Combinatorics of the tropical Torelli map

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A tropical curve $C$ is a triple $(G, l, w)$, where $(G, l)$ is a metric graph, and $w$ is a weight function

$$w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$$

on the vertices of $G$, with the property that every weight zero vertex has degree at least 3.
What is a tropical curve?

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$$w : V(G) \to \mathbb{Z}_{\geq 0}$$

on the vertices of $G$, with the property that every weight zero vertex has degree at least 3.

Its genus is $g(G) + \sum_{v \in V} w(v)$.

Its combinatorial type is the pair $(G, w)$. 
The Jacobian of a tropical curve

Given a genus $g$ tropical curve $C = (G, l, w)$, with edges of $G$ oriented for reference, let $H_1(G, \mathbb{R})$ = formal sums of edges of $G$ with zero boundary.
The Jacobian of a tropical curve

Given a genus $g$ tropical curve $C = (G, l, w)$, with edges of $G$ oriented for reference, let $H_1(G, \mathbb{R}) = \text{formal sums of edges of } G \text{ with zero boundary.}$

Now define a positive semidefinite form $Q$ on $H_1(G, \mathbb{R}) \oplus \mathbb{R} \sum w(v)$ which is 0 on $\mathbb{R} \sum w(v)$ and on $H_1(G, \mathbb{R})$ is

$$Q\left( \sum_{e \in E(G)} \alpha_e \cdot e, \sum_{e \in E(G)} \beta_e \cdot e \right) = \sum_{e \in E(G)} \alpha_e \cdot \beta_e \cdot l(e).$$

Choosing a $\mathbb{Z}$-basis for $H_1(G, \mathbb{Z})$ defines $Q$ as a $g \times g$ positive semidefinite matrix with rational nullspace.
The Jacobian of a tropical curve

\[
\begin{pmatrix} a & e_1 \\ 0 & -b & e_2 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} e_1 - e_2, e_2 - e_3 \\ \end{pmatrix}
\quad
\begin{pmatrix} (a + b & -b) \\ (-b & b + c) \end{pmatrix}
\]

Choosing a different \(\mathbb{Z}\)-basis for \(H_1(G, \mathbb{Z})\) changes \(Q\) by a \(GL_g(\mathbb{Z})\)-action:

\[
\begin{pmatrix} a & e_1 \\ 0 & -b & e_2 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} e_1 - e_2, e_1 - 2e_2 + e_3 \\ \end{pmatrix}
\quad
\begin{pmatrix} (a + b & a + 2b) \\ (-a + 2b & a + 4b + c) \end{pmatrix}
\]
The Jacobian of a tropical curve

\[
\begin{pmatrix}
    a & e_1 \\
    0 & -b & e_2 \\
    c & e_3
\end{pmatrix}
\quad e_1 - e_2, e_2 - e_3
\quad \left(\begin{array}{cc}
a + b & -b \\
-b & b + c
\end{array}\right)
\]

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\quad \left(\begin{array}{cc}
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-a + 2b & a + 4b + c
\end{array}\right)
\]

\[
\begin{pmatrix}
a + b & a + 2b \\
-a + 2b & a + 4b + c
\end{pmatrix}
= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} a + b & -b \\ -b & b + c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.
\]
So we obtain a well-defined element of

$$\tilde{S}_{\geq 0}^g / GL_g(\mathbb{Z}) := \frac{\text{psd matrices with rational nullspace}}{Q \sim X^T Q X \text{ for all } X \in GL_g(\mathbb{Z})},$$

and this point in $\tilde{S}_{\geq 0}^g / GL_g(\mathbb{Z})$ is called the **Jacobian** of the curve.
The tropical Torelli map

Classically, the Torelli map, from the moduli space of curves to the moduli space of principally polarized abelian varieties, sends a curve to its Jacobian.

We will construct a tropical analogue: a tropical Torelli map

\[ t^\text{tr}_g : M^\text{tr}_g \rightarrow A^\text{tr}_g \]

from the moduli space of tropical curves to the moduli space of principally polarized tropical abelian varieties that takes a tropical curve to its Jacobian.

Brannetti-Melo-Viviani arXiv:0907.3324
Towards a moduli space of tropical curves

Warm up: what are the possible combinatorial types of genus 2 tropical curves?
Towards a moduli space of tropical curves

Warm up: what are the possible combinatorial types of genus 2 tropical curves?

