Math 101: Course Summary

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General Information: Math 101 is a first course in real analysis. The main purpose of this class is to introduce real analysis, and a secondary purpose is get you used to the idea of writing rigorous mathematical proofs. I would say that M101 is one of the easier of the upper level course. If you are nervous about starting the upper level math courses, you might want to start with Math 101.

You could say that real analysis is the "nitty gritty behind calculus". When you take calculus courses, there are various technical things – definition of a real number, definition of a limit, definition of continuity, proof of the fundamental theorem of calculus, etc. – that are usually mentioned but not discussed in great detail. Real analysis addresses these topics in detail. So, one of the things you do in a real analysis course is precisely define all the concepts, and prove all the main results, that were (probably) taken for granted in the calculus series.

However, saying that real analysis is the "nitty-gritty behind calculus" does not do justice to this great subject. A better description of real analysis is that it is the "analysis of infinite processes that relate to real numbers." Computing limits, taking derivatives, and calculating integrals are all examples of infinite processes, but real analysis emcompasses a much wider range of topics than just the ones you've already seen in calculus.

What is a Real Number? Most people have known about real numbers since grade school. A rough and ready way to describe a real number is that anything with a decimal expansion is a real number. Numbers like 17 and $\pi = 3.1415926...$ are examples of real numbers. With this definition, you have to be a bit careful. The two expressions .99999... and 1 both describe the same number. So, you would really have to say that a real number is a decimal expansion, but with the proviso that certain decimal expansions name the same number. To be formal about it, you could say that the decimal expansion 3.14159... is the limit of the series

$$3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \dots.$$

So, first of all, you would have to know about about series and limits. Then, you would have to say that a real number is really an *equivalence class* of such expansions. Making the decimal expansion definition work is actually a bit clumsy, and so a real analysis class usually takes different (but closely related) approaches.

A Definition of the Reals: I'll give a definition of a real number that assumes the existence and basic properties of the rational numbers. Defining a real number is not usually the first thing that is done in M 101, but I hope my explanation gives you a feel for what you'll see in M101. Here we go. A sequence of rational numbers $\{r_n\}$ is called *Cauchy* if for every integer $N > 0$, there is another integer $M > 0$ such that

$$\min(m, n) > M \qquad \Rightarrow \qquad |r_n - r_m| < 1/N.$$

Informally this definition says that the sequence $\{r_n\}$ becomes "eventually nearly constant": If you look far enough out (beyond the Mth term) then all the terms are nearly the same (within $1/N$ of each other.) To relate this to what I've already said, the sequence

$$r_1 = 3; \qquad r_2 = 3 + \frac{1}{10}; \qquad r_3 = 3 + \frac{1}{10} + \frac{4}{100}; \dots$$

is a Cauchy sequence. You can turn any decimal expansion into a Cauchy sequence in the way suggested by the example.

Two Cauchy sequences $\{r_n\}$ and $\{s_n\}$ are said to be *equivalent* if the *shuffled sequence* $r_1, s_1, r_2, s_2, r_3, s_3, ..$ is also a Cauchy sequence. For instance, the Cauchy sequences determined by the decimal expansions 1.0000 and 0.9999 are equivalent. By definition, a *real number* is an equivalence class of Cauchy sequences. That is, a real number is an exhaustive collection of Cauchy sequences, all of which are equivalent to each other.

Once we've defined real numbers in this way, we want to recover their basic properties. For instance, how does one define $x + y$, where x and y are real numbers as we've defined them? It works like this, Let $\{r_n\}$ be a Cauchy sequence belonging to the equivalence class x and let $\{s_n\}$ be a Cauchy sequence in the equivalence class y. Then $x+y$ is defined to be the equivalence class of the sequence $\{r_n+s_n\}$. To make sure this works, there are a few little things to check. For instance, you have to check that $\{r_n + s_n\}$ is a Cauchy sequence. Then you have to check that you would get the same answer if you started with $\{r'_n\}$ and $\{s'_n\}$, different Cauchy sequences representing x and y respectively. Once you make these checks, addition is defined. Then you define subtraction, multiplication, etc, and check it all works. When you're done, you have a precise definition of the reals that is solidly built upon the rationals, and you recover all the basic properties of the real number system that you'd known informally for years.

So What? You might wonder what has been gained by this exercise. True, we haven't uncovered any mystical new properties of real numbers. However, we can now say exactly what we mean by "real number". Also, when it comes time to deal with intricate properties of the reals, such as the existence of a continuous but nowhere differentiable function, having a precise definition of the reals (and continuity and differentiation) will give you tools to analyze what is going on.

You might also worry that it will be impossible to do much work with our definition of the reals, since it (perhaps) seems both complicated and strange. This isn't the case. Once you establish the basic properties of real numbers (e.g. the associative law) you can use these properties freely without going back to the original definition every time. Real analysis builds from the ground up, but you don't have to go back to the ground floor every time you want to do something in the subject.

Limits: Once the real numbers are defined, you can talk about limits, continuity, etc. There are several approaches to all these definitions, and the approach I'll take is not necessarily the one you'll see in M101. However, all the approaches are pretty similar to each other.

