

SOLITON DYNAMICS FOR A NON-HAMILTONIAN PERTURBATION OF MKDV

QUANHUI LIN

ABSTRACT. We study the dynamics of soliton solutions to the perturbed mKdV equation $\partial_t u = \partial_x(-\partial_x^2 u - 2u^3) + \epsilon V u$, where $V \in \mathcal{C}_b^1(\mathbb{R})$, $0 < \epsilon \ll 1$. This type of perturbation is non-Hamiltonian. Nevertheless, via symplectic considerations, we show that solutions remain $O(\epsilon t)^{1/2}$ close to a soliton on an $O(\epsilon^{-1})$ time scale. Furthermore, we show that the soliton parameters can be chosen to evolve according to specific exact ODEs on the shorter, but still dynamically relevant, time scale $O(\epsilon^{-1/2})$. Over this time scale, the perturbation can impart an $O(1)$ influence on the soliton position.

1. INTRODUCTION

We consider the modified Korteweg-de Vries (mKdV) equation with a small external potential

$$(1.1) \quad \partial_t u = \partial_x(-\partial_x^2 u - 2u^3) + \epsilon V u.$$

where $0 < \epsilon \ll 1$, $V \in \mathcal{C}_b^1(\mathbb{R})$, i.e. V and V' are continuous and bounded.

The unperturbed case of (1.1),

$$(1.2) \quad \partial_t u = \partial_x(-\partial_x^2 u - 2u^3)$$

is globally well-posed in H^k for $k \geq 1$ (see Kenig-Ponce-Vega [19]), and possesses single soliton solutions $u(x, t) = \eta(x, a + c^2 t, c)$, for $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$, where $\eta(x, a, c) = cQ(c(x - a))$ with $Q(x) = \text{sech}(x)$ (so that $-Q + Q'' + 2Q^3 = 0$). The solitons are orbitally stable as solutions to the unperturbed mKdV (1.2) (see [3, 4, 28, 7]), i.e. the solutions stay close to the soliton manifold

$$M = \{ \eta(x, a, c) \mid a \in \mathbb{R}, c > 0 \}$$

if they are initially close.

Our first main result, Theorem 1.1, shows that this type of orbital stability remains true for the *structurally* perturbed mKdV (1.1), in the following sense: solutions which start an H_x^1 distance ω from the soliton manifold M remain within an H_x^1 distance $(\omega + \epsilon t^{1/2})e^{C\epsilon t}$ up to time $\epsilon^{-1} \log \epsilon^{-1}$. Our second main result, Theorem 1.2, shows that on the shorter time scale $\epsilon^{-1/2} \log \epsilon^{-1}$, we can predict the location on the soliton manifold by solving a system of two ODE for the position parameter a and scale parameter c . Strong agreement between this prediction and the numerical

solution of (1.1) is illustrated in Fig. 1.1 and Fig. 1.2. We prove the global well-posedness of (1.1) in H_x^1 , by adapting the argument of Kenig-Ponce-Vega [19], in Apx. A.

The forced KdV equation

$$(1.3) \quad \partial_t u = \partial_x(-u_{xx} - 3u^2) + \epsilon f$$

is a model for free-surface shallow water flow [20] with contributions to f arising from surface pressure and bottom topography. Numerics and experiments discussed in [20] show that this type of perturbation can effect the evolution of a single soliton by generating a procession of small solitons ahead of, and dispersive waves behind, the primary soliton.

Both (1.1) and (1.3) are specific instances of a family of gKdV equations with general perturbation

$$\partial_t u = \partial_x(-u_{xx} - u^p) + \epsilon f$$

for $p \in \mathbb{N}$, $p \geq 2$, and $f = f(x, t, u)$. The case $p = 3$ (mKdV) is the unique member of the gKdV family that avoids a certain anomaly with the symplectic structure. Specifically, for $p = 3$, one has $\partial_x^{-1} \partial_c \eta \in L^2$ but this fails for $p \neq 3$. For $p = 3$, one can symplectically project onto the tangent space of the soliton manifold M rather than on a skew space. The difference between $p = 3$ and $p \neq 3$ is illustrated in the fact that the local virial estimate of Martel-Merle [21] simplifies for $p = 3$. Nevertheless, we believe that the analysis of the paper carries over in some form to $p \neq 3$ and more general f of the form $f(x, t, u)$. We chose (1.1) as the mathematically simplest case in which to illustrate our method.

1.1. Statements of main results.

Theorem 1.1 (orbital stability). *Let $\delta > 0$ and $a_0, c_0 \in \mathbb{R}$ such that $2\delta \leq c_0 \leq (2\delta)^{-1}$. Suppose $u(x, t)$ solves (1.1) with initial data $u(x, 0)$ such that*

$$\omega \stackrel{\text{def}}{=} \|u(x, 0) - \eta(x, a_0, c_0)\|_{H_x^1} \lesssim \epsilon^{1/2}$$

Then there exist trajectories $a(t)$ and $c(t)$ so that the following hold, where T is the maximum time such that $\delta \leq c(t) \leq \delta^{-1}$ for all $0 \leq t \leq T$ and $w(x, t) \stackrel{\text{def}}{=} u(x, t) - \eta(x, a(t), c(t))$. First, we have the following bounds on the deviation w :

$$(1.4) \quad \|w\|_{L_{[0,t]}^\infty H_x^1} + \|e^{-\alpha|x-a|} w\|_{L_{[0,t]}^2 H_x^1} \leq C(\omega + \epsilon t^{1/2}) e^{C\epsilon t}$$

Second, we have $T \geq C^{-1} \epsilon^{-1}$ and the following estimates for the trajectories $a(t)$ and $c(t)$:

$$(1.5) \quad \|\dot{a} - c^2 - \epsilon c^{-1} \langle V\eta, (x - a)\eta \rangle\|_{L_{[0,t]}^1 \cap L_{[0,t]}^\infty} + \|\dot{c} - \epsilon \langle V\eta, \eta \rangle\|_{L_{[0,t]}^1 \cap L_{[0,t]}^\infty} \leq C(\omega + \epsilon t^{1/2})^2 e^{C\epsilon t}$$

The constants C in (1.4), (1.5) depend on $\|V\|_{C^1}$ and δ .

We remark that the same result holds for $c_0 < 0$, since $\eta(x, a, -c) = -\eta(x, a, c)$.

Theorem 1.2 (exact predictive dynamics). *Suppose $u(x, t)$ solves (1.1) with initial data $u(x, 0)$ satisfying*

$$\omega \stackrel{\text{def}}{=} \|u(x, 0) - \eta(x, a_0, c_0)\|_{H_x^1} \lesssim \epsilon^{1/2}$$

where $c_0 > 0$. Let $(a(t), c(t))$ evolve according to the ODE system

$$(1.6) \quad \begin{cases} \dot{a} = c^2 + \epsilon c^{-1} \langle V\eta, (x - a)\eta \rangle \\ \dot{c} = \epsilon \langle V\eta, \eta \rangle \end{cases}$$

with initial data $a(0) = a_0$, $c(0) = c_0$. Then for

$$0 \leq t \leq T = \sigma \epsilon^{-1/2} \log \epsilon^{-1}, \quad \sigma = \sigma(c_0, \|V\|_{C_b^1}) > 0,$$

we have the following estimates with $w(x, t) = u(x, t) - \eta(x, a(t), c(t))$:

$$(1.7) \quad \|w\|_{L_{[0,t]}^\infty H_x^1} + \|e^{-\alpha|x-a|} w\|_{L_{[0,t]}^2 H_x^1} \leq C(\omega + \epsilon t^{1/2}) e^{C\epsilon^{1/2}t}.$$

where $C = C(c_0, \|V\|_{C_b^1})$.

We remark that if one selects initial data so that $\omega \lesssim \epsilon^{3/4}$, then the two terms on the right-side of the estimate (1.7) balance on the $\epsilon^{-1/2}$ time scale. In this case the bound becomes $\epsilon^{3/4} e^{C\epsilon^{1/2}t}$.