This is the poset of combinatorial types of genus 2 tropical curves, ordered by contraction. Note: contracting a loop at a vertex increases its weight by 1.
Motivation: stratification of $\overline{\mathcal{M}}_g$ by dual graphs

Figure: Posets of cells of $\mathcal{M}^{tr}_2$ (left) and of $\overline{\mathcal{M}}_2$ (right). Vertices record irreducible components, weights record genus, edges record nodes.
Construction of $M_g^{\text{tr}}$

Our goal is to construct a moduli space $M_g^{\text{tr}}$ for genus $g$ tropical curves, that is, a space whose points correspond to tropical curves of genus $g$ and whose geometry reflects the geometry of the tropical curves in a sensible way.
Construction of $M^\text{tr}_g$

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Construction due to B-M-V.

Fix a combinatorial type $(G, w)$ of genus $g$. What is a parameter space for all tropical curves of this type?
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Construction due to B-M-V.

Fix a combinatorial type $(G, \omega)$ of genus $g$. What is a parameter space for all tropical curves of this type?

\[ \frac{\mathbb{R}^3_{\geq 0}}{(a,b,c) \sim (a,c,b)} = \frac{\mathbb{R}^3_{\geq 0}}{S_2} \]
Construction of $M_g^{tr}$ continued

Strategy: each combinatorial type of genus $g$ gets a cell

$$\mathbb{R}_{\geq 0}^{\left|\mathcal{E}(G)\right|} \quad \frac{\left|\mathcal{E}(G)\right|}{\text{Aut}(G, w)}.$$

Now identify two graphs in the disjoint union of all such cells if they are the same after contracting all edges of length zero.
The resulting space, denoted $M^\text{tr}_g$, has points in bijection with genus $g$ tropical curves. It is a Hausdorff topological space (Caporaso 2010).

Figure: Cells of $M^\text{tr}_2$. 
Theorem (C, also Maggiolo-Pagani 2010)

The moduli space $M^\text{tr}_3$ has 42 cells and $f$-vector $(1, 2, 5, 9, 12, 8, 5)$. 

*diagram*
Theorem (C, also Maggiolo-Pagani 2010)

- The moduli space $M^tr_{4}$ has 379 cells and $f$-vector

$$(1, 3, 7, 21, 43, 75, 89, 81, 42, 17).$$

- The moduli space $M^tr_{5}$ has 4555 cells and $f$-vector

$$(1, 3, 11, 34, 100, 239, 492, 784, 1002, 926, 632, 260, 71).$$
Note: does $\mathcal{M}_g^{\text{tr}}$, the moduli space of tropical curves, really deserve to be called that?

That is, we saw a poset correspondence between $\overline{\mathcal{M}}_g$ and $\mathcal{M}_g^{\text{tr}}$, but what about a tropicalization map $\overline{\mathcal{M}}_g \rightarrow \mathcal{M}_g^{\text{tr}}$?

This point is not addressed in my work, but see work on Berkovich spaces by Baker, Payne, and Rabinoff.
What kind of space is $M^\text{tr}_g$?

It consists of rational open polyhedral cones modulo symmetries, glued along boundaries via integral linear maps. We will make this precise by defining a category of stacky fans.
What is a Stacky Fan?

Definition (C) Let

\[ X_1 \subseteq \mathbb{R}^{m_1}, \ldots, X_k \subseteq \mathbb{R}^{m_k} \]
be full-dimensional rational open polyhedral cones and

\[ G_1 \subseteq GL_{m_1}(\mathbb{Z}), \ldots, G_k \subseteq GL_{m_k}(\mathbb{Z}) \]
be subgroups such that the action of each \( G_i \) on \( \mathbb{R}^{m_i} \) fixes \( X_i \). Let

\[ X_i/G_i \quad \text{and} \quad \overline{X_i}/G_i \]
be the topological quotient spaces.
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be subgroups such that the action of each \( G_i \) on \( \mathbb{R}^{m_i} \) fixes \( X_i \). Let \( X_i/G_i \) and \( \overline{X_i}/G_i \) be the topological quotient spaces.

