Math 106: Course Summary

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General Information: Math 106 is a first course on differential geometry. Math 106 has a lot of overlap with Math 20 (or M35), several variable calculus, but the material is covered at a more sophisticated level. You can profitably take M106 after finishing the calculus series and Math 52 (or 54). In Math 106, you apply a combination of linear algebra and several variable calculus to study the geometry of curves in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$, and surfaces in $\boldsymbol{R}^3$. M106 deals both with *local* properties of these objects and *global* properties. The local properties are things that can be measured by taking some derivatives and the global properties are more holistic in nature (but equally precise.) I'll sketch the local parts for each kind of object and then give a sample of some global results.

Plane Curves: Roughly, a plane curve is described as a mapping

$$\gamma : \boldsymbol{R} \to \boldsymbol{R}^2.$$

So, for each parameter t, we have a point $\gamma(t) = (x(t), y(t))$ in the plane. These functions are usually assumed to be have derivatives of all orders. As in calculus, you learn to associate some basic objects to γ:

- The *velocity*, $\gamma'(t) = (x'(t), y'(t))$.
- The *speed*, $\|\gamma'(t)\|$, where $\| \ \|$ denotes the Euclidean norm.
- The *length* of the image $\gamma[a, b]$, given by the integral of the speed:

$$\int_a^b \|\gamma'(t)\| \ dt.$$

- The *unit speed reparametrization* $\gamma(s)$. Here $\gamma(s)$ traces out the same curve as γ, and in the same direction, but $\|\gamma'(s)\| = 1$.
- The *tangent vector* $T = \gamma'/\|\gamma$. This vector points in the direction of the tangent line to γ at each point, and has unit length.
- The *normal vector* N, which is just T rotated counterclockwise by $\pi/2$ degrees.
- The *curvature*, $\|dT/ds\|$. Here we are taking the derivative of T with respect to the unit speed parametrization. That is, we are measuring how fast the tangent vector swings around as we walk along γ at unit speed. The more curved γ is, the faster the tangent vector swings around and the higher the curvature.
- The *acceleration* $\gamma''(t)$.
- The resolution of γ'' into the *tangential and normal components*. That is $\gamma'' = aT + bN$, where a and b are functions that have a physical meaning.

All of these things are typically defined in e.g. Math 20. In M106 they are done in a similar way though in more depth.

Curves in Space: A curve in space has the same definition as a curve in the plane, except that there is one extra variable. The velocity, acceleration, speed, and length of a space curve have essentially the same definition as in the planar case. The tangent vector T also has the same defnition, but the normal requires a new definition because "rotate by $\pi/2$ degrees" is not a well-defined operation in space. The idea is to define the normal as

$$N(s) = \frac{dT/ds}{\|dT/ds\|}.$$

Here s is the unit speed parameter. Physically, if we are running along the curve at unit speed, we will feel a pull in the direction of N. The pull is dT/ds. We divide by the length $\|dT/ds\|$ to get a unit vector. N isn't defined when $dT/ds = 0$, as it would be for a straight line, but otherwise the definition works.

Now we start to see some of the geometry. The curves T and N span a plane at each point along γ, called the *osculating plane*. This is the plane that, in some sense, comes closest to containing γ. As you move along γ at unit speed, these planes spin around, like a revolving door. The rate of spin is called the *torsion*. In terms of formulas, the vector

$$B = T \times N$$

(the cross product) is normal to the osculating plane. The torsion is given by $\|dB/ds\|$. That is, we can measure how fast the osculating planes are spinning around by measuring the rate of change of B.

The Frenet-Serret Equations Continuing with the discussion for space curves, the vectors $\{T, N, B\}$ form an orthonormal basis at each point along γ (as long as everything is defined). This frame is called the *Frenet* frame. Since the Frenet frame is a basis at each point, you can make money expressing dT/ds and dN/ds and dB/ds in terms of this basis. The result is known as the *Frenet-Serret equations*:

$$\begin{bmatrix} dT/ds \\ dN/ds \\ dB/ds \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

On the right side of this equation, we're doing matrix multiplication. For instance $dN/ds = -\kappa T + \tau B$. Here κ and τ denote curvature and torsion respectively. The F.S. equations have many geometrical consequences for space curves, and these are explored in M106. To give an example, one can use the F.S. equations to efficiently prove that a curve with nonzero curvature and constant nonzero torsion is a helix.