1.2. Relation to recent work. The energy-Lyapunov based methods for proving orbital stability of solitons subject to perturbations (of the data, as opposed to the structural perturbations considered here) were developed by Benjamin [3], Bona [4], Weinstein [28], Grillakis-Shatah-Strauss [11, 12]. In the last decade several results have emerged using the same basic framework to address the dynamics of solitons for equations subject to structural perturbations [6, 9, 10, 13, 14, 16, 17, 18, 8, 1, 2, 23, 24]. The nonlinear Schrödinger equation (NLS) with slowly varying potential was considered by Fröhlich-Gustafson-Jonsson-Sigal [9] and a result of “orbital stability” type was obtained, however the estimates were not strong enough to obtain “exact predictive dynamics”. Holmer-Zworski [18] obtained exact predictive dynamics plus refined accuracy by adopting the conceptual perspective of symplectic projection, but also, at the technical level, finding an appropriate distortion of the soliton manifold that enabled refined Lyapunov estimates. This “symplectic projection plus correction term method” has been subsequently pursued in different contexts in Datchev-Ventura [8], Holmer-Lin [14], Holmer-Perelman-Zworski [16], and Pocovnicu [25]. To treat a problem in which the perturbation gives rise to significant dispersive radiation, a different approach was employed by Holmer [13]. He treated the KdV equation with a slowly varying potential, and used the Martel-Merle local virial estimate [23, 24] to supplement the energy Lyapunov estimate. In this paper, we follow this approach as well. We show the method is sufficiently robust to handle small non-Hamiltonian

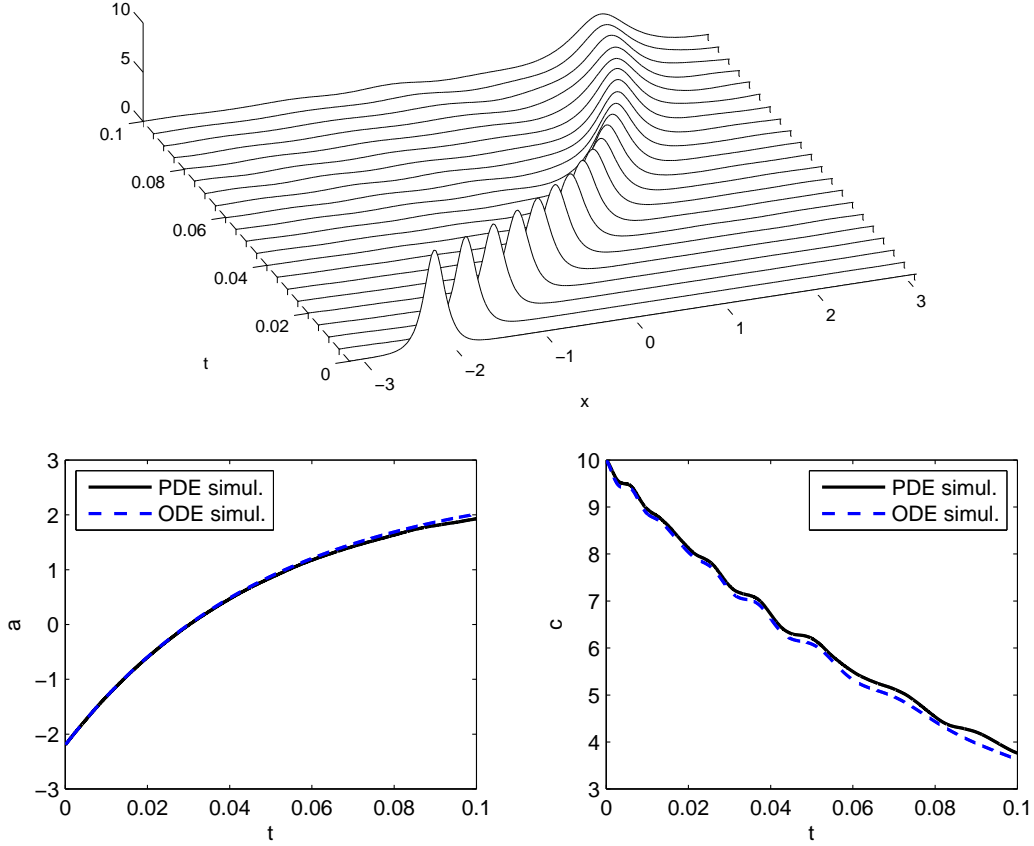


FIGURE 1.1. With external potential given by V_1 as in (1.8), the top plot gives the rescaled evolution $U(X, T)$, the bottom two plots give the comparison between the evolution of the parameters obtained solving the ODE system and exact PDE evolution, i.e. we fit the solution to $\eta(X, \tilde{A}, \tilde{C})$, and plot T versus \tilde{A} and \tilde{C} respectively.

perturbations, which had not been considered in any of the above papers. A stochastic variant of the problem we consider has been addressed by de Bouard–Debussche [5] without the use of the local virial estimate. Work in progress by Holmer–Setayeshgar [15] will adapt the present paper to the stochastic setting and obtain a refinement of the results of [5].

1.3. Numerics. To solve (1.1) numerically we adapt the method in [26] which is based on the fast fourier transform in x , then fourth-order Runge-Kutta for the resulting ODE in t . We use the rescaled coordinate frame $X = \epsilon^{-1/3}x$, $T = \epsilon^{-1}t$, and consider the equation on $[-\pi, \pi)$. If $U(X, T)$ solves

$$\partial_T U = \partial_X (-\partial_X^2 U - 2U^3) + V(X)U,$$

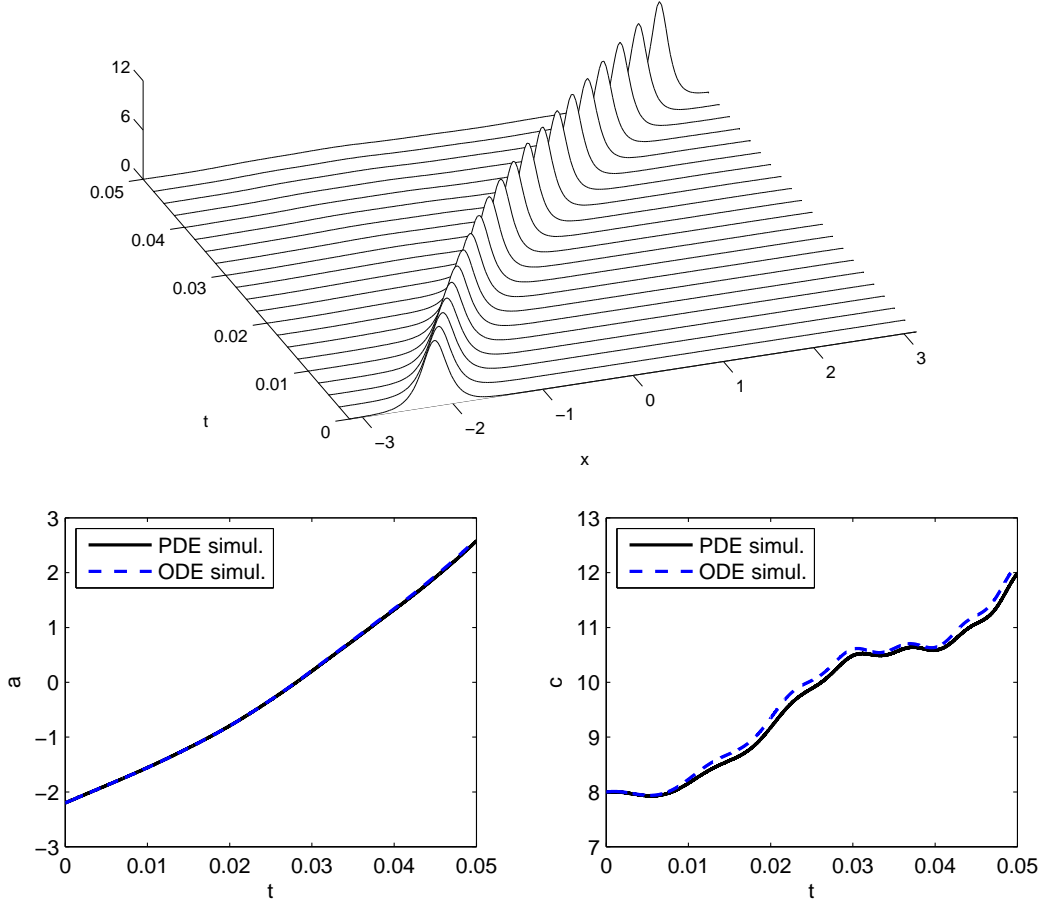


FIGURE 1.2. These plots are analogs of Fig. 1.1, the external potential is given by V_2 as in (1.9).

with initial data

$$U(0, X) = \eta(X, A_0, C_0) = \eta(X, \epsilon^{1/3}a_0, \epsilon^{-1/3}c_0),$$

then $u(x, t) = \epsilon^{1/3}U(\epsilon^{1/3}x, \epsilon t)$ gives a solution of (1.1) on $[-\pi/\epsilon^{1/3}, \pi/\epsilon^{1/3}]$ with initial data $u(0, x) = \eta(x, a_0, c_0)$, and periodic boundary conditions. Fig. 1.1 and Fig 1.2 plot the evolution of the soliton initial data (after rescaling) in the following external potential respectively

$$(1.8) \quad V_1 = -10 \cos^2(6X) + 6 \sin(10X),$$

$$(1.9) \quad V_2 = 8 \cos^2(4X) - 4 \sin(2X).$$

Note that to examine the solution $u(x, t)$ on time interval $0 \leq t \leq C\epsilon^{-1/2}$ (or $C\epsilon^{-1}$), we should let $U(X, T)$ evolve for time $C\epsilon^{1/2}$ (or C).