A sequence $\{x_n\}$ of real numbers is said to *converge* to the number x if for every integer N, there is some M such that

$$n > M \qquad \Rightarrow \qquad |x_n - x| < 1/N.$$

The fact that $|x_n - x|$ makes sense for real numbers is one of the basic properties alluded to above. Here some common ways this limit relationship is denoted.

$$\lim_{n\to\infty} x_n = x \qquad \{x_n\} \to x; \qquad x_n \to x.$$

Once limits are defined, their basic properties are established. For instance, if $x_n \to x$ and $y_n \to y$ then $x_n y_n \to xy$.

Continuity: Let $\boldsymbol{R}$ be the set of real numbers. A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is said to be *continuous* at a point x if the following holds for all sequences $\{x_n\}$ that converge to x.

$$x_n \to x \qquad \Rightarrow \qquad f(x_n) \to f(x).$$

In other words, f maps sequence converging to x over to sequences converging to $f(x)$. The function f is said to be *continuous* if it is continuous at all real numbers. Once you have the definition of continuity, you establish basic properties. For instance, if f and g are continuous, then so are

$$f+g; \qquad f-g; \qquad fg; \qquad f\circ g.$$

The last one is the composition of f and g.

Differentiability: Here is a related definition, dealing with the case where f is not necessarily defined on all of $\boldsymbol{R}$. Assuming that f is defined on all points y such that $0 < |x-y| < \epsilon$, we say that

$$\lim_{y\to x} f(y) = A$$

if the following holds. If $x_n \to x$ then $f(x_n) \to A$. This is supposed to hold for all sequence $\{x_n\}$ that converge to x such that $|x_n - x| < \epsilon$.

With this related definition, we can talk about differentiability. A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is said to be *differentiable at* x if there is some C such that

$$\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = C.$$

This definition makes sense because the function

$$g(y) = \frac{f(y)-f(x)}{y-x}$$

is defined for all points except x. f is said to be *differentiable* if f is differentiable at every point.

Open Sets After going through and recovering all the basic results from a calculus course, M 101 goes in a different direction and introduces a subject called *point set topology*. The idea is to define open sets of real numbers and to carefully study how they interact with continuity and related notions. I'll give a small sample of this subject.

An *open interval* in $\boldsymbol{R}$ is a set of the form

$$\{y|\ |y-x|<\epsilon\}.$$

A subset $U \subset \boldsymbol{R}$ is *open* if every point $x \in U$ is contained in an open interval I_x such that $I_x \subset U$ as well. A set S is *closed* if the complement $\boldsymbol{R} - S$ is open.

Open sets are closely related to continuous functions. Suppose $f: \boldsymbol{R} \to \boldsymbol{R}$ is a function. For each set U we have the inverse image $f^{-1}(U)$, consisting of points x such that $f(x) \in U$. Let's temporarily call f *nice* if it has the following property. If U is open then so if $f^{-1}(U)$. One of the basic results in point set topology is that f is continuous if and only if f is nice.

Accumulation Points: Let S be an infinite subset of $\boldsymbol{R}$. A point $x \in \boldsymbol{R}$ is said to be an *accumulation point* of S if it has the following property: Every open set containing x intersects S in infinitely many points. So, no matter how much we shrink our open set around x, we can't avoid points of S. One of the theorems you learn in M101 is the *Bolzano-Weierstrass Theorem*: Every bounded infinite set in $\boldsymbol{R}^n$ has an accumulation point. If you put an infinite number of points in a bounded region on the line, they have to pile up somewhere. Here *bounded* means that K is contained in some interval $[-N, N]$.

Compactness: A subset K of $\boldsymbol{R}$ is said to be *compact* if K has the following property. If $\{U_n\}$ is any (possibly infinite collection) of open sets whose union contains K, then there is a finite subcollection $U_1, ..., U_n$ whose union contains K. This definition is a bit hard to work with, and the *Heine-Borel Theorem* from M 101 comes in handy. The Heine-Borel Theorem says that K is compact if and only if K is closed and bounded.

Exotic Examples: Here are a few exotic examples of things you'll understand after taking M101. These are things you most likely wouldn't learn in a calculus class.

- There exists a continuous function that is not differentiable at any point.

- There exists a continuous curve $f : \boldsymbol{R} \to \boldsymbol{R}^2$ such that $f(\boldsymbol{R}) = \boldsymbol{R}^2$. Such a curve is called a *plane filling curve* for obvious reasons.

- There exists a continuous function $f : [0, 1] \to [0, 1]$ whose derivative exists and is 0 at every point except a set of length 0. Yet $f(0) = 0$ and $f(1) = 1$. Somehow, f manages to increase from 0 to 1 even though its derivative is (almost) always 0.

Beyond the Line: One of the advantages of the material in M101, especially the point-set topology, is that it easily generalizes to higher dimensions. For instance, in $\boldsymbol{R}^n$, an open ball is defined just as we defined an open interval in $\boldsymbol{R}$, but with $\|y - x\|$ replacing $|y - x|$. Once open balls are defined, one can talk rigorously about open sets, continuity, closed sets, accumulation points, the Heine-Borel theorem, the Bolzano-Weierstrass theorem, etc. M101 serves as a good warm-up for real analysis in higher dimensions, the subject of M113 and M114.