Math 104: Course Summary

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General Information: M104 is a topics course on geometry. The idea of the course is to introduce you to some of the beautiful classic geometric results and constructions. M104 is a good class to take if you want to teach mathematics at the K-12 level, because it will give you a large supply of interesting things to tell bright students about. I would imagine that M104 varies somewhat from year to year. I've never actually taught M104, but I'll summarize the kind of class I would teach.

The Euclidean Plane: The Euclidean plane is the set $\boldsymbol{E}^2$ of pairs (x_1, x_2) where x_1, x_2 are real numbers. The distance in $\boldsymbol{E}^2$ is given by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Here, I've set $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The distance formula in $\boldsymbol{E}^2$ is a reflection of the famous Pythagorean Theorem: $a^2 + b^2 = c^2$, where a, b are the lengths of the short sides of a right triangle and c is the length of the long side. In M104 you'll see a proof of the Pythagorean Theorem.

Lines in $\boldsymbol{E}^2$ are given by equations of the form $a_1x_1 + a_2x_2 = a_3$. Polygons, triangles, squares, etc., are as usual. Here is a sample of some nice result about Euclidean polygons that one might see in M104:

- The sum of the interior angles of an n-gon is $2\pi(n-2)$.
- The three angle bisectors of a triangle meet at a point.
- The three altitudes of a triangle meet at a point.
- The Law of Sines: In any triangle, the quantity $s/\sin(\theta)$ is independent of the side. Here θ is the angle opposite the side of length s.

- Heron's formula: The area of a triangle having side length a, b, c is given by
$$\sqrt{s(s-a)(s-b)(s-b)}; \qquad s = \frac{a+b+c}{2}.$$

- Bramagupta's formula: Let Q be a quadrilateral that is inscribed in a circle. If Q has side lengths a, b, c, d, the area of Q is
$$\sqrt{(s-a)(s-b)(s-b)(s-d)} \qquad s = \frac{a+b+c+d}{2}.$$

- The isoperimetric inequality for polygons: If P and P' are polygons having the same side lengths, and P is inscribed in a circle, then $\text{area}(P) \geq \text{area}(P')$.

- Pick's Theorem: Say that a *lattice point* is a point of the form (m, n) where m and n are integers. Say that a *lattice polygon* is a polygon having vertices that are lattice points. If P is a lattice polygon then
$$\text{area}(P) = L_1 + \frac{L_2}{2} - 1.$$
Here L_1 is the number lattice points in the interior of P and L_2 is the number of lattice points on the boundary of P.

- Dissection Theorem: Let P_1 and P_2 be unit area polygons. Then P_1 can be cut into triangles, and those triangles can be rearranged (without overlaps) to make P_2.

Hyperbolic Geometry: The Euclidean plane is a model for the classical Euclidean geometry of points and lines that is described in terms of 5 axioms:

- Any two distinct points can be joined by a line segment.

- Any line segment can be extended indefintely to a straight line.

- There is a circle of any radius through any point.

- All right angles are equal to each other.

- Given a straight line L and a point p not on L, there is a unique line L' through P that does not intersect L. (That is L and L' are parallel.)

For about 2000 years, people wondered if the first four axioms implied the fifth one. Finally, Lobachevski, Bolyai, and Gauss discovered a non-Euclidean geometry, now known as hyperbolic geometry, in which the first four axioms hold and the fifth one fails.

There are many useful and interesting models for the hyperbolic plane. Here we will describe the *Poincare model*. In this model, the hyperbolic plane is the interior of the unit disk. The straight lines are arcs of circles that meet the unit circle at right angles. When two such straight lines intersect, their angle of intersection is measured in the Euclidean sense, as the angle between the tangents to the circles at the intersection point.