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2. BACKGROUND ON HAMILTONIAN STRUCTURE

Let $J = \partial_x$, and consider $L^2(\mathbb{R} \mapsto \mathbb{R})$ as a manifold with metric $\langle v_1, v_2 \rangle = \int v_1 v_2 dx$, we can define the symplectic form as

$$(2.1) \quad \omega(v_1, v_2) = \langle v_1 J^{-1} v_2 \rangle = \langle v_1, \partial_x^{-1} v_2 \rangle,$$

where J^{-1} is given by

$$J^{-1} f(x) = \partial_x^{-1} f(x) \stackrel{\text{def}}{=} \frac{1}{2} \left(\int_{-\infty}^x - \int_x^{+\infty} \right) f(y) dy.$$

The mKdV equation (1.2) is the Hamiltonian flow associated with

$$H_0(u) = \frac{1}{2} \int (u_x^2 - u^4),$$

i.e. we can write (1.2) as

$$(2.2) \quad \partial_t u = J H'_0(u).$$

Solutions to mKdV also satisfy conservation of mass $M(u)$ and momentum $P(u)$, where

$$M(u) = \int u dx, \quad P(u) = \frac{1}{2} \int u^2 dx.$$

We define 2-dimensional manifold of solitons M as

$$M = \{ \eta(\cdot, a, c) \mid a \in \mathbb{R}, c \in \mathbb{R} \setminus \{0\} \}.$$

The symplectic form (2.1) restricted to M is given by $\omega|_M = da \wedge dc$. We denote $\eta = \eta(\cdot, a, c)$, the dependence of (a, c) on η is always meant implicitly. The tangent space at η is given by

$$T_\eta M = \text{span}\{ \partial_a \eta, \partial_c \eta \}.$$

Note that $J H'_0(\eta) \in T_\eta M$, thus the flow associated to (1.2) will remain on M if it is initially. Specifically, direct computation shows

$$(2.3) \quad J H'_0(\eta) = c^2 \partial_a \eta.$$

which, together with (2.2), explains the form of the expression for single solitons. This is equivalent to saying that the flow (2.2) restricted to M (and thus stays on M) is given by

$$(2.4) \quad \begin{cases} \dot{a} = c^2 \\ \dot{c} = 0 \end{cases}$$

One can also get (2.4) by first restricting H_0 to M to obtain

$$H_0(\eta) = -\frac{1}{3}c^3,$$

and then noticing that (2.4) is just the solution to the Hamilton equations of motion for $H_0(\eta)$ with respect to $\omega|_M$:

$$(2.5) \quad \begin{cases} \dot{a} = -\frac{\partial H_0}{\partial c} = c^2 \\ \dot{c} = \frac{\partial H_0}{\partial a} = 0 \end{cases}$$

Note that we can write (2.3) as

$$(2.6) \quad JH'_0(\eta) + c^2JP'(\eta) = 0.$$

From this, we learned that $L'(\eta) = 0$, where

$$(2.7) \quad L(u) \stackrel{\text{def}}{=} H_0(u) + c^2P(u).$$

which is the Lyapunov functional used in the classical orbital stability theory, see [28].

Next, we define the symplectic orthogonal projection operator at (a, c) :

$$\Pi_{a,c} : L^2 \cong T_\eta L^2 \rightarrow T_\eta M,$$

by requiring that

$$\langle \Pi_{a,c}^\perp f, J^{-1}\partial_a \eta \rangle = \langle \Pi_{a,c}^\perp f, J^{-1}\partial_c \eta \rangle = 0,$$

where $\Pi_{a,c}^\perp = I - \Pi_{a,c}$, equivalently,

$$\Pi_{a,c} f = \langle f, J^{-1}\partial_c \eta \rangle \partial_a \eta - \langle f, J^{-1}\partial_a \eta \rangle \partial_c \eta.$$

Note that for mKdV,

$$J^{-1}\partial_a \eta = -\eta \text{ and } J^{-1}\partial_c \eta = c^{-1}(x - a)\eta.$$

3. DECOMPOSITION OF THE FLOW

We can arrange the modulation parameters $a(t)$ and $c(t)$ so that

$$\Pi_{a(t),c(t)} [u(x, t) - \eta(x, a(t), c(t))] = 0.$$

This is a standard fact and we recall it in the following

Lemma 3.1. *Given \tilde{a} , \tilde{c} , there exist $\delta_1 > 0$, $C > 0$, such that if $u = \eta(\cdot, \tilde{a}, \tilde{c}) + \tilde{w}$ with $\|\tilde{w}\|_{H_x^1} \leq \delta_1$, then there exist unique a, c such that*

$$(3.1) \quad w(x, t) \stackrel{\text{def}}{=} u(x, t) - \eta(x, a(t), c(t))$$

satisfies the symplectic orthogonality conditions

$$(3.2) \quad \langle w, J^{-1}\partial_a \eta \rangle = \langle w, J^{-1}\partial_c \eta \rangle = 0.$$

Moreover,

$$|a - \tilde{a}| \leq C \|\tilde{w}\|_{H_x^1}, \quad |c - \tilde{c}| \leq C \|\tilde{w}\|_{H_x^1}.$$

Proof. Define $\phi : H_x^1 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$\phi(v, a, c) = \begin{bmatrix} \langle v - \eta, \eta \rangle \\ \langle v - \eta, (x - a)\eta \rangle \end{bmatrix}$$

Using $\omega|_M = da \wedge dc$, we can get the Jacobian matrix of ϕ with respect to (a, c) at $(\eta(\cdot, \tilde{a}, \tilde{c}), \tilde{a}, \tilde{c})$

$$(D_{a,c}\phi)(\eta(\cdot, \tilde{a}, \tilde{c}), \tilde{a}, \tilde{c}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which implies, by the implicit function theorem, that the equation $\phi(u, a, c) = 0$ can be solved for (a, c) in terms of u in a neighborhood of $\eta(\cdot, \tilde{a}, \tilde{c})$. \square

Now since $u = w + \eta$ and u solves (1.1), we compute

$$\begin{aligned} \partial_t w &= \partial_x(-\partial_x^2 w - 6\eta^2 w - 6\eta w^2 - 2w^3) + \epsilon V w - F_0 \\ (3.3) \quad &= \partial_x(\mathcal{L}w - c^2 w - 6\eta w^2 - 2w^3) + \epsilon V w - F_0, \end{aligned}$$

where

$$\mathcal{L} = -\partial_x^2 - 6\eta^2 + c^2,$$

and F_0 results from the perturbation and ∂_t landing on the parameters:

$$F_0 = (\dot{a} - c^2)\partial_a \eta + \dot{c}\partial_c \eta - \epsilon V \eta.$$

Next, decompose F_0 into the symplectically parallel part $\Pi_{a,c} F_0$ and symplectically orthogonal part $\Pi_{a,c}^\perp F_0$, explicitly,

$$(3.4) \quad \Pi_{a,c} F_0 = (\dot{a} - c^2 - \epsilon \langle V \eta, J^{-1} \partial_c \eta \rangle) \partial_a \eta + (\dot{c} + \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle) \partial_c \eta,$$

$$(3.5) \quad \Pi_{a,c}^\perp F_0 = -\epsilon V \eta + \epsilon \langle V \eta, J^{-1} \partial_c \eta \rangle \partial_a \eta - \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle \partial_c \eta.$$

We now obtain the equations for the parameters:

Lemma 3.2 (effective dynamics). *Given $V \in \mathcal{C}_b^1$, suppose that w defined by (3.1) satisfies the orthogonality conditions (3.2). Then there exists $\alpha > 0$ such that*

$$(3.6) \quad \|\partial_t \eta - c^2 \partial_a \eta - \epsilon \Pi_{a,c}(V \eta)\|_{T_{a,c}M} \lesssim \|e^{-\alpha|x-a|} w\|_{H^1}^2 + \epsilon \|e^{-\alpha|x-a|} w\|_{H^1}.$$

Explicitly,

$$\begin{aligned} (3.7) \quad |\dot{a} - c^2 - \epsilon \langle V \eta, J^{-1} \partial_c \eta \rangle| &\lesssim \|e^{-\alpha|x-a|} w\|_{H^1}^2 + \epsilon \|e^{-\alpha|x-a|} w\|_{H^1}, \\ |\dot{c} + \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle| &\lesssim \|e^{-\alpha|x-a|} w\|_{H^1}^2 + \epsilon \|e^{-\alpha|x-a|} w\|_{H^1}. \end{aligned}$$

As all norms on a finite dimensional space are equivalent, we can take

$$\|\alpha \partial_a \eta + \beta \partial_c \eta\|_{T_{a,c}M} = |\alpha| + |\beta|$$

