Math 126: Course Summary

Rich Schwartz

August 19, 2009

General Information: Math 126 is a course on complex analysis. You might say that complex analysis is the study of what happens when you combine calculus and complex numbers. Complex analysis contains some of the most beautiful theorems in undergraduate mathematics. It is a course that you can take right after the calculus series, but if you want extra grounding in real analysis before taking complex analysis, you could take M 101 first.

A Quick Review of Calc 3: Let's start with something from several variable calculus. Suppose that $F : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a (sufficiently) differentiable map from the plane to itself. Given point p in the domain of F, the behavior of F is well approximated near p by the matrix dF_p of first partial derivatives of F, evaluated at p. This is to say that

$$F(p + tv) = F(p) + t\ dF_p(v) + O(t^2).$$

Here v is a unit vector and t is a small positive number. Here $O(t^2)$ means that the error term is at worst proportional t^2. I am being a bit lazy in using the term "sufficiently differentiable". To be precise, all one needs is that the first partials of F exist and vary continuously.

In this context, dF_p is a linear transformation from the plane to itself. We know geometrically that a linear transformation maps ellipses to ellipses. So, geometrically, the approximation result above says that F maps very small ellipses centered at p to curves which look very much like ellipses. Put still another way, the approximation result above says that (sufficiently) differentiable maps look like linear transformations on small scales.

Similarities: There is a special kind of linear transformation called a *similarity*. Such a linear transformation has the effect of dilating and/or rotating the plane. If we identify the plane with $\boldsymbol{C}$, the set of complex numbers, then every similarity has the form $T(z) = wz$, where w is some complex number. To give an example, if $w = 2i$, then T has the effect of rotating counter-clockwise by 90 degrees and dilating by a factor of 2. Notice that complex numbers enter into the picture when we think of the plane as $\boldsymbol{C}$ rather than as $\boldsymbol{R}^2$.

Holomorphic Functions: Now for the big idea: The map F is said to be *holomorphic* if the map dF_p is a similarity at each point p in the domain. (The conditions on dF_p that make this true are called the *Cauchy-Riemann equations*.) Geometrically, the condition means that F maps tiny circles to curves that are nearly circles. So, on small scales, F practically maps circles to circles. To tie this even more closely to complex numbers, it turns out that the condition that F is holomorphic is equivalent to the condition that the limit

$$\lim_{h \to 0} \frac{F(p+h) - F(p)}{h}$$

exists for all points where everything is defined. Here we think of F as a map from $\boldsymbol{C}$ to $\boldsymbol{C}$ and all the numbers involved in the limit are complex numbers.

It turns out that many of the functions you know and love are holomorphic. For instance.

- The polynomials $f(x) = a_0 + a_1 z + ... + a_n z^n$ are holomorphic.
- Convergent power series: You can stick complex numbers into Taylor series. If the result converges, the resulting function is holomorphic. For instance, the series

  $$1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

  converges for every complex number z, and defines the exponential function e^z on all of $\boldsymbol{C}$.
- Using the same trick as in the previous example, all the trig functions are defined and holomorphic on all of $\boldsymbol{C}$. There are some great identities, such as $\cos(iz) = \cosh(z)$. If you ever wondered how the hyperbolic cosine relates to the ordinary cosine, the identity I just mentioned tells you. There is a single holomorphic function which equals the cosine along the real axis and the hyperbolic cosine along the imaginary axis. There is a similar formula for sines.

- Up to rotations, there is a unique holomorphic map that bijectively carries the regular n-gon to the unit circle in such a way that the center gets mapped to the center. This result is part of the famous Riemann mapping theorem.

Properties: Once holomorphic functions are defined in M126, their properties are explored. There is some preliminary material on limits and continuity, just as in a real analysis class, but then the good stuff comes. Holomorphic functions have some amazing properties. Here are 3 such properties.

- A holomorphic function is infinitely differentiable. The definition, in terms of similarities, only requires that the first partials exist and vary continuously, but then it automatically happens that all higher derivatives exist.

- Two holomorphic functions agree everywhere provided that they agree on any bounded infinite collection of points. For instance, if two holomorphic functions agree on the points $(1/n, 0)$ for $n = 1, 2, 3...$ then they agree. This is to say, in some sense, that a holomorphic function is like a hologram: You can reconstruct the whole thing if you know (well enough) how it behaves in even the tiniest of regions.

- The property we just mentioned can be turned on its head: If you know the values of a holomorphic function on some big circle, you can figure out the value of the function at the center of the circle. This is sort of like saying that the exact pattern of your life can be determined by looking at the distant stars. The method of determining the value of a holomorphic function at the center of a circle based on the values on the circle is known as the *Cauchy integral formula*. It is one of the important theorems you learn in M 126. The Cauchy integral formula is expressed in terms of something called a contour integral.

Contour Integrals: In Calc 3, you may have learned about line integrals. These are integrals of the form

$$\int_\gamma f(x,y)dx + g(x,y)dy.$$

Here γ is some curve in the plane and f and g are real valued functions. In complex analysis, a contour integral is a very similar gadget. Contour integrals have the form

$$\int_\gamma F(z)dz.$$

To make this look more like a line integral, we go back to working in $\boldsymbol{R}^2$ and write

$$F(z) = u(x,y) + iv(x,y).$$

Then

$$\int_\gamma F(z)dz = \int_\gamma (u(x,y) + iv(x,y)) \times (dx + idy) =$$

$$= \int_\gamma u(x,y)dx - v(x,y)dy + iu(x,y)dy + iv(x,y)dx.$$

We are just multiplying things out formally. This is not the rigorous way to do things, but it gives you an expression you might recognize from calculus, except for the appearance of i in the final answer.

With this notation, the Cauchy integral formula looks like

$$f(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z} dz.$$

Here C is a circle centered at the origin. A similar result holds for circles not centered at 0, but we chose circles centered at 0 for convenience.

It turns out that many integrals, which are quite hard to evaluate using just ordinary calculus, can be evaluated using contour integration. In M 126 you'll learn a basic result about how to evaluate contour integrals, known as the residue theorem.

Power Series: We already mentioned above that convergent power series give examples of holomorphic functions. That is, the function

$$f(z) = a_0 + a_1 z + a_2 z^2 ...$$

is holomorphic provided that the series converges. One purpose of Math 126 is to make this idea precise. Another purpose is to prove the converse result: Any holomoprhic function can be expressed as a convergent power series in a small disk about any point where it is defined.

In a certain sense, the convergent power series give all the examples of holomorphic functions. Another way to think about complex analysis is that it is the study of what happens when complex numbers are plugged into Taylor series. However, this description somehow doesn't capture the richness of the subject. In any case, writing out a holomorphic function in a power series and analyzing that series is one of the basic techniques you'll learn in M 126.

Harmonic Functions: When discussing contour integrals, we wrote

$$F(z) = u(x, y) + iv(z, y).$$

The functions u and v are real valued functions on the plane. It turns out that these functions have special properties when F is holomorphic. Namely, u and v are *harmonic functions*. One way to say this is that the value of u at any point is the same as the average of u on a disk centered at that point. (The same goes for v.) Another way to say this is that $\Delta u = \Delta v = 0$, where

$$\Delta u = u_{xx} + u_{yy}$$

is the Laplacian of u. Exploring the basic properties of harmonic functions, and their relation to holomorphic functions, is another topic in M 126.