The 4×4 minors of a 5 × n matrix are a tropical basis

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joint work with Anders Jensen and Elena Rubei
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The **tropical semiring** $\left( \mathbb{R}, \oplus, \otimes \right)$ consists of the real numbers equipped with tropical addition and multiplication:

\[
\begin{align*}
    x \oplus y &:= \min(x, y) \\
    x \otimes y &:= x + y.
\end{align*}
\]

**Example:**

\[
\begin{align*}
    3 \oplus 4 &= 3 \\
    3 \otimes 4 &= 7
\end{align*}
\]
Let $K$ be the field of well-ordered power series in a variable $t$

$$\{ \alpha = \sum_{n \in S} a_n t^n : S \text{ a well-ordered subset of } \mathbb{R}, \ a \in \mathbb{C} \}.$$
Background: Tropical Hypersurfaces

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The tropicalization of a polynomial $f$ with coefficients in $K$ is the tropical polynomial $F$ obtained by replacing each coefficient with its valuation (lowest exponent) and replacing all classical operations with tropical ones.

Example: $f = t^3 x + 4 i y - 5 z$ yields $F = 3 \, \circ \, X \, \oplus \, 1 \, \circ \, Y \, \oplus \, Z$. 

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The tropical hypersurface $T(f)$ of a polynomial $f \in K[x_1, \ldots, x_n]$ is the set of points in $\mathbb{R}^n$ at which $F$ attains its minimum at least twice.

Example: $T(f)$ is a tropical line centered at $(-3, -1, 0)$. 
Background: Tropical Prevarieties and Varieties

Fix polynomials $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$. Their tropical prevariety is

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**Theorem ("Fundamental Theorem of Tropical Geometry")**

For $I \subseteq K[x_1, \ldots, x_n]$, the tropical variety $T(I)$ consists of those real points which lift (coordinate-wise) to the classical variety $V(I)$. 
Definition 1: Tropical Rank

An \( n \times n \) real matrix \( A \) is **tropically singular** if the minimum, over all permutations \( \pi \in S_n \), of \( a_{1\pi(1)} + \cdots + a_{n\pi(n)} \) occurs at least twice.

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Example: \[
\begin{pmatrix}
0 & 1 & 2 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\] has tropical rank 2.

The set of $d \times n$ matrices of tropical rank $< r$ is the prevariety of the $r \times r$ minors of a $d \times n$ matrix.
Definition 2: Kapranov rank

Given a matrix $\mathcal{A}$ over the field $K$, let $A$ be the real matrix of lowest exponents appearing in each entry of $\mathcal{A}$. We say that $\mathcal{A}$ is a lift of $A$.

Example: $\mathcal{A} = \begin{pmatrix} 1 & t & t^2 \\ 2t & 3t & 5t \\ 1 + 2t & 4t & 5t + t^2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.
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The set of $d \times n$ matrices of Kapranov rank $< r$ is the variety of the $r \times r$ minors of a $d \times n$ matrix.
These notions of rank were studied by Develin, Santos, Sturmfels; also Akian, Gaubert, Izhakian, Rowen, Kim-Roush, . . .

**Today:** Proof of a conjecture made by [Develin-Santos-Sturmfels]: the $4 \times 4$-minors of a $5 \times n$ matrix form a tropical basis
Tropical Rank versus Kapranov Rank

Question: Does every matrix of tropical rank $< r$ have Kapranov rank $< r$?

Equivalently: are the $r \times r$-minors of an $d \times n$ matrix a tropical basis? That is, are the prevariety and the variety of the $r \times r$ minors equal?

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- Yes, if $r \leq 3$ or $r = \min\{d, n\}$ (Develin, Santos, Sturmfels 2006)
- No, if $r = 4$ and $d = n = 7$ (Fano plane)
- Challenge posed for $r = 4, d = n = 5$ (50€, Berlin, 2007)
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Theorem

The $4 \times 4$-minors of a $5 \times n$ matrix are a tropical basis.
Computational proof for the $5 \times 5$ case

The tropical prevariety of the $25 \ 4 \times 4$-minors is a pure 21-dimensional fan with 9-dimensional lineality space, and $f = (1450, 28450, 257300, \ldots, 2521800)$.

The tropical variety of the ideal $\langle 4 \times 4$-minors $\rangle$ is a pure 21-dimensional fan with 9-dimensional lineality space, and $f = (3250, 53650, 421750, \ldots, 2894400)$.

Same Euler characteristic $\chi = -3120$

Careful computations in \texttt{gfan} (Anders Jensen) show that the supports agree.
Combinatorial Proof for a $5 \times n$ Matrix

Suppose

$$W = \left[ \begin{array} { c | c | c | \cdots | c } w_1 & w_2 & w_3 & \cdots & w_n \end{array} \right]$$

has tropical rank $\leq 3$; want to lift it to a matrix in $K^{5 \times n}$ of rank 3.
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**Idea:** Delete last row of $W$, get $n$ coplanar points in $\mathbb{T}P^3$. They lie on a plane $a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus a_3 \odot x_3 \oplus a_4 \odot x_4$. So columns of $W$ lie on hyperplane

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Similarly for other rows: Get five special hyperplanes $H_1, \ldots, H_5$. 

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**Lemma:** If the stable intersection $H_i \cap_{\text{stab}} H_j$ of some pair contains $W$, then $W$ lifts to a matrix of rank 3 as desired.
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Otherwise, for each pair $i, j$, there must exist a witness pair $k, l$: a pair such that some column $w_s$ lies in the closed sectors $k$ and $l$, and no other closed sectors, for both hyperplanes $H_i$ and $H_j$. 

![Diagram]

This gives, for each $i, j$, a geometric condition on the hyperplane arrangement. Combinatorial case analysis shows that no hyperplane arrangement can satisfy these conditions.

In fact no tropical oriented matroid can satisfy these conditions (Ardila and Develin).
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What next?

- $4 \times 4$-minors and $5 \times 5$-minors of $6 \times n$ matrices
- Topology, e.g. shellability, schönness of these spaces.
- Matrices with special structure: symmetric, Hankel, 