z}$

   $m\mathbb{Z} + n\mathbb{Z}, \quad (m\mathbb{Z})(n\mathbb{Z}), \quad m\mathbb{Z} \cap n\mathbb{Z}$

   in the form $d\mathbb{Z}$ for some number $d$.


5. This is a problem in beginning algebraic geometry. Given an ideal $I \subseteq \mathbb{R}[x, y]$, we let the vanishing locus or variety of $I$ be the subset of $\mathbb{R}^2$

   \[ V(I) = \{(a, b) \in \mathbb{R}^2 : f(a, b) = 0 \text{ for all } f \in I\}. \]

   (a) Prove that if $I = (f_1, \ldots, f_n)$ then $V(I) = \{(a, b) \in \mathbb{R}^2 : f_i(a, b) = 0 \text{ for all } i = 1, \ldots, n\}$.
   
   (b) Draw pictures of $V(I)$ for $I = (y(y - x^2))$ and for $I = (x - y, y - x^3)$.
   
   (c) Prove that if $I_1 \subseteq I_2$ are ideals of $\mathbb{R}[x, y]$ then $V(I_1) \supseteq V(I_2)$.
   
   (d) Using part (c) to help, prove the identities

   \[ V(I + J) = V(I) \cap V(J) \quad \text{and} \quad V(IJ) = V(I \cap J) = V(I) \cup V(J). \]

   Check your results in part (b) accordingly.

For more, please see the book *Ideals, varieties, and algorithms* (Cox, Little, O’Shea).