Suppose that we have a topological space \( X \) and, for each \( i = 1, \ldots, k \), a continuous map \( \alpha_i : \overline{X_i}/G_i \to X \).
Then $X$ is a **stacky fan**, with cells $X_i/G_i$, if the following four properties hold:

1. The restriction of $\alpha_i$ to $\frac{X_i}{G_i}$ is a homeomorphism onto its image,

   \[ \overline{X}_i \quad \overline{G}_i \]

   \[ \alpha_i \quad X \]

2. We have an equality of sets $X = \coprod \alpha_i(X_i/G_i)$,
3. For each face $F$ of any cone $\overline{X_i}$, there exists $k$ such that $\alpha_i(F) = \alpha_k(\overline{X_k}/G_k)$, and an invertible, lattice point-preserving linear map $L$ taking $F$ to $\overline{X_k}$, such that the following diagram commutes:

We say that $\overline{X_k}/G_k$ is a **stacky face** of $\overline{X_i}/G_i$ in this situation.
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4. For each pair $i, j$,

$$\alpha_i(\overline{X_i}/G_i) \cap \alpha_j(\overline{X_j}/G_j) = \alpha_{k_1}(\overline{X_{k_1}}/G_{k_1}) \cup \cdots \cup \alpha_{k_t}(\overline{X_{k_t}}/G_{k_t})$$

where the union ranges over the common stacky faces.
Theorem (B-M-V,C)

The moduli space $M_g^{tr}$ is a stacky fan with cells corresponding to combinatorial types of genus $g$. 
Theorem (B-M-V,C)

*The moduli space $M^\text{tr}_g$ is a stacky fan with cells corresponding to combinatorial types of genus $g$.*

We have constructed the moduli space $M^\text{tr}_g$ and shown that it is a stacky fan. Next, we will construct the moduli space of principally polarized tropical abelian varieties, denoted $A^\text{tr}_g$, and then show that the tropical Torelli map is a stacky morphism.
Construction of the moduli space $A^\text{tr}_g$

A principally polarized tropical abelian variety is a point in

$$\frac{\mathcal{S}_g^g_{\geq 0}}{GL_g(\mathbb{Z})} := \frac{\text{psd matrices with rational nullspace}}{Q \sim X^T QX \text{ for all } X \in GL_g(\mathbb{Z})}.$$
Construction of the moduli space $A_{g}^{tr}$

A **principally polarized tropical abelian variety** is a point in

$$\frac{\tilde{S}_{g}^{\geq 0}}{GL_{g}(\mathbb{Z})} := \frac{\text{psd matrices with rational nullspace}}{Q \sim X^{T}QX \text{ for all } X \in GL_{g}(\mathbb{Z})}.$$ 

What is a good moduli space of principally polarized tropical abelian varieties?

$$\tilde{S}_{g}^{g}/GL_{g}(\mathbb{Z}) \text{ itself?}$$
Construction of the moduli space $A^\text{tr}_g$

A **principally polarized tropical abelian variety** is a point in

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What is a good moduli space of principally polarized tropical abelian varieties?

$\tilde{S}_g^g / GL_g(\mathbb{Z})$ itself?

Not good enough: it’s not even Hausdorff, and does not admit stacky fan structure.

Instead, we will use the beautiful combinatorics of **Voronoi reduction theory** (Voronoi, 1908) to break $\tilde{S}_g^g / GL_g(\mathbb{Z})$ into a finite number of polyhedral pieces, then glue them back together.
Given $Q \in \tilde{S}^g_{\geq 0}$, the **Delone subdivision** $\text{Del}(Q)$ is the infinite-periodic regular subdivision of $\mathbb{R}^g$ obtained by lifting each lattice point $x \in \mathbb{Z}^g$ to the height $x^T Q x$, then taking lower faces of the convex hull of the lifted points.
Given \( Q \in \tilde{S}^g_{\geq 0} \), the **Delone subdivision** \( \text{Del}(Q) \) is the infinite-periodic regular subdivision of \( \mathbb{R}^g \) obtained by lifting each lattice point \( x \in \mathbb{Z}^g \) to the height \( x^T Q x \), then taking lower faces of the convex hull of the lifted points.

Now, given a Delone subdivision \( D \), let

\[
\sigma_D = \{ Q \in \tilde{S}^g_{\geq 0} : \text{Del}(Q) = D \}.
\]

Then \( \sigma_D \) is an open rational polyhedral cone, called the **secondary cone** of \( D \).
Voronoi reduction theory

Figure: Infinite decomposition of $\tilde{S}_0^2$ into secondary cones.
Theorem (Main theorem of Voronoi reduction theory)

The set of closed secondary cones

\[ \{ \overline{\sigma_D} : D \text{ is a Delone subdivision of } \mathbb{R}^g \} \]

yields an infinite polyhedral fan whose support is \( \tilde{S}_g \geq 0 \). There are only finitely many \( GL_g(\mathbb{Z}) \)-orbits of this set.
Theorem (Main theorem of Voronoi reduction theory)

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yields an infinite polyhedral fan whose support is \( S_{\geq 0}^g \). There are only finitely many \( GL_g(\mathbb{Z}) \)-orbits of this set.