Global Theorems about Curves: Here I'll give several examples of global theorems about curves. The most famous one is the *isoperimetric inequality* a closed embedded loop in the plane (think of a lasso) of length 1 encloses the most possible area if and only if the loop is a circle. The isoperimetric theorem has many proofs, and you'll probably get to see at least one in M106.

Here is another example. The total curvature of a curve is

$$\int \kappa(s) ds,$$

the integral of the curvature, with respect to the unit speed parameter, over the whole curve. One global result is that the total curvature of any loop is at least 2π. You get exactly 2π for the circle, so you might say that the total curvature of any loop is at least that of the circle. A deeper theorem, known as the *Fary-Milnor theorem*, says that the total curvature of a knotted loop in space is at least 4π. That is, a loop needs at least twice the curvature of a circle in order to make a knot.

Surfaces: The material on surfaces in M106 is meatier than the material on curves, and my discussion will consequently be a bit sketchier. A surface is defined in M106 pretty much as it is defined in M20. It is the graph of a function $h(u,v) = (x(u,v), y(u,v), z(u,v))$. As usual, all derivatives are assumed to exist. In M20 you learn about the first and second derivative tests, which detect local extema and saddle points of a surface. In M106 you go much further in this direction.

Here is an example construction. A surface S has a tangent plane at each point, and the normal vector to this plane is called the *normal to the surface*. Given a point p on the surface, the normal N at p, and any tangent vector T at p, You can form the plane $\Pi(T)$ that is spanned by N and T. The intersection $S \cap \Pi(T)$ is a curve, and you can measure its curvature, $\kappa(T)$. The function $T \to \kappa(T)$ is takes a tangent vector at p and spits out a number. In M106 you study this function.

The function I am talking about is either constant or has two extrema at each point. These extrema are called the *principle curvatures*. The product of these numbers is called the *Gauss curvature* of the surface. The Gauss curvature is a single number that you assign to each point of the surface. It doesn't completely capture how the surface is curving, but it does capture some interesting geometric features of of the surface.

For example, suppose that you cut a tennis ball in half and flex one of the halves, keeping your finger on some point of the half-ball. Then the individual curvatures (coming from the slices) at the point of interest change but the Gauss curvature does not change. That is, the product of the extrema are constant. This fact, when formalized, is Gauss' famous *Theorema Egregium*. It's a pretty hard theorem to prove in M106. Sometimes you'll see it and sometimes not.

Curves and Surfaces As we have already discussed, there are lots of interesting curves on a surface. One can look at the slicing construction described above, or else simply draw a curve on a surface and study its geometry. In M106, one studies the way the local properties of the curve – e.g., the Frenet-

Serret equations – interact with the local properties of the surface – e.g., its Gauss curvature. Similar to the Frenet-Serret equations, there are various "master equations" that describe these relationships.

One of the most commonly studied kind of curve on a surface is a *geodesic*. Roughly speaking, the geodesics are the curves that join points together in a way that takes the shortest possible length. Physically, one could think of pulling a rubber band over a greased surface. The (idealized) shape taken by the rubber band will be a geodesic. In M106 (and in later differential geometry classes) you will learn about the *geodesic equations*. These are differential equations that describe the geodesics without resorting to global properties like length-minimization.

Gauss-Bonnet Theorem Probably the most famous global theorem about surfaces is the *Gauss-Bonnet Theorem*. The *total curvature* of a surface is defined as

$$\int \kappa dS,$$

where κ denotes the Gauss curvature discussed above and where dS is the area element. The Gauss-Bonnet theorem says that the total curvature of a closed surface (like a sphere or the surface of a donut) only depends on the topology of the surface. Put in more concrete terms, the result says for instance that a sphere and an egg have the same total curvature, no matter what the specific shape of the egg. The only requirement is that the egg is *generally spherical*. (If you know some topology, you'll understand that I am trying to say that the egg should be homeomorphic to a sphere.) To give another example of the Gauss-Bonnet theorem: The total curvature of the surface of a donut is 0, no matter how lumpy the thing is. You'll learn about the Gauss-Bonnet Theorem in M106.