Math 127: Course Summary

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General Information: M127 is a course in functional analysis. Functional analysis deals with normed, infinite dimensional vector spaces. Usually, these vector spaces are spaces of functions, and the norm describes a particular way of measuring the size of the function. By considering a particular function as a point in such a vector space, it is often possibly to analyze the function in a deep way using tools from linear algebra, real analysis, and geometry.

There doesn't seem to be a set syllabus for M127; different people teach different topics. In this summary, I'll try to list some topics that would probably appear in any version of M127, but you should take this list with a grain of salt. Though M127 seems to build on some concepts from M113, you can take M127 without taking M113. The M113 course goes into much more depth on these common topics. In any case, you might want to read my summary for M113 before reading this one.

Banach Spaces: Let V be a real vector space. A *norm* on V is a function $v \to \|v\|$ which satisfies the following properties:

- $\|v\| \geq 0$, with equality iff $v = 0$.
- $\|av\| = |a|\|v\|$ for all $a \in \boldsymbol{R}$.
- $\|v + w\| \leq \|v\| + \|w\|$.

The same definition works if V is a complex vector space, provided that $|a|$ is interpreted as the usual absolute value of $a \in \boldsymbol{C}$. The norm on V turns V into a metric space, with distance function $d(v, w) = \|v - w\|$. If V is complete (meaning that every Cauchy sequence converges) then V is called a *Banach space*. Here are some examples.

- $\boldsymbol{R}^n$, equipped with the standard norm, is a Banach space.
- Let V denote the space of real valued functions f such that

$$\|f\|_p := \left(\int_{\infty}^{\infty} |f|^p\right)^{1/p} < \infty.$$

  Write $f \sim g$ if f and g agree except on a set of measure 0. Let $L^p(\boldsymbol{R}, \boldsymbol{R}) = V/\sim$, the quotient space. This is a Banach space known as an L^p space. An obvious variant is $L_p(\boldsymbol{R}, \boldsymbol{C})$, the L^p space of complex valued functions.

- Let $C^0(I)$ denote the set of continuous functions on a closed interval $I \subset \boldsymbol{R}$. For a norm, we can use

$$\|f\|_\infty = \sup_I |f|.$$

Contraction Mapping Principle: Here is a basic fact about a complete metric space X. A map $T : X \to X$ is called a *contraction* if

$$d(T(x), T(y)) < \alpha d(x, y)$$

for some $\alpha < 1$ and all $x, y \in X$. A contraction map has a unique fixed point. The uniqueness is easy. For existence, one chooses an arbitrary starting point x and considers the sequence $x, T(x), T(T(x))$, etc. This sequence is easily seen to be Cauchy, and the limit is fixed by T. The contraction mapping principle applies to Banach spaces, and often provides a tool for finding functions with special properties.

Existence of First Order ODE's: I'll give a quick application of the contraction mapping principle, as applied to a Banach space. I'll prove a toy version of Picard's theorem on the existence of ordinary differential equations. The formulation here is definitely not the most general one. Suppose that $\phi : \boldsymbol{R} \to \boldsymbol{R}$ is a smooth function such that $|d\phi/dx| < M$ for all points. We seek a solution $F : [0, \infty) \to \boldsymbol{R}$ to the differential equation

$$\frac{dF}{dx} = \phi \circ F(x); \qquad F(0) = A.$$

The argument I give will show that F exists on the interval

$$I_0 = [0, 1/(2M)].$$

A repetition of the argument shows that F exists on the interval

$$I_1 = [1/(2M), 2/(2M)].$$

And so on.

Let $X = C^0(I_0)$, as defined above. Consider the map $T : X \to X$ defined as follows:

$$Tf(x) = A + \int_0^x \phi \circ f(t)\ dt. \tag{1}$$

Since $|d\phi/dx| < M$, the map ϕ stretches distances by at most a factor of M. This means that

$$|\phi \circ f(t) - \phi \circ g(t)\| < M\|f(t) - g(t)\|.$$

Hence

$$\|Tf - Tg\| \leq \int_0^{1/(2M)} M\|f - g\|dt \leq \frac{1}{2}\|f - g\|.$$

Hence T is a contraction. Let F be the unique fixed point of T.

Since F is the integral of a continuous function, F is once differentiable. But then F is the integral of a once-differentiable function. So, F is twice differentiable. And so on. In short, F is smooth. Moreover, F satisfies our differential equation, by the Fundamental Theorem of Calculus. That's the proof.

Hilbert Spaces: A Hilbert space is a special case of a Banach space. An *inner product* on a vector space V is a binary operation $\langle\ ,\ \rangle$ such that

- $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$.
- $\langle v, w\rangle = \langle w, v\rangle$.
- $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$.

The inner product defines a metric on V as follows:

$$d(v, w) = \langle v - w, v - w\rangle^{1/2}.$$

V is called a *Hilbert space* if V is complete with this metric. The L^p spaces above are Hilbert spaces for $p = 2$. The inner product is given by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g}(x)\ dx. \tag{2}$$

Here $\overline{g}$ is the complex conjugate of g. This case is especially interesting.