For example, when \( g = 2 \), there are four \( GL_g(\mathbb{Z}) \)-classes of Delone subdivisions, with representatives shown below. They give rise to secondary cones of dimensions 3, 2, 1, and 0, respectively.

\[
\begin{align*}
\text{D}_1 & \quad \text{D}_2 & \quad \text{D}_3 & \quad \text{D}_4 \\
\end{align*}
\]
The moduli space $A_g^{tr}$

Pick Delone subdivisions $D_1, \ldots, D_k$ that are representatives for the $GL_g(\mathbb{Z})$-equivalence classes. Let $\text{Stab}(\sigma_D)$ denote the subgroup of elements of $GL_g(\mathbb{Z})$ that fix $\sigma_D$ as a set.
The moduli space $A^\text{tr}_g$

Pick Delone subdivisions $D_1, \ldots, D_k$ that are representatives for the $GL_g(\mathbb{Z})$-equivalence classes. Let $\text{Stab}(\sigma_D)$ denote the subgroup of elements of $GL_g(\mathbb{Z})$ that fix $\sigma_D$ as a set.

Then define the moduli space of principally polarized tropical abelian varieties, denoted $A^\text{tr}_g$, to be the topological space

$$A^\text{tr}_g = \left( \prod_{i=1}^{k} \frac{\sigma_{D_i}}{\text{Stab}(\sigma_{D_i})} \right) / \sim,$$

where $\sim$ denotes gluing by $GL_g(\mathbb{Z})$-equivalence.
The moduli space $A^\text{tr}_g$

Theorem (B-M-V, C)

The moduli space $A^\text{tr}_g$ is a stacky fan. Its cells correspond to $\text{GL}_g(\mathbb{Z})$-equivalence classes of Delone subdivisions.

Theorem (C)

$A^\text{tr}_g$ is a Hausdorff topological space. It is independent of the choice of representative Delone subdivisions in its construction. That is, choosing different representatives produces an isomorphic stacky fan.
Example: $A_2^{tr}$

When $g = 2$, we have four $GL_g(\mathbb{Z})$-classes of Delone subdivisions, with secondary cones of dimensions 3, 2, 1, and 0, respectively.

$A_2^{tr}$ is homeomorphic to a closed, 3-dimensional simplicial cone.
The tropical Torelli map

Definition
We define the **tropical Torelli map**

\[ t_g^{\text{tr}} : M_g^{\text{tr}} \rightarrow A_g^{\text{tr}} \]

to send a tropical curve \( C \in M_g^{\text{tr}} \) to its Jacobian \( Jac(C) \in A_g^{\text{tr}} \).
The tropical Torelli map

Definition
We define the tropical Torelli map

$$t_g^{\text{tr}} : M^\text{tr}_g \rightarrow A^\text{tr}_g$$

to send a tropical curve $C \in M^\text{tr}_g$ to its Jacobian $\text{Jac}(C) \in A^\text{tr}_g$.

Theorem (B-M-V)

The map $t_g^{\text{tr}}$ is a morphism of stacky fans. That is, it takes each cell of $M^\text{tr}_g$ to a cell of $A^\text{tr}_g$, and this map is induced by an integral-linear map on the relevant cones.
Figure: Cells of $M_3^{\text{tr}}$ and of $A_3^{\text{tr}}$, color-coded according to $t_g^{\text{tr}}$. 
The tropical Schottky locus

The tropical Torelli map $t^\text{tr}_g$ is surjective when $g = 2, 3$, but not when $g \geq 4$.

Thus, it becomes interesting to study the tropical Schottky locus, i.e. the image of $t^\text{tr}_g$ inside $A^\text{tr}_g$. 
The tropical Schottky locus

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Theorem (C)

We obtained the following computational results:

1. The tropical Schottky locus $A_3^{\text{cogr}}$ has nine cells and $f$-vector $(1, 1, 1, 2, 2, 1, 1)$.
2. The tropical Schottky locus $A_4^{\text{cogr}}$ has 25 cells and $f$-vector $(1, 1, 1, 2, 3, 4, 5, 4, 2, 2)$.
3. The tropical Schottky locus $A_5^{\text{cogr}}$ has 92 cells and $f$-vector $(1, 1, 1, 2, 3, 5, 9, 12, 15, 17, 15, 7, 4)$. 
The tropical Schottky locus: computations

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<th>$A_g^{cogr}$</th>
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Number of maximal cells and total number of cells in the stacky fans $M_g^{tr}$, the Schottky locus $A_g^{cogr}$, and $A_g^{tr}$.

