MATH 1530 ABSTRACT ALGEBRA
PROBLEM SET 12 DUE TUESDAY MAY 2 1PM IN CLASS

1. Let $R = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$, i.e., R is the ring of infinite tuples of $\mathbb{Z}$, indexed by the positive integers, with coordinatewise addition and multiplication. For each $j = 1, 2, \ldots$ let

$$I_j = \{(a_1, a_2, a_3 \ldots) \in R : a_i = 0 \text{ for all } i \geq j.\}$$

   (a) Show that the I_j are principal ideals forming an ascending chain $I_1 \subsetneq I_2 \subsetneq \cdots$ that doesn't stabilize. Conclude that $I = \bigcup_{j \geq 1} I_j$ an ideal that is not finitely generated.

   (b) Is I prime?

   (c) (extra, just for fun) Show R/I contains a copy of $\mathbb{Z}[x]$ inside it. That is, R/I has a subring isomorphic to $\mathbb{Z}[x]$.

2. Dummit and Foote p. 306 problem 2

3. Convince yourself that the polynomial $x^3 + x + 1$ is irreducible in $\mathbb{F}_2[x]$. Write out the multiplication table for the 8-element field $K = \mathbb{F}_2[x]/(x^3 + x + 1)$, and check that the multiplicative group of nonzero elements in K is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.

4. For K a field of characteristic $p > 0$, we define the *Frobenius map* $e \colon K \to K$ by $e(a) = a^p$. Show that e is a homomorphism. (To show that $e(a+b) = e(a) + e(b)$ you may wish to appeal to the Binomial Theorem.)

   Also, compute what e does to each element of the field K from Problem 3.