1. Let \( R = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\} \), i.e., \( R \) is the ring of infinite tuples of \( \mathbb{Z} \), indexed by the positive integers, with coordinatewise addition and multiplication. For each \( j = 1, 2, \ldots \) let

\[ I_j = \{(a_1, a_2, a_3 \ldots) \in R : a_i = 0 \text{ for all } i \geq j\} \]

(a) Show that the \( I_j \) are principal ideals forming an ascending chain \( I_1 \subseteq I_2 \subseteq \cdots \) that doesn’t stabilize. Conclude that \( I = \bigcup_{j \geq 1} I_j \) an ideal that is not finitely generated.

(b) Is \( I \) prime?

(c) (extra, just for fun) Show \( R/I \) contains a copy of \( \mathbb{Z}[x] \) inside it. That is, \( R/I \) has a subring isomorphic to \( \mathbb{Z}[x] \).

2. Dummit and Foote p. 306 problem 2

3. Convince yourself that the polynomial \( x^3 + x + 1 \) is irreducible in \( \mathbb{F}_2[x] \). Write out the multiplication table for the 8-element field \( K = \mathbb{F}_2[x]/(x^3 + x + 1) \), and check that the multiplicative group of nonzero elements in \( K \) is isomorphic to \( \mathbb{Z}/7\mathbb{Z} \).

4. For \( K \) a field of characteristic \( p > 0 \), we define the Frobenius map \( e: K \rightarrow K \) by \( e(a) = a^p \). Show that \( e \) is a homomorphism. (To show that \( e(a+b) = e(a) + e(b) \) you may wish to appeal to the Binomial Theorem.)

Also, compute what \( e \) does to each element of the field \( K \) from Problem 3.