We can also define similar spaces for sequences. For instance, $L_2(\boldsymbol{Z}, \boldsymbol{C})$ is the space of bi-infinite sequences $\{c_n\}$ of complex numbers such that

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

The inner product is defined as in Equation **??**, except that we do a sum instead of an integral.

Going in another direction, we can replace $\boldsymbol{R}$ with $\boldsymbol{R}/\boldsymbol{Z}$, the unit circle. That is, we can look at 1-periodic functions and define the inner product by integrating over a single period. This leads to the space $L_2(\boldsymbol{R}/\boldsymbol{Z}, \boldsymbol{C})$.

Fourier Series: Let $L_2 = L_2(\boldsymbol{R}/\boldsymbol{Z}, \boldsymbol{C})$ and $l_2 = L_2(\boldsymbol{Z}, \boldsymbol{C})$. Let g_n be the function

$$g_n(x) = \exp(2\pi i n x).$$

We define

$$c_n = \int_0^1 f(x) g_{-n}(x)\ dx \tag{3}$$

We have

$$f \sim \sum_{i=-\infty}^{\infty} c_n g_n.$$

The convergence (of the partial sums of the series) takes place in L_2. This sum is called the *Fourier series* of F.

One of the most beautiful facts about the Fourier series is Parseval's Identity:

$$\int_{\boldsymbol{R}} |f|^2 = \sum_{n=-\infty}^{\infty} |c_i|^2. \tag{4}$$

That is, the map $f \to \{c_n\}$ is an *isometry* (distance preserving map) from L_2 to l_2. In M127, you see a proof.

Fourier series are a central topic in M127. One main kind of theorem about Fourier series are *convergence results*. One is interested in how well the truncated series

$$S_N = \sum_{n=-N}^{N} c_n g_n$$

converges to f. Here are some sample results.

- Riesz-Fischer Theorem: $\|S_n - f\| \to 0$ in the L_2 norm.
- $S_n(x)$ converges pointwise to $f(x)$ when f is differentiable at x. In particular, S_n converges pointwise to f if f is differentiable.
- Carlson's Theorem: If f is continuous then S_N converges to f except perhaps on a set of measure 0. (This result is outside the scope of M127.)

An Application of Fourier Series: In M127, you also see various applications of Fourier series. For example, consider the function

$$g(x) = x - \text{floor}(x).$$

We think of $g \in L_2$, as in the previous section. We have

$$\|g\|^2 = 1/3; \qquad c_0 = \frac{1}{2}; \qquad c_{-n} = -c_n = fraci2n\pi.$$

Applying Parsival's identity, one obtains

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}.$$

Rearranging this, we get the famous sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Analysis of the Heat Equation: Consider the *Heat Equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}. \tag{5}$$

Here we think of $f(t, *)$ as a real valued periodic function. Call this function f_t. Starting with e.g. a differentiable f, we can expand f out in a Fourier series:

$$f_t = \sum_{n=0}^{\infty} c_{n,t}\cos(2\pi nx).$$

Here we are using the fact that f is real-valued to simplify the sum.

Since the heat equation is linear, we can analyze f_t simply by analyzing the behavior of the heat equation on the sin function $c_t \cos(2\pi nx)$. In order for this function to satisfy the heat equaton, we have

$$\frac{dc_{n,t}}{dt} = -(2\pi n)^2 c_{n,t}.$$

This implies that $c_{n,t}$ decays exponentially fast as a function of t. The larger the value of n, the faster the decay. So, if we think of f as a superposition of waves of different frequencies, the heat equation damps out the high frequency waves at a very fast rate. This phenomenon explains various things about f. For instance, it turns out that f_t is real-analytic for $t > 0$.

The Fourier Transform: Closely related to Fourier Series is the Fourier Transform. Given $f : \boldsymbol{R} \to \boldsymbol{C}$, one defines

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi inx) dx. \tag{6}$$

In M127 you will see that the map $f \to \widehat{f}$ is an isometry of $L_2(\boldsymbol{R}, \boldsymbol{C})$ to itself. The inverse map is given by

$$f(x) = \int_{-\infty}^{\infty} f(\xi) \exp(2\pi intxi) d\xi. \tag{7}$$

The Fourier transform is closely related to the operation of *convolution*. One defines

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy. \tag{8}$$

The function $f * g$ is called the *convolution* of f and g. When g is supported in a small interval about 0 and has unit integral, then $f * g$ is a kind of "blurred" version of f that is close to f. If one takes f to be a temperature profile of a uniformly thick wire and g to be a Gaussian distribution,

$$g(x) = c_1 \exp(-c_2 x^2)$$

for suitably chosen constants, then $f * g$ represents the temperature profile described by letting the heat diffuse for a certain time. The amount of time depends on c_2. The constant c_1 is a normalizing constant designed to make the total integral 1.

Setting $h = f * g$, one has the *convulution formula*.

$$\widehat{h} = \widehat{f}\widehat{g}. \tag{9}$$

That is, the Fourier transform converts convolution to multiplication. This result is important for the analysis of linear differential equations such as heat flow.