Additional problems for Math 0540

December 4, 2015

A1. Write out the addition and multiplication tables for $\mathbb{F}_7$, the finite field with 7 elements.

A2. Let $V = {\mathbb{F}_2}^2$, a vector space over the finite field $\mathbb{F}_2$. How many linear maps $V \to V$ are there?

A3. Let U be the vector space of real polynomials $f(x)$ of degree at most 4 such that

$$\int_0^1 f(x)dx = 0.$$

By considering the integration map $\mathcal{P}_4(\mathbb{R}) \to \mathbb{R}$ that sends $f(x)$ to $\int_0^1 f(x)dx$, compute the dimension of U.

A4. For each of the following linear maps $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, write down the matrix of T, with respect to the standard basis of $\mathbb{R}^2$ on both copies of $\mathbb{R}^2$. (You should convince yourself that each map is linear, but you do not need to prove anything in this problem. Just compute the answer.)

1. Dilation by a factor of 2 with respect to the origin; that is, the map T that sends each vector v to $2v$.
2. Reflection across the line $x = y$. (Here x and y denote the usual coordinates (x, y) of $\mathbb{R}^2$.)
3. Projection to the line $x = y$. That is, T sends v to the point on the line $x = y$ that is closest to v.
4. The identity map.

A5. Do the same for each of the linear maps in problem A4, but now with respect to the basis $(1, 0), (1, 1)$ on *both* copies of $\mathbb{R}^2$.

A6. Compute each of the following matrix products, or explain why they are not defined:

1.
$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 11 & 13 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$$

2.
$$\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 11 & 13 \end{pmatrix}$$

3.
$$\begin{pmatrix} 1+i \\ 0 \\ i \end{pmatrix} \begin{pmatrix} -i & 0 & i \end{pmatrix}$$

4.
$$\begin{pmatrix} -i & 0 & i \end{pmatrix} \begin{pmatrix} 1+i \\ 0 \\ i \end{pmatrix}$$

A7. Let $T_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by counterclockwise rotation by θ about the origin. Let A be the matrix of T_θ with respect to the standard bases. Compute A^2 and A^3; then interpret these products geometrically to deduce formulas for $\cos(2\theta), \sin(2\theta), \cos(3\theta)$, and $\sin(3\theta)$.

A8. Let W be the subset of $\mathbb{F}^\infty$ given by

$$W = \{(x_1, x_2, \ldots) : \text{there exists a number } N \text{ such that } x_i = x_j \text{ for all } i, j \geq N\}.$$

1. Prove W is a subspace.
2. Prove that $\mathbb{F}^\infty / W$ is infinite-dimensional.

A9. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $D \colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ be a function such that:

(P1) $D(v_1, \ldots, \alpha v_k, \ldots, v_n) = \alpha D(v_1, \ldots, v_k, \ldots, v_n)$ for all $v_i \in \mathbb{F}^n$, $\alpha \in \mathbb{F}$;

(P2) $D(v_1, \ldots, v_k + v_k', \ldots, v_n) = D(v_1, \ldots, v_k, \ldots, v_n) + D(v_1, \ldots, v_k', \ldots, v_n)$ for all $v_i \in \mathbb{F}^n$.

Show that D satisfies

(P3) $D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n) = -D(v_1, \ldots, v_k, \ldots, v_j, \ldots, v_n)$ for all $v_i \in \mathbb{F}^n$

if and only if D satisfies

(P3') $D(v_1, \ldots, v_j + \alpha v_k, \ldots, v_k, \ldots, v_n) = D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n)$ for any $v_i \in \mathbb{F}^n$ and $\alpha \in \mathbb{F}$.

A10. Apply the Gram Schmidt procedure to the linearly independent list

$$(1, 0, 0), (1, 1, 1), (1, 1, 2) \text{ in } \mathbb{R}^3.$$

A11. Apply the Gram Schmidt procedure to the linearly independent list

$$(1, 1 + i), (1, i) \in \mathbb{C}^2.$$

A12. Consider the map $T_\theta \in \mathcal{L}(\mathbb{R}^2)$ that is counterclockwise rotation by angle θ.

1. Verify that $T_\theta^* = T_\theta^{-1}$.
2. For which values of $\theta \in [0, 2\pi)$ is T_θ self-adjoint? for which values is it normal?

A13. Classify the isometries of $\mathbb{R}^2$, as follows. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be an isometry. Show that

- T is a rotation, and $\det \mathcal{M}(T) = 1$, or
- T is a reflection across a line through the origin, and $\det \mathcal{M}(T) = -1$,

where the matrices are taken with respect to the standard basis of $\mathbb{R}^2$.