Proof. Recall that

$$\partial_t w = JH_0''(\eta)w - J(6\eta w^2 - 2w^3) + \epsilon Vw - F_0.$$

Write \mathcal{R} for the error terms of the same order as the right hand side of (3.6), take derivative with respect to t for $\langle w, J^{-1}\partial_a\eta \rangle$, we have

$$\begin{aligned} 0 &= \langle \partial_t w, J^{-1}\partial_a\eta \rangle + \langle w, J^{-1}\partial_a\partial_t\eta \rangle \\ (3.8) \quad &= -\langle F_0, J^{-1}\partial_a\eta \rangle + \langle JH_0''(\eta)w, J^{-1}\partial_a\eta \rangle + \langle w, J^{-1}\partial_a\partial_t\eta \rangle + \mathcal{R} \\ &= -\langle F_0, J^{-1}\partial_a\eta \rangle + \langle w, J^{-1}\partial_a(\partial_t\eta - JH_0'(\eta)) \rangle + \mathcal{R} \\ &= -\langle F_0, J^{-1}\partial_a\eta \rangle + \langle w, J^{-1}\partial_a(\Pi_{a,c}F_0) \rangle + \mathcal{R}, \end{aligned}$$

where for the penultimate equality we have used $J^*J^{-1} = -I$ and the self-adjointness of H_0'' , and for the last that

$$\partial_t\eta - JH_0'(\eta) = (\dot{a} - c^2)\partial_a\eta + \dot{c}\partial_c\eta = \Pi_{a,c}F_0 + O(\epsilon)\partial_a\eta + O(\epsilon)\partial_c\eta.$$

Taking derivative for $\langle w, J^{-1}\partial_c\eta \rangle$, similar computation gives

$$0 = -\langle F_0, J^{-1}\partial_c\eta \rangle + \langle w, J^{-1}\partial_c(\Pi_{a,c}F_0) \rangle + \mathcal{R}.$$

Combining with (3.8), and applying the orthogonality conditions for the second terms when ∂_a and ∂_c land on the coefficients of $\Pi_{a,c}F_0$, the lemma follows from Cauchy-Schwarz and the smallness of w . \square

4. LOCAL VIRIAL ESTIMATE

In this section we review, and then apply, part of the local virial estimates due to Martel-Merle. Let $\Phi \in \mathcal{C}(\mathbb{R})$, $\Phi(x) = \Phi(-x)$, $\Phi' \leq 0$ on $(0, \infty)$, such that

$$\Phi(x) = 1 \text{ on } [0, 1], \quad \Phi(x) = e^{-x} \text{ on } [2, \infty), \quad e^{-x} \leq \Phi(x) \leq 3e^{-x} \text{ on } [0, \infty).$$

Let $\Psi(x) = \int_0^x \Phi(y) dy$, and for $A \gg 0$, set $\Psi_A(x) = A\Psi(x/A)$, we have following

Lemma 4.1 (Martel-Merle [21, 22] local virial spectral estimate). *There exists A sufficiently large and λ_0 sufficiently small, such that if w satisfies the orthogonal condition (3.2), then*

$$-\langle \Psi_A(x-a)w, \partial_x \mathcal{L}w \rangle \geq \lambda_0 \int (w_x^2 + w^2) e^{-|x-a|/A} dx.$$

Denoting $\psi(\cdot)$ for $\Psi_A(\cdot - a)$, we now proceed as in [21]:

Lemma 4.2 (local virial estimate). *Suppose V is bounded, then there exist $\alpha > 0$ and $\kappa_j > 0$, $j = 1, 2$, such that if w solves (3.3) and satisfies the orthogonality conditions (3.2), then*

$$(4.1) \quad \|e^{-\alpha|x-a|}w\|_{H_x^1}^2 \leq -\kappa_1 \partial_t \int \psi w^2 dx + \kappa_2 \epsilon^2 + \kappa_2 \epsilon \|w\|_{H_x^1}^2.$$

Proof. From the equation for $\partial_t w$, we have

$$\begin{aligned}
\partial_t \int \psi w^2 &= -\dot{a} \int \psi' w^2 + 2 \int \psi w \partial_t w \\
&= -\dot{a} \int \psi' w^2 + 2 \int \psi w \partial_x (\mathcal{L}w) && \leftarrow \text{I} + \text{II} \\
&\quad - 2c^2 \int \psi w \partial_x w - 12 \int \psi w \partial_x (\eta w^2) && \leftarrow \text{III} + \text{IV} \\
&\quad - 4 \int \psi w \partial_x (w^3) + 2\epsilon \int \psi V w^2 - 2 \int \psi w F_0 && \leftarrow \text{V} + \text{VI} + \text{VII}
\end{aligned}$$

Using integration by parts,

$$\text{III} = c^2 \int \psi' w^2,$$

hence

$$(4.2) \quad |\text{I} + \text{III}| = |-(\dot{a} - c^2) \int \psi' w^2| \lesssim \epsilon \|w\|_{H_x^1}^2 + \|e^{-\alpha|x-a|} w\|_{H_x^1}^2 \|w\|_{H_x^1}^2$$

by (3.7). Following from the boundedness of ψ and V , and the estimate $\|w\|_{L_x^\infty} \lesssim \|w\|_{H_x^1}$, we obtain

$$\begin{aligned}
|\text{IV}| &\lesssim \|e^{-\alpha|x-a|} w\|_{H_x^1}^2 \|w\|_{H_x^1}, \\
|\text{V}| &= \left| 3 \int \psi' w^4 \right| \lesssim \|w\|_{H_x^1}^2 \|e^{-|x-a|/(2A)} w\|_{L_x^2}^2, \\
|\text{VI}| &\lesssim \epsilon \|w\|_{H_x^1}^2,
\end{aligned}
\tag{4.3}$$

where for the second estimate we have used $\psi' = \Phi((x-a)/A)$ and the definition of Φ . Decomposing VII term as

$$\text{VII} = -2 \int \psi w \Pi F_0 - 2 \int \psi w \Pi^\perp F_0 = \text{VIIA} + \text{VIIB},$$

we have by Lemma 3.2 that

$$(4.4) \quad \text{VIIA} \lesssim \epsilon \|w\|_{H_x^1}^2 + \|e^{-\alpha|x-a|} w\|_{H_x^1}^2 \|w\|_{H_x^1},$$

and by $\Pi^\perp F_0 \sim \epsilon \eta$ (see (3.5)) that for any $\mu > 0$,

$$(4.5) \quad \text{VIIB} \lesssim \epsilon \|e^{-\alpha|x-a|} w\|_{H_x^1} \lesssim \mu^{-1} \epsilon^2 + \mu \|e^{-\alpha|x-a|} w\|_{H_x^1}^2$$

Note in above estimates the value of α may change from one line to the next, but we can choose one single small enough α that works for all.

By Lemma 4.1, we have

$$\text{II} = 2 \langle \psi w, \partial_x (\mathcal{L}w) \rangle \leq -\lambda_0 \int (w_x^2 + w^2) e^{-|x-a|/A} dx.$$

Combining with (4.2), (4.3), (4.4) and (4.5), the estimate (4.1) follows by the smallness of $\|w\|_{H_x^1}$, taking A large enough so that $1/(2A) < \alpha$, and $\mu > 0$ suitably small. \square

5. ENERGY ESTIMATE

In this section we formulate the energy estimate necessary for the estimation of the error term w . Recall $\mathcal{L} = -\partial_x^2 - 6\eta^2 + c^2$. Let

$$\mathcal{E} = \frac{1}{2} \langle \mathcal{L}w, w \rangle - 2 \int \eta w^3 dx - \frac{1}{2} \int w^4 dx,$$

Note that $\mathcal{L} = H_0''(\eta) + c^2 = L''(\eta)$, see (2.6) and (2.7). We have classical coercivity properties for \mathcal{L} (for a proof, see e.g. [27, Prop 2.9] or [17, Prop 4.1] for a more direct proof – note that \mathcal{L} is the operator L_+ considered there):

Lemma 5.1 (energy spectral estimate). *Suppose that w satisfies the orthogonality condition (3.2). Then*

$$(5.1) \quad \langle \mathcal{L}w, w \rangle \gtrsim \|w_x\|_{L^2}^2 + c^2 \|w\|_{L_x^2}^2,$$

Since we impose a lower bound on c in Theorems 1.1, it follows from (5.1) that if $\|w\|_{H_x^1}$ is smaller than some (ϵ independent) constant, then

$$\|w(t)\|_{H_x^1}^2 \sim \mathcal{E}(t)$$

Lemma 5.2 (energy estimate). *Suppose we are given $V \in \mathcal{C}_b^1$, $\delta_0 > 0$ and $w(x, t)$, such that $\delta_0 < c(t) < \delta_0^{-1}$, w solves (3.3) and satisfies the orthogonality conditions (3.2), then*

$$(5.2) \quad |\partial_t \mathcal{E}| \lesssim \epsilon \|w\|_{H_x^1}^2 + \epsilon \|e^{-\alpha|x-a|} w\|_{H_x^1} + \|w\|_{H_x^1}^2 \|e^{-\alpha|x-a|} w\|_{H_x^1}^2 + \|w\|_{H_x^1}^6.$$

where the implicit constant depends on δ_0 , σ_0 and the bounds on V and V' .

