Additional problems for Math 0540

December 4, 2015

A1. Write out the addition and multiplication tables for $\mathbb{F}_7$, the finite field with 7 elements.

A2. Let $V = \mathbb{F}_2^2$, a vector space over the finite field $\mathbb{F}_2$. How many linear maps $V \to V$ are there?

A3. Let $U$ be the vector space of real polynomials $f(x)$ of degree at most 4 such that

$$\int_0^1 f(x)dx = 0.$$ 

By considering the integration map $\mathcal{P}_4(\mathbb{R}) \to \mathbb{R}$ that sends $f(x)$ to $\int_0^1 f(x)dx$, compute the dimension of $U$.

A4. For each of the following linear maps $T:\mathbb{R}^2 \to \mathbb{R}^2$, write down the matrix of $T$, with respect to the standard basis of $\mathbb{R}^2$ on both copies of $\mathbb{R}^2$. (You should convince yourself that each map is linear, but you do not need to prove anything in this problem. Just compute the answer.)

1. Dilation by a factor of 2 with respect to the origin; that is, the map $T$ that sends each vector $v$ to $2v$.

2. Reflection across the line $x = y$. (Here $x$ and $y$ denote the usual coordinates $(x, y)$ of $\mathbb{R}^2$.)

3. Projection to the line $x = y$. That is, $T$ sends $v$ to the point on the line $x = y$ that is closest to $v$.

4. The identity map.

A5. Do the same for each of the linear maps in problem A4, but now with respect to the basis $(1, 0), (1, 1)$ on both copies of $\mathbb{R}^2$. 

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A6. Compute each of the following matrix products, or explain why they are not defined:

1. $$\begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 11 & 13 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$$

2. $$\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 7 \\ 11 & 13 \end{pmatrix}$$

3. $$\begin{pmatrix} 1+i \\ 0 \\ i \end{pmatrix} \begin{pmatrix} -i & 0 & i \end{pmatrix}$$

4. $$\begin{pmatrix} -i & 0 & i \end{pmatrix} \begin{pmatrix} 1+i \\ 0 \\ i \end{pmatrix}$$

A7. Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by counterclockwise rotation by $\theta$ about the origin. Let $A$ be the matrix of $T_\theta$ with respect to the standard bases. Compute $A^2$ and $A^3$; then interpret these products geometrically to deduce formulas for $\cos(2\theta), \sin(2\theta), \cos(3\theta),$ and $\sin(3\theta)$.

A8. Let $W$ be the subset of $\mathbb{F}^\infty$ given by

$$W = \{(x_1, x_2, \ldots) : \text{there exists a number } N \text{ such that } x_i = x_j \text{ for all } i, j \geq N\}$$

1. Prove $W$ is a subspace.

2. Prove that $\mathbb{F}^\infty/W$ is infinite-dimensional.

A9. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $D : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ be a function such that:

(P1) $D(v_1, \ldots, \alpha v_k, \ldots, v_n) = \alpha D(v_1, \ldots, v_k, \ldots, v_n)$ for all $v_i \in \mathbb{F}^n$, $\alpha \in \mathbb{F}$;
(P2) \[ D(v_1, \ldots, v_k + v'_k, \ldots, v_n) = D(v_1, \ldots, v_k, \ldots, v_n) + D(v_1, \ldots, v'_k, \ldots, v_n) \]
for all \( v_i \in \mathbb{F}^n \).

Show that \( D \) satisfies

(P3) \[ D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n) = -D(v_1, \ldots, v_k, \ldots, v_j, \ldots, v_n) \]
for all \( v_i \in \mathbb{F}^n \) if and only if \( D \) satisfies

(P3') \[ D(v_1, \ldots, v_j + \alpha v_k, \ldots, v_k, \ldots, v_n) = D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n) \]
for any \( v_i \in \mathbb{F}^n \) and \( \alpha \in \mathbb{F} \).

A10. Apply the Gram Schmidt procedure to the linearly independent list

\((1, 0, 0), (1, 1, 1), (1, 1, 2)\) in \( \mathbb{R}^3 \).

A11. Apply the Gram Schmidt procedure to the linearly independent list

\((1, 1 + i), (1, i)\) in \( \mathbb{C}^2 \).

A12. Consider the map \( T_\theta \in \mathcal{L}(\mathbb{R}^2) \) that is counterclockwise rotation by angle \( \theta \).

1. Verify that \( T_\theta^* = T_\theta^{-1} \).

2. For which values of \( \theta \in [0, 2\pi) \) is \( T_\theta \) self-adjoint? for which values is it normal?

A13. Classify the isometries of \( \mathbb{R}^2 \), as follows. Let \( T \in \mathcal{L}(\mathbb{R}^2) \) be an isometry. Show that

- \( T \) is a rotation, and \( \det \mathcal{M}(T) = 1 \), or
- \( T \) is a reflection across a line through the origin, and \( \det \mathcal{M}(T) = -1 \),

where the matrices are taken with respect to the standard basis of \( \mathbb{R}^2 \).