A closer look at the tropical Schottky locus

There is a close relationship between the tropical Schottky locus and cographic matroids.

Let $M$ be a simple regular matroid of rank at most $g$, and let $A$ be a $g \times n$ totally unimodular matrix that represents $M$. Let $v_1, \ldots, v_n$ be the columns of $A$. Then let $\sigma_A \subseteq \mathbb{R}^{g+1 \choose 2}$ be the rational open polyhedral cone

$$\mathbb{R}_{>0} \langle v_1 v_1^T, \ldots, v_n v_n^T \rangle.$$
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$$\mathbb{R} > 0 \langle v_1 v_1^T, \ldots, v_n v_n^T \rangle.$$

**Example.** Let $M$ be the uniform matroid $U_{2,3}$. Then

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

represents $M$, and $\sigma_A$ is the open cone generated by matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$
A closer look at the tropical Schottky locus

Proposition (B-M-V)

The cone $\sigma_A$ is a secondary cone in $\tilde{S}_g^{\geq 0}$. Choosing a different matrix $A'$ to represent $M$ produces a cone $\sigma_{A'}$ that is $GL_g(\mathbb{Z})$-equivalent to $\sigma_A$. Thus, we may associate to $M$ a unique cell of $A^t_{g}$, denoted $C(M)$.

Proposition (B-M-V)

The tropical Schottky locus is the union of cells

\[ \{ C(M) : M \text{ a simple cographic matroid of rank } \leq g \} \]
A closer look at the tropical Schottky locus

What permutations on the rays of $\sigma_A$ are realized by $\text{Stab}(\sigma_A)$?
A closer look at the tropical Schottky locus

What permutations on the rays of $\sigma_A$ are realized by $\text{Stab}(\sigma_A)$?

Theorem (Gerritzen 1980s, C)

*The subgroup of permutations on the rays of $\sigma_A$ that are realized by $\text{Stab}(\sigma_A)$ is isomorphic to $\text{Aut}(M)$.***
A closer look at the tropical Schottky locus

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The subgroup of permutations on the rays of $\sigma_A$ that are realized by $\text{Stab}(\sigma_A)$ is isomorphic to $\text{Aut}(M)$.

Example. Each cell of $A_3^{tr}$ is cographic, and $A_3^{tr}$ is a 6-dimensional closed simplicial cone modulo the automorphisms of the matroid $M(K_4)$, plus some additional identifications along the boundary.
One problem with the spaces $M_{g}^{\text{tr}}$ and $A_{g}^{\text{tr}}$ is that although they are tropical moduli spaces, they do not “look” very tropical: they do not satisfy a tropical balancing condition. In other words: stacky fans, so far, are not tropical varieties.

But what if we allow ourselves to consider finite-index covers of our spaces – can we then produce a more tropical object?

We can do this for $A_{3}^{\text{tr}}$, using the Fano matroid $F_{7}$. 
A tropical cover for $A_3^{tr}$

**Theorem (C)**

Let $\mathbb{F}P^6$ denote the complete polyhedral fan in $\mathbb{R}^6$ usually associated to the toric variety $\mathbb{P}^6$, e.g. with rays $e_1, \ldots, e_6, e_7 := -e_1 - \cdots - e_6$.

Then there is a surjective morphism of stacky fans

$$\mathbb{F}P^6 \rightarrow A_3^{tr}$$

mapping each of the seven maximal cells of $\mathbb{F}P^6$ surjectively onto the maximal cell of $A_3^{tr}$. 
A tropical cover for $A_3^{tr}$

Proof Sketch.

We would like to send each maximal cone of $\mathbb{FP}^6$ to the unique maximal cell of $A_3^{tr}$, with maps that agree on the lower-dimensional cones of $\mathbb{FP}^6$. The only possible obstacle is that not all 3-dimensional and 4-dimensional cells of $A_3^{tr}$ look alike.

However, the Fano matroid precisely gives a way to coherently identify each 6-element set of $\{1, \ldots, 7\}$ with the matroid $M(K_4)$.