Proof. We compute

$$\begin{aligned} \partial_t \mathcal{E} &= \langle \mathcal{L}w, \partial_t w \rangle + \dot{c}c \|w\|_{L_x^2}^2 - 6 \langle (\dot{a}\partial_a \eta + \dot{c}\partial_c \eta) \eta w, w \rangle && \leftarrow \text{I} + \text{II} + \text{III} \\ &\quad - \langle \partial_t w, 6\eta w^2 + 2w^3 \rangle - 2 \langle (\dot{a}\partial_a \eta + \dot{c}\partial_c \eta), w^3 \rangle && \leftarrow \text{IV} + \text{V} \end{aligned}$$

Substitute (3.3) into I:

$$\begin{aligned} \text{I} &= \langle \mathcal{L}w, \partial_x(\mathcal{L}w) \rangle - c^2 \langle \mathcal{L}w, \partial_x w \rangle - \langle \mathcal{L}w, \partial_x(6\eta w^2 + 2w^3) \rangle + \langle \mathcal{L}w, \epsilon V w \rangle - \langle \mathcal{L}w, F_0 \rangle \\ &= \text{IA} + \text{IB} + \text{IC} + \text{ID} + \text{IE} \end{aligned}$$

First, $\text{IA} = 0$. Integration by parts yields $\text{IB} = -6c^2 \langle \eta \eta_x, w^2 \rangle$. By the boundedness of V and V' ,

$$\text{ID} \lesssim \epsilon \|w\|_{H_x^1}^2,$$

and since $\mathcal{L}(TM) \subset TM$ (by direct computation), we have

$$\text{IE} = -\langle \mathcal{L}w, \Pi F_0 \rangle - \langle \mathcal{L}w, \Pi^\perp F_0 \rangle = -\langle \mathcal{L}w, \Pi^\perp F_0 \rangle,$$

but by (3.5)

$$|\langle \mathcal{L}w, \Pi^\perp F_0 \rangle| \lesssim \epsilon \|e^{-\alpha|x-a|} w\|_{H_x^1},$$

hence

$$|\mathbf{IE}| \lesssim \epsilon \|e^{-\alpha|x-a|}w\|_{H_x^1}.$$

Combining, we obtain

$$(5.3) \quad \begin{aligned} \mathbf{I} &= \mathbf{IB} + \mathbf{IC} + O\left(\epsilon \|w\|_{H_x^1}^2 + \epsilon \|e^{-\alpha|x-a|}w\|_{H_x^1}\right) \\ &= -6c^2 \langle \eta \eta_x, w^2 \rangle - \langle \mathcal{L}w, \partial_x(6\eta w^2 + 2w^3) \rangle + O\left(\epsilon \|w\|_{H_x^1}^2 + \epsilon \|e^{-\alpha|x-a|}w\|_{H_x^1}\right). \end{aligned}$$

Substituting (3.3) into IV, we have

$$(5.4) \quad \mathbf{IC} + \mathbf{IV} = -\langle \partial_x(-c^2w - 6\eta w^2 - 2w^3) + \epsilon Vw - F_0, 6\eta w^2 + 2w^3 \rangle.$$

By (3.7), we have

$$(5.5) \quad |\dot{a} - c^2| \lesssim \epsilon + \|e^{-\alpha|x-a|}w\|_{H_x^1}^2, \quad |\dot{c}| \lesssim \epsilon + \|e^{-\alpha|x-a|}w\|_{H_x^1}^2,$$

hence

$$|\langle F_0, 6\eta w^2 + 2w^3 \rangle| \lesssim \epsilon \|w\|_{H_x^1}^2 + \|w\|_{H_x^1}^2 \|e^{-\alpha|x-a|}w\|_{H_x^1}^2.$$

Note

$$-\langle \partial_x(-c^2w), 6\eta w^2 + 2w^3 \rangle = -2c^2 \int \eta' w^3 dx.$$

Estimating the rest of the terms in (5.4) using Cauchy-Schwarz and that $\|w\|_{L_x^\infty} \lesssim \|w\|_{H_x^1}$, we obtain

$$(5.6) \quad \mathbf{IC} + \mathbf{IV} = -2c^2 \langle \eta', w^3 \rangle + O(\epsilon \|w\|_{H_x^1}^2 + \|e^{-\alpha|x-a|}w\|_{H_x^1}^2 \|w\|_{H_x^1}^2 + \|w\|_{H_x^1}^6).$$

By (5.5) again, and that $\partial_x \eta = -\partial_a \eta$, we have

$$(5.7) \quad \mathbf{II} + \mathbf{V} = 2\dot{a} \langle \eta', w^3 \rangle + O(\epsilon \|w\|_{H_x^1}^2 + \|e^{-\alpha|x-a|}w\|_{H_x^1}^2 \|w\|_{H_x^1}^2),$$

and

$$(5.8) \quad \begin{aligned} \mathbf{IB} + \mathbf{III} &= 6(\dot{a} - c^2) \langle \eta \eta_x, w^2 \rangle - 6 \langle \dot{c}(\partial_c \eta) \eta, w^2 \rangle \\ &\lesssim \epsilon \|w\|_{H_x^1}^2 + \|e^{-\alpha|x-a|}w\|_{H_x^1}^2 \|w\|_{H_x^1}^2. \end{aligned}$$

Apply (5.5) again to the sum of (5.6) and (5.7), then combine with (5.3) and (5.8), we can obtain (5.2). □

6. PROOF OF THE MAIN THEOREMS

First, we give the proof of Theorem 1.1.

Let $[0, T']$ be the maximal time interval so that

$$(6.1) \quad \|w\|_{L_{[0,T']}^\infty H_x^1} \leq \mu \langle t \rangle^{-1/4}$$

for $\mu > 0$ chosen small enough to ensure the validity of Lemmas 3.1, 3.2, 4.2, and 5.2, and also small enough to beat some constants in the estimates that follow (as explained below).

Let

$$\mathcal{V}(t) \stackrel{\text{def}}{=} \int_0^t \|e^{-\alpha|x-a(s)|} w(s)\|_{H_x^1}^2 ds, \quad \mathcal{F}(t) \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} \|w(s)\|_{H_x^1}^2.$$

Integrating the local virial estimate (4.1) gives

$$(6.2) \quad \mathcal{V}(t) \lesssim \mathcal{F}(t) + \epsilon^2 t + \epsilon \int_0^t \mathcal{F}(s) ds.$$

Integrating (5.2) over $0 \leq t \leq \tau$ yields

$$\mathcal{E}(\tau) \leq \mathcal{E}(0) + \epsilon \int_0^\tau \mathcal{F}(s) ds + \epsilon \tau^{1/2} \mathcal{V}(\tau) + \mathcal{F}(\tau) \mathcal{V}(\tau) + \tau \mathcal{F}(\tau)^3.$$

Using that $\mathcal{E}(\tau) \sim \|w(\tau)\|_{H_x^1}^2$, and then taking the sup of the above estimate over $0 \leq \tau \leq t$, we obtain

$$\mathcal{F}(t) \lesssim \mathcal{F}(0) + \epsilon \int_0^t \mathcal{F}(s) ds + \epsilon t^{1/2} \mathcal{V}(t)^{1/2} + \mathcal{F}(t) \mathcal{V}(t) + t \mathcal{F}(t)^3$$

By (6.1) and the estimate $\epsilon t^{1/2} \mathcal{V}(t)^{1/2} \leq \mu^{-1} \epsilon^2 t + \mu \mathcal{V}(t)$ this implies

$$\mathcal{F}(t) \lesssim \epsilon \int_0^t \mathcal{F}(s) ds + \mathcal{F}(0) + \mu^{-1} \epsilon^2 t + \mu \mathcal{V}(t)$$

Substituting (6.2) into here, taking μ (introduced in (6.1) above) small enough to beat the implicit constants,

$$(6.3) \quad \mathcal{F}(t) \lesssim \epsilon \int_0^t \mathcal{F}(s) ds + \mathcal{F}(0) + \epsilon^2 t.$$

Hence, for some $\kappa > 0$,

$$\frac{d}{dt} \left(e^{-\kappa \epsilon t} \int_0^t \mathcal{F}(s) ds \right) \leq e^{-\kappa \epsilon t} (\mathcal{F}(0) + \epsilon^2 t)$$

Integrating yields

$$\int_0^t \mathcal{F}(s) ds \lesssim (e^{\kappa \epsilon t} - 1)(\epsilon^{-1} \mathcal{F}(0) + 1)$$

Substituting this back into (6.3),

$$\mathcal{F}(t) \lesssim e^{\kappa \epsilon t} \mathcal{F}(0) + \epsilon((e^{\kappa \epsilon t} - 1) + \epsilon t)$$

For the second term, we might as well bound $(e^{\kappa \epsilon t} - 1) + \epsilon t \lesssim \epsilon t e^{\kappa \epsilon t}$, so

$$\mathcal{F}(t) \lesssim e^{\kappa \epsilon t} (\mathcal{F}(0) + \epsilon^2 t)$$

This enables us to reach time $\sigma \epsilon^{-1} \log \epsilon^{-1}$, for $\sigma > 0$ small, while still reinforcing the bootstrap assumption (6.1). Returning to (6.2), we obtain the bound for $\mathcal{V}(t)$, thus completing the proof of (1.4). The $L_{[0,T]}^1$ estimates (1.5) follow from integrating (3.7) in time and applying (1.4). The $L_{[0,T]}^\infty$ estimates also follow from (3.7) by dropping the spatial localization in the terms on the right-hand side of (3.7) and applying the bound on $\|w\|_{L_{[0,T]}^\infty H_x^1}$ given by (1.4).