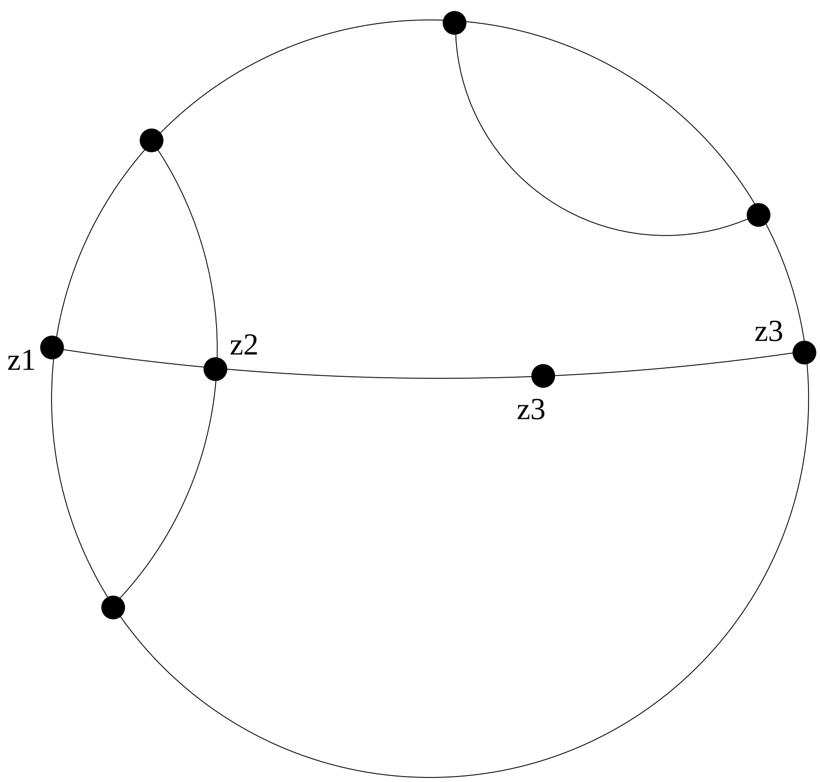

Figure 1: Some lines and points in the hyperbolic plane

Once you have the model for the hyperbolic plane, you can consider many of the same objects as in the Euclidean plane, e.g. circles and polygons. One (initially) shocking feature of the hyperbolic plane is that the sum of the angles of a triangle is always less than π, and can take on all values between 0 and π.

Hyperbolic Geometry In the Poincare model, there is a nice formula for distance. One thinks of the unit disk as a subset of $\boldsymbol{C}$, the set of complex numbers. Given points z_2 and z_3, one finds the points z_1 and z_4 on the disk boundary and then defines

$$d(z_2, z_3) = \log \chi; \qquad \chi = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}.$$

The quantity χ is defined for any quadruple of distinct complex numbers. It is called the *cross ratio*. When the points all lie on the same circle χ is real. The length of any line in the hyperbolic plane is infinite, in the sense that there are points on the same line that are further than N apart for any N.

Given the distance function, one can define the lengths of curves in the hyperbolic plane by an integration procedure that is similar to what one sees in calculus for curves in the Euclidean plane. One shocking thing about the geometry of the hyperbolic plane is that the circumference of a circle of radius R is about $\exp(R)$. Contrast this with the Euclidean case, where the circumference would be πR^2. In the hyperbolic plane, circles spread out exponentially, whereas in the Euclidean plane, circles spread out quadratically.

One can also define areas of regions in the hyperbolic plane by an integration procedure. One has the beautiful hyperbolic Gauss-Bonnet Theorem: Let T be a hyperbolic triangle having interior angles a, b, c. Then

$$\text{area}(T) = \pi - a - b - c.$$

So, the larger the triangle, the smaller the sum of the angles. One consequence of this result is that all triangles have area less than π. This is quite different from the situation in the Euclidean plane. In the hyperbolic plane, large triangles look like tripods. A similar result holds for any polygon.

It turns out that the hyperbolic plane is "as symmetric" as the Euclidean plane. Given any two points in the hyperbolic plane, there is a distance-preserving map that carries the one point to the other. Moreover, there is a 1-parameter family of rotations about any point. So, to an observer living in the hyperbolic plane, any location looks the same as any other and any direction looks the same as any other. In the Poincare model, these distance-preserving maps, or *isometries* have the form

$$T(z) = \frac{az + b}{cz + d}. \tag{1}$$

One can characterize the quadruples $(a, b, c, d) \in \boldsymbol{C}^4$ which give rise to actual hyperbolic isometries. In general, maps as we have defined are called *Mobius transformations*.

