Math 111 – Ordinary Differential Equations

Ordinary Differential Equations (ODEs for short) come up whenever you have an exact relationship between variables and their rates. Therefore you can find them in geometry, economics, engineering, ecology, mechanics, physiology, and many other subjects. For instance, they describe geodesics in geometry, and competing species in ecology.

Math 111 is an introductory course in the subject. The prerequisites are calculus and linear algebra. No other prerequisites are needed. It's not a very difficult course so it's a good one to take immediately after taking linear algebra.

(Applied Math 33-34 is a year course with content that is so similar that you should not take both it and Math 111. The difference is that AM 33-34 has just a little linear algebra (there being no linear algebra prerequisite) but a lot of applications to science. Math 111 is briefer on the science but gives more emphasis to the geometry.)

In this subject one (or more) variables depends on another "independent" variable. There is just one independent variable so the derivatives are not partials. A solution of an ODE is always some kind of integral. The simplest ODE is $\frac{dy}{dx} = f(x)$, which is just standard calculus. Remember however that even ordinary integrals, such as $\int \exp(x^2)\,dx$, might not be able to be evaluated explicitly. Much more interesting is $\frac{dy}{dx} = f(x, y)$, whose solutions might or might not be expressible in terms of integrals. Yet more interesting are equations that involve second or higher derivatives, as well as systems of several equations with several dependent variables.

In your calculus course you saw a few simple ODEs. Math 111 begins with a quick introduction to these and a few other fairly easy equations. Early in the course, we study *linear ODEs with constant coefficients.* They can always be solved!!! Such ODEs could look like

$$\frac{dy}{dx} = Ay. \tag{1}$$

Here x is the independent variable, $y = (y_1, y_2, \dots y_n)$ is a vector, and A is an $n \times n$ matrix. Each component of y can be regarded as a dependent variable. Such an ODE is called a *system* because there is more than one scalar dependent variable. (For instance, in economics, x may be time and $y_1, y_2, \dots$ may be the cost of labor, the overhead, the price of goods, etc., each of which changes with time.) The solutions of the system (1) are entirely expressible in terms of exponentials, sines and cosines, and polynomials! In terms of the matrix A, the solution is expressed as e^{xA}. We will discuss the meaning of the exponential of a matrix. Naturally this part of the course involves a lot of linear algebra, especially eigenvalues and eigenvectors.

Now the moment we consider *variable* coefficients, such as

$$\frac{dy}{dx} = A(x)y, \tag{2}$$

where $A(x)$ is a family of matrices *that depend on* x, there might or might not be an explicit solution. Even worse, is a *nonlinear* system, like this one:

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

with general "nonlinear" dependence on y. (Or better, because such systems are much richer mathematically?)

Fortunately, we always know that *there is a unique solution* of the general system (3) with any initial condition $y(0) = y_0$, where y_0 is any given vector. This is the central theorem of the course. The proof involves approximating the solution by iteration, as in

$$\frac{dy^m}{dx} = f(x, y^{m-1}), \quad y^m(0) = y_0 \quad (m = 1, 2, 3, \dots),$$

where m is a superscript. It is really simple to get our hands on the sequence of functions $y^1(x), y^2(x), \dots$. Then we prove that it converges and that its limit satisfies (3). This involves a bit of analysis, namely uniform convergence. Actually, there are a couple of caveats to this theorem. First, the solution is only guaranteed to exist for a short interval of x ("for a short time"). Secondly, it has to be assumed that the function $f(x, y)$ in the equation is not too bad (continuously differentiable is enough).

Okay, now we know how to solve the linear, constant-coefficient systems, and we know that there is always a solution for any initial condition for a little while. What happens to the more complicated equations after that little while? Well, it can get very, very complicated. The solutions could be chaotic, like fractals.

If the system is linear but has variable coefficients, the linear structure is still very useful but there might be no explicit solution formula. For instance, Bessel's equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

is an equation that comes up in most problems with circular or spherical symmetry. Its space of solutions is two-dimensional. One of its solutions is the Bessel function, which oscillates like $\sin x$ but gradually decreases.

A very interesting type of nonlinear system is

$$\frac{dy_1}{dx} = f(y_1, y_2), \qquad \frac{dy_2}{dx} = g(y_1, y_2) \tag{4}$$

where f and g are functions that depend on y_1 and y_2 but not on x. This is called an *autonomous* 2×2 *system*. The nice thing about it is that you can draw pictures of the solutions in the y_1, y_2 plane. This is called the *phase plane* and the pictures of the solutions are called the *phase portraits*. Typical solutions are called nodes, saddles and centers.

Some of them are stable and some are unstable. "Stable" means that when you wiggle the initial condition a little, the solution only changes a little. Stability is an important concept because it tells you something about what happens beyond that guaranteed little interval of x. Stability is also a crucial concept in most applications. For instance, the planets satisfy a differential equation coming from Newton's law of motion under the influence of gravity. They move in stable orbits around the sun; if they didn't, they would fly off and disappear from the solar system or crash into another planet or the sun. For system (4) there may be a *limit cycle*, which is a closed orbit (a periodic solution) that is a limit of spiraling solutions. The Poincaré-Bendixson theorem states that 2×2 systems typically have limit cycles.

It is when we get to 3×3 systems that we may find chaos. In n dimensions, that is, where there are n dependent variables, the solutions are curves in n-dimensional euclidean space. Then other, more sophisticated geometrical and topological concepts become useful.