Now we discuss the proof of Theorem 1.2.

Let \tilde{a}, \tilde{c} solve the ODE system

$$\begin{cases} \dot{\tilde{a}} - \tilde{c}^2 - \epsilon \tilde{c}^{-1} \langle V\tilde{\eta}, (x - \tilde{a})\tilde{\eta} \rangle = 0 \\ \dot{\tilde{c}} - \epsilon \langle V\tilde{\eta}, \tilde{\eta} \rangle = 0 \end{cases}$$

with initial data $\tilde{a}(0) = a_0, \tilde{c}(0) = c_0$, where $\tilde{\eta} = \tilde{c}Q(\tilde{c}(x - \tilde{a}))$. Since $|\dot{c}|, |\dot{\tilde{c}}| \lesssim \epsilon$, we can assume $\delta_0 < c, \tilde{c} < \delta_0^{-1}$ on $[0, T]$. Define

$$\bar{a} = a - \tilde{a}, \quad \bar{c} = c - \tilde{c},$$

we have

$$\langle V\eta, (x - a)\eta \rangle - \langle V\tilde{\eta}, (x - \tilde{a})\tilde{\eta} \rangle = \beta_1(a - \tilde{a}) + \beta_2(c - \tilde{c}),$$

where we have defined

$$\begin{aligned} \beta_1 &= \frac{1}{a - \tilde{a}} \int \left(V\left(\frac{x}{\tilde{c}} + a\right) - V\left(\frac{x}{\tilde{c}} + \tilde{a}\right) \right) x \eta^2 dx, \\ \beta_2 &= \frac{1}{c - \tilde{c}} \int \left(V\left(\frac{x}{c} + a\right) - V\left(\frac{x}{\tilde{c}} + a\right) \right) x \eta^2 dx, \end{aligned}$$

similarly,

$$\frac{1}{c} \langle V\eta, \eta \rangle - \frac{1}{\tilde{c}} \langle V\tilde{\eta}, \tilde{\eta} \rangle = \gamma_1(a - \tilde{a}) + \gamma_2(c - \tilde{c}),$$

where

$$\begin{aligned} \gamma_1 &= \frac{1}{a - \tilde{a}} \int \left(V\left(\frac{x}{\tilde{c}} + a\right) - V\left(\frac{x}{\tilde{c}} + \tilde{a}\right) \right) \eta^2 dx, \\ \gamma_2 &= \frac{1}{c - \tilde{c}} \int \left(V\left(\frac{x}{c} + a\right) - V\left(\frac{x}{\tilde{c}} + a\right) \right) \eta^2 dx, \end{aligned}$$

Denote $\mathcal{R}_1, \mathcal{R}_2$ for the error terms in Lemma 3.2, i.e.

$$\begin{cases} \dot{a} - c^2 - \epsilon c^{-1} \langle V\eta, (x - a)\eta \rangle - \mathcal{R}_1 = 0 \\ \dot{c} - \epsilon \langle V\eta, \eta \rangle - \mathcal{R}_2 = 0, \end{cases}$$

Apply (1.5) to (3.7), we obtain

$$(6.4) \quad \|\mathcal{R}_j\|_{L^1_{[0,t]}} \leq C(\omega + \epsilon t^{1/2})^2 e^{C\epsilon^{1/2}t}, \quad j = 1, 2.$$

Note

$$\frac{\dot{c}}{c} - \frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{\bar{c}}}{c} - \frac{\dot{\bar{c}}}{c\tilde{c}} \bar{c},$$

and

$$c\dot{a} - \tilde{c}\dot{\tilde{a}} = c\dot{\bar{a}} + (c - \tilde{c})\dot{\bar{a}},$$

denoting

$$\theta_1 = \frac{1}{c} \left[(c^2 + \tilde{c}^2 + c\tilde{c}) - (\tilde{c}^2 + \epsilon \tilde{c}^{-1} \langle V\tilde{\eta}, (x - \tilde{a})\tilde{\eta} \rangle) + \epsilon \beta_2 \right],$$

and

$$\theta_2 = \frac{1}{\epsilon} \frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\tilde{c}} \langle V\tilde{\eta}, \tilde{\eta} \rangle,$$

we can obtain the equation for (\bar{a}, \bar{c}) ,

$$(6.5) \quad \begin{bmatrix} \bar{a} \\ \bar{c} \end{bmatrix}' = \begin{bmatrix} \epsilon\beta_1 c^{-1} & \theta_1 \\ \epsilon c\gamma_1 & \epsilon(\theta_2 + c\gamma_2) \end{bmatrix} \begin{bmatrix} \bar{a} \\ \bar{c} \end{bmatrix} + \begin{bmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{bmatrix}.$$

Writing

$$A(t) = \begin{bmatrix} \epsilon\beta_1 c^{-1} & \theta_1 \\ \epsilon c\gamma_1 & \epsilon(\theta_2 + c\gamma_2) \end{bmatrix}.$$

From the boundedness of $\beta_j, \gamma_j, \theta_j, j = 1, 2$, which is a result of the boundedness of V, V', c and \tilde{c} , we have the estimate

$$(6.6) \quad |A(t)| \lesssim \begin{bmatrix} \epsilon & 1 \\ \epsilon & \epsilon \end{bmatrix}.$$

Writing $p(s) = (\epsilon\bar{a}^2 + \bar{c}^2)^{1/2}$, then by above estimate

$$\begin{aligned} |\dot{p}| &\lesssim \frac{1}{p} [\epsilon|\bar{a}|(\epsilon|\bar{a}| + |\bar{c}| + |\mathcal{R}_1|) + |\bar{c}|(\epsilon|\bar{a}| + \epsilon|\bar{c}| + |\mathcal{R}_2|)] \\ &\lesssim \frac{1}{p} [\epsilon(\epsilon\bar{a}^2 + \bar{c}^2) + \epsilon^{1/2}(\epsilon\bar{a}^2 + \bar{c}^2) + \epsilon|\bar{a}||\mathcal{R}_1| + |\bar{c}||\mathcal{R}_2|] \\ &\lesssim \epsilon^{1/2}p + \epsilon^{1/2}|\mathcal{R}_1| + |\mathcal{R}_2|. \end{aligned}$$

By Gronwall and $p(0) = 0$, we obtain

$$p(t) \leq C e^{C\epsilon^{1/2}t} \int_0^t (\epsilon^{1/2}|\mathcal{R}_1| + |\mathcal{R}_2|)(s) ds.$$

Applying (6.4), we obtain

$$p(t) \leq C e^{C\epsilon^{1/2}t} (\omega + \epsilon t^{1/2})^2,$$

recalling the bounds on t and ω in Theorem 1.2, this gives

$$p(t) \leq C\epsilon^{1/2}(\omega + \epsilon t^{1/2})e^{C\epsilon^{1/2}t}.$$

The bounds on \bar{a} and \bar{c} now follow from the definition of p :

$$|\bar{a}| \leq C(\omega + \epsilon t^{1/2})e^{C\epsilon^{1/2}t},$$

$$|\bar{c}| \leq C\epsilon^{1/2}(\omega + \epsilon t^{1/2})e^{C\epsilon^{1/2}t}.$$

Compare the above two estimates with (1.7), we can conclude the proof of Theorem 1.2.

Remark 6.1. The $\epsilon^{-1/2}$ constraint on the time scale stems from the fact that the eigenvalues of $\begin{bmatrix} \epsilon & 1 \\ \epsilon & \epsilon \end{bmatrix}$ are only of order $\epsilon^{1/2}$.

APPENDIX A. LOCAL AND GLOBAL WELL-POSEDNESS

The global well-posedness for gKdV in energy space was obtained by Kenig-Ponce-Vega in [19], where they introduced new and powerful local smoothing and maximal function estimates, especially, they proved the local well-posedness for (1.2) in $H^s(\mathbb{R})$ for $s \geq 1/4$. To prove well-posedness for (1.1) at H^1 level of regularity, the full strength of these estimates is not needed, we here follow the presentation of [16] Apx. A and make necessary modifications.

