1. Let $p$ be a prime number. Find all solutions to $x^3 \equiv x \mod p$ and prove that these are the only solutions.

2. (a) Calculate the greatest common divisor of 200 and 88 using the Euclidean Algorithm. Show all calculations for full credit.
(b) Find all integer solutions to $200x + 88y = 24$ or show that none exist.

3. Solve the given system of congruences. Show all calculations for full credit.
\[
x \equiv 3 \mod 11
\]
\[
x \equiv 2 \mod 9
\]

4. Prove the following: Let $p$ and $q$ be distinct odd primes with $(p-1)|(q-1)$. Let $a$ be an integer such that $\gcd(a, pq) = 1$. Then $a^{\varphi(q)} \equiv 1 \mod pq$.

5. Evaluate $18^{19^{20}} \mod 21$.
   Your answer should be an integer between 0 and 20.
   Note that $18^{19^{20}} = 18^{(19^{20})}$ and NOT $(18^{19})^{20}$.

6. (a) Is 2 a primitive root modulo 31? Explain all work to justify your answer.
(b) Find all incongruent solutions to $x^5 \equiv 1 \mod 31$ or show that none exist. You may use the fact that 3 is a primitive root modulo 31. You do not need to simplify your final answer(s).
(c) Find all incongruent solutions to $x^5 \equiv 1 \mod 37$ or show that none exist. You may use the fact that 2 is a primitive root modulo 37. You do not need to simplify your final answer(s).

7. Prove the following: Let $a$ be a positive integer and let $p$ be an odd prime number with $p \nmid a$. Then
\[
\left( \frac{a}{p} \right) + \left( \frac{2a}{p} \right) + \left( \frac{3a}{p} \right) + \cdots + \left( \frac{(p-1)a}{p} \right) = 0.
\]

8. Compute the Legendre symbol $\left( \frac{15}{101} \right)$. Note that 101 is a prime number.

9. (Bonus: 5 points) Prove that the number $a$ is divisible by 11 if and only if the alternating sum of the digits of $a$ is divisible by 11 (for example, if $a = 7390902$, then the alternating sum of the digits is $7 - 3 + 9 - 0 + 9 - 0 + 2$). Hint: write $a$ as a sum of multiples of powers of 10.