Math 112 – Partial Differential Equations

Partial Differential Equations (PDEs for short) come up in most parts of mathematics and in most sciences. For instance, complex analysis is the study of the Cauchy-Riemann equations

$$u_x = v_y \ , \quad u_y = -v_x \ . \tag{1}$$

Another example is the recent resolution of the celebrated Poincaré conjecture in topology, which uses diffusion-type PDEs to analyze the singularities of surfaces. Brownian motion is a random process that is described by a PDE. Other examples of PDEs occur in the flow of fluids, diffusion of chemicals, conduction of neural impulses along an axon, radiation of electromagnetic waves, quantum mechanics, spread of heat, propagation of sound, spread of epidemics, etc. PDE is a vast subject.

Math 112 is an introductory course in the subject. All you really have to know is calculus and linear algebra. The key prerequisite is several-variable calculus. However, it is not one of the easier 100-level courses, so it is recommended (but not required) that you take at least one other 100-level course before taking Math 112. Very good preparatory courses would be Math 101 (Introduction to Analysis), 126 (Complex Analysis), 113 (Real Analysis) or 111 (Ordinary Differential Equations). Math 111 is not a prerequisite.

A few PDEs are really easy to solve, such as

$$u_t + u_x = 0, \tag{2}$$

which has the general solution $u(x,t) = f(x-t)$ for any function f of one variable. But most of them are difficult or notoriously difficult or impossible to solve. (It's something like indefinite integrals, most of which, like $\int \exp(x^2)\,dx$, do not have explicit solutions.)

In Math 112 we will study the most important equations that are in the easy-to-difficult category. The notoriously difficult ones are subjects of current mathematical research or computational experimentation. Most of the ones studied in this course have explicit solutions in the form of fairly complicated formulas involving integrals or infinite series. Just about all PDEs have *infinitely many* solutions, often an infinite-dimensional space of solutions, as in the very simple example above. After a while, one realizes that one has to rethink what one means by "solving" a PDE. Many PDEs have no solution formula at all. Instead, one has to find properties of the solutions.

A PDE like $u_t + u_x = 0$ has *order* one because it involves only first derivatives. The PDE $u_t = u_{xx}$ has order two because there is a second derivative, etc. After a very short introduction to first-order PDEs, the focus of the course rests on the three most fundamental PDEs of order two:

$$u_{xx} + u_{yy} = 0, \quad u_t = u_{xx} \quad \text{and} \quad u_{tt} = u_{xx}. \tag{3}$$

These are called the Laplace equation, the diffusion (or heat) equation and the wave equation, respectively. There are also the higher-dimensional versions of them, namely:

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0, \quad u_t = \Delta u \quad \text{and} \quad u_{tt} = \Delta u \tag{4}$$

in three spatial dimensions. Each of these three basic equations has its own "personality". These equations are linear, and therefore the space of solutions is a vector space! So the concepts of linear algebra come into play, particularly the concepts of eigenvector and orthogonality.

There are an infinite number of solutions, so ... how do we pick out any particular one? Here's the standard way. For the diffusion or wave equations we impose *initial conditions*, which specify the solution at $t = 0$. In the space variables, we take a domain D and specify *boundary conditions* on its boundary.

For instance, for the diffusion equation $u_t = u_{xx}$, we could take the interval $0 \leq x \leq 1$ and look for a solution which vanishes at both endpoints $x = 0, 1$. With the initial condition $u(x, 0) = \phi(x)$, this problem (with these extra conditions)

$$u_t = u_{xx}\ , \quad u(x,0) = \phi(x)\ , \quad u(0,t) = u(1,t) = 0 \tag{5}$$

has a *unique* solution! It turns out to be given by the complicated formula

$$u(x,t) = \sum_{n=1}^{\infty} a_n\ e^{-n^2\pi^2 t}\ \sin(n\pi x). \tag{6}$$

How in blazes to we get to this? Well, take the course and you'll find out. Put $t = 0$ in the formula and we see that

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x). \tag{7}$$

where the coefficients a_n are given in terms of the initial function by

$$a_n = 2\int_0^1 \phi(x)\sin(n\pi x)dx. \tag{8}$$

This formula (7) is a Fourier series. Fourier used it around 1820 to explain heat conduction in a material. An important part of this course is devoted to studying Fourier series. The central concepts are orthogonality of functions, completeness of a collection of orthogonal functions, and especially convergence of infinite series of functions. Actually, it turns out that the convergence of Fourier series is somehow much better than the convergence of Taylor series. It's better because Fourier series converge for many more functions ϕ. A function has a Taylor series expansion only if it is analytic, but even discontinuous functions have Fourier series expansions. We study different types of convergence (pointwise, uniform, L^2).

Later in the course, we study two and three dimensional problems, where more interesting geometry comes into play. For instance, the three-dimensional wave equation is the basic equation of the theory of relativity, where time can perhaps be regarded as a fourth "space" variable. Directly from the PDE, we can see how all waves travel at less than the speed of light.

Here are some other questions that may be considered in the course:

In three dimensions, what functions take the place of the sine function (as in the example above)? A partial answer: the Bessel functions come into play if the domain is a sphere.

How do we discretize solutions so that we can compute them numerically? A warning: unless we discretize very carefully, we could get junk.

How do we solve the Schrödinger equation of quantum mechanics?

How do we solve the Maxwell equations of electromagnetism?

How do we handle other PDEs, especially *nonlinear* ones? Example: the KdV equation $u_t + u_{xxx} + uu_x = 0$ has a nonlinear term uu_x. For nonlinear PDEs, one finds new phenomena such as shock waves and solitons.