Let $Q_n = [n - \frac{1}{2}, n + \frac{1}{2}]$, and $\tilde{Q}_n = [n - 1, n + 1]$. An example of notation is:

$$\|u\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} = \sup_n \|u\|_{L_{[0,T]}^2 L_{Q_n}^2}.$$

Note that due to the finite incidence of overlap, we have

$$\|u\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} \sim \|u\|_{\ell_n^\infty L_T^2 L_{\tilde{Q}_n}^2}.$$

We omit the ϵ in (1.1), and consider

$$(A.1) \quad \partial_t u = \partial_x(-\partial_x^2 u - 2u^3) + Vu, \quad V \in \mathcal{C}_b^1.$$

As in [16], we first prove a local smoothing estimate and a maximal function estimate (weak versions), by an integrating factor method:

Lemma A.1. *Suppose that*

$$(A.2) \quad v_t + v_{xxx} - Vv = f,$$

then there exists $C > 0$, such that if

$$T \leq C(1 + \|V\|_{L_x^\infty})^{-1},$$

we have the energy and local smoothing estimates

$$(A.3) \quad \|v\|_{L_T^\infty L_x^2} + \|v_x\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} \lesssim \|v_0\|_{L_x^2} + \begin{cases} \|\partial_x^{-1} f\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \\ \|f\|_{L_T^1 L_x^2} \end{cases}$$

and the maximal function estimate

$$(A.4) \quad \|v\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \lesssim \|v_0\|_{L_x^2} + T^{1/2} \|v\|_{L_T^2 H_x^1} + T^{1/2} \|f\|_{L_T^2 L_x^2}.$$

The implicit constants are independent of V .

Proof. Let $\phi(x) = -\tan^{-1}(x - n)$, and set $w(x, t) = e^{\phi(x)} v(x, t)$. By (A.2),

$$\partial_t w + w_{xxx} - 3\phi' w_{xx} + 3(-\phi'' + (\phi')^2)w_x + (-\phi''' + 3\phi''\phi' - (\phi')^3)w - Vw = e^\phi f,$$

integrating its product with $\frac{1}{2}w$ over x ,

$$\partial_t \|w\|_{L_x^2}^2 = -6\langle \phi', w_x^2 \rangle + \langle -\phi''' + 2(\phi')^3, w^2 \rangle + 2\langle V, w^2 \rangle + 2\langle e^\phi f, w \rangle,$$

integrating this identity over $[0, T]$, and using $\phi'(x) = -\langle x - n \rangle^{-2}$, we obtain

$$\begin{aligned} & \|w(T)\|_{L_x^2}^2 + 6\|\langle x - n \rangle^{-1} w_x\|_{L_T^2 L_x^2}^2 \\ & \leq \|w_0\|_{L_x^2}^2 + C_1 T(1 + \|V\|_{L_x^\infty}) \|w\|_{L_T^\infty L_x^2}^2 + C_1 \int_0^T \left| \int e^\phi f w \, dx \right| dt, \end{aligned}$$

for some constant $C_1 > 0$, replace T by t , and take supremum over $t \in [0, T]$, we obtain, for $T \leq \frac{1}{2} C_1^{-1} (1 + \|v\|_{L_x^\infty})$, the estimate

$$\|w\|_{L_T^\infty L_x^2}^2 + \|\langle x - n \rangle^{-1} w_x\|_{L_T^2 L_x^2}^2 \lesssim \|w_0\|_{L_x^2}^2 + \int_0^T \left| \int e^\phi f w \, dx \right| dt,$$

note that $0 < e^{-\pi/2} \leq e^{\phi(x)} \leq e^{\pi/2} < \infty$, we can convert the above estimate back to an estimate for v :

$$\|v\|_{L_T^\infty L_x^2}^2 + \|v_x\|_{L_T^2 L_{Q_n}^2}^2 \lesssim \|v_0\|_{L_x^2}^2 + \int_0^T \left| \int e^{2\phi} f v \, dx \right| dt.$$

Estimating as

$$\int_0^T \left| \int e^{2\phi} f v \, dx \right| dt \lesssim \|f\|_{L_T^1 L_x^2} \|v\|_{L_T^\infty L_x^2},$$

and then taking the supremum in n yields the second estimate in (A.3). Estimating instead as

$$\begin{aligned} \int_0^T \left| \int e^{2\phi} f v \, dx \right| dt &= \int_0^T \left| \int (\partial_x^{-1} f \partial_x (e^{2\phi} v)) \, dx \right| dt \\ &\leq \sum_m \|\partial_x^{-1} f\|_{L_T^2 L_{Q_m}^2} \|\langle \partial_x \rangle v\|_{L_T^2 L_{Q_m}^2} \\ &\leq \|\partial_x^{-1} f\|_{\ell_m^1 L_T^2 L_{Q_m}^2} \|\langle \partial_x \rangle v\|_{\ell_m^\infty L_T^2 L_{Q_m}^2} \end{aligned}$$

and then taking the supremum in n yields the second estimate in (A.3).

For the proof of estimate (A.4), take $\phi(x) = 1$ on $[n - \frac{1}{2}, n + \frac{1}{2}]$ and 0 outside $[n - 1, n + 1]$, set $w = \phi v$, and compute similarly as the above.

□

Using estimates in the above lemma, we can prove:

Theorem A.2 (local well-posedness in H_x^1). *Suppose that*

$$(A.5) \quad M \stackrel{\text{def}}{=} \|V\|_{L_x^\infty} + \|V'\|_{L_x^\infty} < \infty.$$

For any $R \geq 1$, take

$$T \lesssim \min(M^{-1}, R^{-2}),$$

we have

- (1) If $\|u_0\|_{H^1} \leq R$, there exists a solution $u(t) \in \mathcal{C}([0, T]; H_x^1)$ to (A.1) on $[0, T]$ with initial data $u_0(x)$ satisfying

$$\|u\|_{L_T^\infty H_x^1} + \|u_{xx}\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} \lesssim R.$$

- (2) *This solution $u(t)$ is unique among all solutions in $\mathcal{C}([0, T]; H_x^1)$.*
 (3) *The data-to-solution map $u_0 \mapsto u(t)$ is continuous as a mapping $H^1 \rightarrow \mathcal{C}([0, T]; H_x^1)$.*

Proof. We prove the existence by contraction in the space X , where

$$X = \{ u \mid \|u\|_{\mathcal{C}([0, T]; H_x^1)} + \|u_{xx}\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} + \|u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \leq CR \},$$

where the constant C is chosen large enough to (10 times, say) exceed the implicit constants in Lemma A.1. Given $u \in X$, let $\varphi(u)$ denote the solution to

$$(A.6) \quad \partial_t \varphi(u) + \partial_x^3 \varphi(u) - V \varphi(u) = -2\partial_x(u^3).$$

with initial condition $\varphi(u)(0) = u_0$. A fixed point $\varphi(u) = u$ in X will solve (A.1).

The local smoothing estimate (A.3) applied to $v = \varphi(u)$ and the estimate

$$\|(u^3)_x\|_{L_T^1 L_x^2} \lesssim T \|u\|_{L_T^\infty H_x^1}^3$$

give the estimate

$$(A.7) \quad \|\varphi(u)\|_{L_T^\infty L_x^2} \lesssim \|u_0\|_{H_x^1} + T \|u\|_{L_T^\infty H_x^1}^3,$$

The maximal function estimate (A.4) applied to $v = \varphi(u)$ and the estimate

$$\|(u^3)_x\|_{L_T^2 L_x^2} \lesssim T^{1/2} \|u\|_{L_T^\infty H_x^1}^3$$

imply the estimate

$$(A.8) \quad \|\varphi(u)\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \lesssim \|u_0\|_{L_x^2} + T \|\varphi(u)\|_{L_T^\infty H_x^1} + T \|u\|_{L_T^\infty H_x^1}^3.$$

Now applying ∂_x to (A.6), and denoting $v = \varphi(u)_x$ instead:

$$v_t + v_{xxx} - Vv = -2(u^3)_{xx} + V'\varphi(u).$$

By (A.3) again,

$$(A.9) \quad \|\varphi(u)_x\|_{L_T^\infty L_x^2} + \|\varphi(u)_{xx}\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} \lesssim \|u_0\|_{H_x^1} + \|(u^3)_x\|_{\ell_n^1 L_T^2 L_{Q_n}^2} + \|V'\varphi(u)\|_{L_T^1 L_x^2}.$$

Applying Gagliardo-Nirenberg inequality to $\phi(x)u$, where $\phi(x) = 1$ on $[n - \frac{1}{2}, n + \frac{1}{2}]$ and 0 outside $[n - 1, n + 1]$, we obtain (writing Q for Q_n and \tilde{Q} for \tilde{Q}_n for the following):

$$\|u\|_{L_Q^\infty}^2 \lesssim (\|u\|_{L_Q^2} + \|u_x\|_{L_Q^2}) \|u\|_{L_Q^2},$$

hence

$$\|(u^3)_x\|_{L_Q^2} \lesssim \|u_x\|_{L_Q^2} \|u\|_{L_Q^\infty}^2 \lesssim \|u_x\|_{L_Q^2} \|u\|_{L_Q^2} (\|u\|_{L_Q^2} + \|u_x\|_{L_Q^2}).$$

Taking L_T^2 norm and applying the Hölder inequality, we obtain

$$\|(u^3)_x\|_{L_T^2 L_Q^2} \lesssim \|u_x\|_{L_T^\infty L_Q^2} \|u\|_{L_T^\infty L_Q^2} (\|u\|_{L_T^2 L_Q^2} + \|u_x\|_{L_T^2 L_Q^2}).$$

