1. Let $R$ be a commutative ring, let $R'$ be a commutative $R$-algebra. Prove that if an $R$-algebra $A$ is finite étale over $R$, then $A \otimes_R R'$ is finite étale over $R'$.

2. Prove that $A = \{(n, m) \in \mathbb{Z}^2 \mid n \equiv m \mod 2\}$ is not finite étale as $\mathbb{Z}$-algebra.

3. Give coordinate free definitions of determinant and trace, and prove that they coincide with the ‘usual definition’ that picks a basis first.

4. Let $g \geq 1$ be an integer. Prove that
   \[
   \mathbb{Z}_g \cong \prod_{p \mid g} \mathbb{Z}_p
   \]
as topological rings.

5. Prove that $\hat{\mathbb{Z}}$ is flat as $\mathbb{Z}$-module.

6. Prove that there is an isomorphism $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ of topological rings. Here, the product ranges over all prime numbers $p$.

7. A Steinitz number or supernatural number is a formal expression
   \[
a = \prod_p p^{a(p)},
   \]
where $p$ ranges over the set of all prime numbers, and $a(p) \in \{0, 1, 2, \ldots, \infty\}$ for each $p$. If $a = \prod_p p^{a(p)}$ is a Steinitz number, then we write $a\hat{\mathbb{Z}}$ for the subgroup of $\hat{\mathbb{Z}}$ that under the isomorphism $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ from the previous exercise corresponds to $\prod_p p^{a(p)}\mathbb{Z}_p$, with $p^\infty\mathbb{Z}_p = \{0\}$.
   (a) Let $a$ be a Steinitz number. Prove that $a\hat{\mathbb{Z}}$ is the intersection of all groups $n\hat{\mathbb{Z}}$, with $n$ ranging over the positive integers that divide $a$ (definition obvious).
   (b) Prove that the map sending $a$ to $a\hat{\mathbb{Z}}$ gives a bijection from the set of Steinitz numbers to the set of closed subgroups of $\hat{\mathbb{Z}}$. For which Steinitz numbers $a$ is $a\hat{\mathbb{Z}}$ an open subgroup of $\hat{\mathbb{Z}}$?
   (c) Let $k$ be a finite field. Explain how Steinitz numbers classify $k$-isomorphism classes of algebraic field extensions $l/k$.
   (d) A profinite group $\pi$ is called procyclic if there exists $\sigma \in \pi$ such that the subgroup generated by $\sigma$ is dense in $\pi$. Prove that $\pi$ is procyclic if and only if $\pi \cong \hat{\mathbb{Z}}/a\hat{\mathbb{Z}}$ for some Steinitz number $a$. Is $a$ unique?
8. Lang, Exercise III.14.

9. An $R$-module $M$ is called *projective* if and only if the functor $\text{Hom}_R(M, -)$ is exact. Suppose that $M$ is finitely generated and projective. Prove that there exists a finitely generated projective $R$-module $Q$ with $M \oplus Q \cong R^n$ for some non-negative integer $n$.

10. An $R$-module $M$ is called *injective* if and only if the functor $\text{Hom}_R(-, M)$ is exact. Prove that a $\mathbb{Z}$-module is injective if and only if it is divisible as abelian group (meaning that $x \mapsto nx$ is surjective for all integers $n$).

11. Let $R$ be commutative, and suppose that $R[X]$ is Noetherian. Prove that $R$ is Noetherian. (This is the converse of Hilbert’s basis theorem.)