Inversive Geometry: Let $\boldsymbol{C}$ denote the complex plane. We add an extra point, called ∞, and call the union $\boldsymbol{C} \cup \infty$ the *Riemann sphere*. We think of ∞ as being "close" to points in $\boldsymbol{C}$ that are far from the origin. When this is done rigorously, one can simply interpret $\boldsymbol{C} \cup \infty$ as a sphere. $\boldsymbol{C} \cup \infty$ is not "round" as the ordinary sphere, but it "topologically equivalent" to the sphere in the sense that there is a homeomorphism between $\boldsymbol{C} \cup \infty$ and the round sphere S^2. A *homeomorphism* is a bijection f such that both f and f^{-1} are continuous. In fact, there is a beautiful homeomorphism from S^2 to $\boldsymbol{C} \cup \infty$ called *stereographic projection*. This map is shown, one dimension down, in Figure 2. The red point gets mapped to ∞.

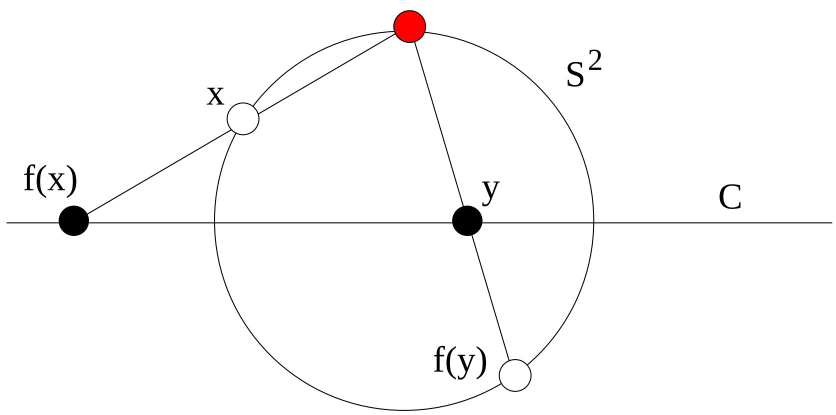

Figure 2: Stereographic Projection

A *circle* in $\boldsymbol{C} \cup \infty$ is defined to be either an ordinary circle in $\boldsymbol{C}$ or a straight line. It is a classical theorem in geometry that stereographic projection maps circles on S^2 to circles in $\boldsymbol{C} \cup \infty$.

We have already described the isometries of the hyperbolic plane in the Poincare model. Here we can re-purpose these maps. Dropping the "hyperbolic isometry condition" (which we didn't state) and considering all maps of the form given by in Equation 1, subject only to the constraint that

$$ad - bc \neq 0, \tag{2}$$

we get a nice family of transformations of $\boldsymbol{C} \cup \infty$. It turns out that these maps permute the circles of $\boldsymbol{C} \cup \infty$. A natural topic in M104 would be to explore patterns of circles in $\boldsymbol{C} \cup \infty$ and how they interact with Mobious transformations. This subject is known as *inversive geometry*.

Projective Geometry: A close relative of $\boldsymbol{C} \cup \infty$ is the *real projective plane*, $\boldsymbol{RP}^2$. This is the space of lines through the origin in $\boldsymbol{R}^3$. A *line* in the projective plane is the set of lines through the origin that are contained in a plane through the origin.

A linear isomorphism of $\boldsymbol{R}^3$ permutes both the lines through the origin and the planes through the origin. Thus, any linear isomorphism gives rise to a transformation of $\boldsymbol{RP}^2$ known as a *projective transformation*. The projective transformations are homeomorphisms of $\boldsymbol{RP}^2$ that carry lines to lines. One should compare this with the statement that the Mobius transformations are homeomorphisms of $\boldsymbol{C} \cup \infty$ that carry circles to circles.