Taking ℓ_n^1 norm and applying the Hölder inequality again yields

$$\|(u^3)_x\|_{\ell_n^1 L_T^2 L_Q^2} \lesssim \|u_x\|_{\ell_n^\infty L_T^\infty L_Q^2} \|u\|_{\ell_n^2 L_T^\infty L_Q^2} (\|u\|_{\ell_n^2 L_T^2 L_Q^2} + \|u_x\|_{\ell_n^2 L_T^2 L_Q^2}).$$

Using the bounds $\|u_x\|_{\ell_n^\infty L_T^\infty L_Q^2} \lesssim \|u_x\|_{L_T^\infty L_x^2}$,

$$\|u\|_{\ell_n^2 L_T^2 L_Q^2} \lesssim \|u\|_{L_T^2 L_x^2} \lesssim T^{1/2} \|u\|_{L_T^\infty L_x^2}$$

and

$$\|u_x\|_{\ell_n^2 L_T^2 L_Q^2} \lesssim \|u_x\|_{L_T^2 L_x^2} \lesssim T^{1/2} \|u_x\|_{L_T^\infty L_x^2},$$

we obtain

$$\|(u^3)_x\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \lesssim T^{1/2} \|u\|_{L_T^\infty H_x^1}^2 \|u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2},$$

inserting into (A.9),

$$\begin{aligned} (A.10) \quad & \|\varphi(u)_x\|_{L_T^\infty L_x^2} + \|\varphi(u)_{xx}\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} \\ & \lesssim \|u_0\|_{H_x^1} + T^{1/2} \|u\|_{L_T^\infty H_x^1}^2 \|u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} + T \|V'\|_{L_x^\infty} \|\varphi(u)\|_{L_T^\infty L_x^2}. \end{aligned}$$

Summing (A.7), (A.8) and (A.10), we obtain that $\|\varphi(u)\|_X \leq CR$ if $\|u\|_X \leq CR$ provided $T \leq C_0 \min(M^{-1}, R^{-2})$, with C_0 small enough. Thus $\varphi : X \rightarrow X$. A similar argument establishes that φ is a contraction on X .

Now suppose $u, v \in \mathcal{C}([0, T]; H_x^1)$ solve (A.1). By (A.4),

$$\begin{aligned} (A.11) \quad & \|u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \lesssim \|u_0\|_{L_x^2} + T \|u\|_{L_T^\infty H_x^1} + T \|u\|_{L_T^\infty H_x^1}^3, \\ & \|v\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \lesssim \|v_0\|_{L_x^2} + T \|v\|_{L_T^\infty H_x^1} + T \|v\|_{L_T^\infty H_x^1}^3, \end{aligned}$$

Set $w = u - v$. Then, with $g = (u^3 - v^3)/(u - v) = u^2 + uv + v^2$, we have

$$w_t + w_{xxx} + 2(gw)_x - Vw = 0.$$

Apply (A.3) to $v = w_x$, we obtain

$$(A.12) \quad \|w_x\|_{L_T^\infty L_x^2} \lesssim \|(gw)_x\|_{\ell_n^1 L_T^2 L_{Q_n}^2} + \|V'w\|_{L_T^1 L_x^2}.$$

The terms of $\|(gw)_x\|_{\ell_n^1 L_T^2 L_{Q_n}^2}$ can be bounded in the following manner:

$$\begin{aligned} (A.13) \quad & \|u_x v w\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \lesssim \|u_x\|_{\ell_n^\infty L_T^\infty L_{Q_n}^2} \|v w\|_{\ell_n^1 L_T^2 L_{Q_n}^\infty} \\ & \lesssim \|u_x\|_{\ell_n^\infty L_T^\infty L_{Q_n}^2} (\|v w\|_{\ell_n^1 L_T^2 L_{Q_n}^1} + \|(v w)_x\|_{\ell_n^1 L_T^2 L_{Q_n}^1}) \end{aligned}$$

The term in the parentheses is bounded by

$$\|v\|_{\ell_n^2 L_T^2 L_{Q_n}^2} \|w\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} + \|v_x\|_{\ell_n^2 L_T^2 L_{Q_n}^2} \|w\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} + \|v\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \|w_x\|_{\ell_n^2 L_T^2 L_{Q_n}^2}$$

which by (A.11) and

$$\|u_x\|_{\ell_n^\infty L_T^\infty L_{Q_n}^2} \lesssim \|u\|_{L_T^\infty H_x^1}, \quad \|v\|_{\ell_n^2 L_T^2 L_{Q_n}^2} \lesssim T^{1/2} \|v\|_{L_T^\infty L_x^2}$$

implies

$$\|u_x v w\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \lesssim \|u\|_{L_T^\infty H_x^1} \|v\|_{L_T^\infty H_x^1} T^{1/2} (\|w\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} + \|w\|_{L_T^\infty H_x^1}).$$

Same bounds follow for other terms in $\|(gw)_x\|_{\ell_n^1 L_T^2 L_{Q_n}^2}$, combined with $\|V'w\|_{L_T^1 L_x^2} \lesssim T\|V'\|_{L_x^\infty}\|w\|_{L_T^\infty H_x^1}$, this establishes the estimate

$$\|w_x\|_{L_T^\infty L_x^2} \lesssim T^{1/2}(\|w\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} + \|w\|_{L_T^\infty H_x^1}),$$

where the implicit constant depends on $\|u\|_{L_T^\infty H_x^1}$ and $\|v\|_{L_T^\infty H_x^1}$. Same estimate follows for $\|w\|_{L_T^\infty L_x^2}$ by applying (A.3) to $v = w$. Hence

$$(A.14) \quad \|w\|_{L_T^\infty H_x^1} \lesssim T^{1/2}(\|w\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} + \|w\|_{L_T^\infty H_x^1}),$$

but applying (A.4) to $v = w$ yields

$$(A.15) \quad \|w\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \lesssim T\|w\|_{L_T^\infty H_x^1}$$

since e.g.

$$\|uvw_x\|_{L_T^2 L_x^2} \lesssim T^{1/2}\|w\|_{L_T^\infty H_x^1}\|u\|_{L_T^\infty H_x^1}\|v\|_{L_T^\infty H_x^1}$$

which can be proved by the same method as in (A.13), and thus $\|(gw)_x\|_{L_T^2 L_x^2} \lesssim T^{1/2}\|w\|_{L_T^\infty H_x^1}$. Substituting (A.15) into (A.14) implies $w \equiv 0$ for T sufficiently small, which then establishes the uniqueness of solutions in $\mathcal{C}([0, T]; H_x^1)$. The continuity of the data-to-solution map can be proved by a similar argument. \square

We now prove the global well-posedness in H^1 by (almost) conservation laws.

Theorem A.3 (global well-posedness). *Suppose $M < \infty$, where M is defined in (A.5), for $u_0 \in H^1$, there is a unique global solution $u \in C_{loc}([0, \infty); H_x^1)$ to (A.1) with $\|u\|_{L_T^\infty H_x^1}$ controlled by $\|u_0\|_{H^1}$, T and M .*

Proof. First, note from Gagliardo-Nirenberg inequality, $\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^3 \|u_x\|_{L^2}$, we have

$$\|u_x\|_{L^2}^2 - \|u\|_{L^2}^3 \|u_x\|_{L^2} \leq H_0(u) \leq \|u_x\|_{L^2}^2.$$

Applying Peter-Paul inequality to the $\|u\|_{L^2}^3 \|u_x\|_{L^2}$ term gives us

$$\|u_x\|_{L^2}^2 + \|u\|_{L^2}^6 \sim H_0(u) + \|u\|_{L^2}^6.$$

Suppose u solves (A.1), then

$$(A.16) \quad \begin{aligned} \left| \frac{d}{dt} H_0(u) \right| &= |\langle H'_0(u), JH'_0(u) + Vu \rangle| = |\langle H'_0(u), Vu \rangle| \\ &\lesssim M(\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u\|_{L^4}^4) \lesssim M(\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u\|_{L^2}^3 \|u_x\|_{L^2}) \\ &\lesssim M(\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u\|_{L^2}^6) \lesssim M(H_0(u) + \|u\|_{L^2}^2 + \|u\|_{L^2}^6), \end{aligned}$$

on the other hand, by

$$\left| \frac{d}{dt} P(u) \right| = |\langle u, Vu \rangle| \lesssim MP(u),$$

and Gronwall inequality, we obtain a bound on $\|u\|_{L_T^\infty L_x^2}$ in terms of $\|u_0\|_{L^2}$ and M , combine this with (A.16), and apply Gronwall again, we obtain a bound on $H_0(u)$ and hence $\|u\|_{H_x^1}$. \square

Remark A.4. A global well-posedness in H_x^k for $k \geq 1$ can in fact be proved, provided $V \in \mathcal{C}_b^k$, by similar arguments.

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