Indeed, $\boldsymbol{C} \cup \infty$ also has an interpretation as the space of complex lines through the origin in $\boldsymbol{C}^2$, and the Mobius transformations are none other than the actions of complex linear isomorphisms of $\boldsymbol{C}^2$ on these lines. Thus the relationship between the Riemann sphere and the projective plane is analogous to the relationship between $\boldsymbol{C}^2$ and $\boldsymbol{R}^3$.

It is possible to see the Euclidean plane inside $\boldsymbol{RP}^2$. One considers all the lines through the origin in $\boldsymbol{R}^2$ that are not contained in the (x, y)-plane. Any such line intersects the plane $\{z = 1\}$. Conversely, any point in the plane $\{z = 1\}$ determines a unique line through the origin in $\boldsymbol{R}^3$. Thus, we can identity the plane $\{z = 1\}$ with a subset of $\boldsymbol{RP}^2$.

It is also possible to see the hyperbolic plane in $\boldsymbol{RP}^2$. Working in the Eucliean subset of $\boldsymbol{RP}^2$ we have just discussed, we say that the *Klein model* for the hyperbolic plane is the open unit disk in (the Euclidean plane inside) $\boldsymbol{RP}^2$. This time the straight lines are ordinary line segments in the open unit disk. The isometries in this case are exactly the real projective transformations that map the unit disk to itself. There is a formula for distance that is similar to what we gave above for the Poincare model. The one funny thing about this model is that the angles are distorted. The hyperbolic angle between two straight lines is different from the Euclidean angle.

Though the real projective transformations do not preserve this planar subset of $\boldsymbol{RP}^2$, they do map large portions of it back into itself. In an informal way, we can speak of the real projective transformations as acting on the plane. It is a beautiful fact that the real projective transformations permute the conic sections. Thus, one can map a circle to an hyperbolic using a real projective transformation. In this way, one sees the conic sections in a uniform way: The various conic sections in the plane are all images of a single object in $\boldsymbol{RP}^2$ under various real projective transformations.

Configuration Theorems: There are a number of classical configuration theorems in projective geometry. Oneof these is Pappus's theorem: Referring to Figure 3, the points A_3, B_3, C_3 are collinear provided that the points A_1, B_1, C_1 are collinear and the points A_2, B_2, C_3 are collinear.

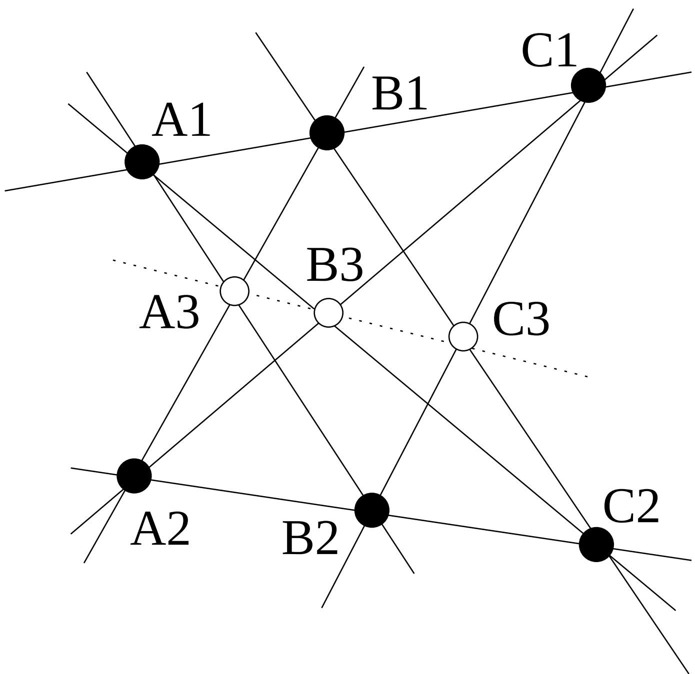

Figure 3: Pappus's Theorem

Pappus's Theorem holds not only when the 6 black points lie in a pair of lines, but also when these 6 points lie on a conic section. This generalization is known as Pascal's Theorem. Pascal's Theorem in turn is equivalent to Brianchon's Theorem: If a hexagon is inscribed in a conic section, then the three main diagonals of the hexagon meet at a point. M104 is a perfect place to explore these sorts of results.