Math 141 Summary

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September 18, 2009

General Information: Math 141 is a first course in topology. Generally speaking, topology deals with properties of spaces (e.g., curves, graphs, and surfaces) that only depend on notions of continuity and not on rigid notions in geometry like length and angle. On the other hand, topology and geometry are very closely intertwined, and part of the beauty of the two subjects is discovering how one subject influences the other. In M141, you learn some of the basic principles of topology and also prove some of the famous classic results in the subject. The exposition is "combinatorial" in the sense that most of the spaces involved are described in terms of collections of triangles (or higher dimensional tetrahedra) that have been glued together. In these notes, I'll give an account of selected topics that typically appear in M141.

Homeomorphism and Shape: Suppose that X and Y are two spaces on which one has a well defined notion of continuous maps. For example, X and Y might be curves in the plane, or surfaces in space. A map $f : X \to Y$ is called a *homeomorphism* if f is a bijection and if both f and f^{-1} are continuous. The spaces X and Y are said to be *homeomorphic* if there is a homeomorphism between them. Here are some examples:

- An ellipse is homeomorphic to a circle.
- The surface of a cube is homeomorphic to sphere of the same dimension.
- The open unit ball in $\boldsymbol{R}^n$ is homeomorphic to all of $\boldsymbol{R}^n$.
- The surface of a donut (a torus) is not homeomorphic to a sphere.
- $\boldsymbol{R}^2 - \{0\}$ is homeomorphic to $\boldsymbol{R}^2 - B$, where B is the closed unit ball.

Note, for instance, that the surface of a cube has a rather different geometric shape than a sphere, but there is something "generally spherical" about it. This something is captured by the notion of a homeomorphism. On the other hand, the notion of homeomorphism is tight enough so that it distinguishes between the surface of a sphere and the surface of a donut.

There are other, related, notions of equivalence that are similar to homeomorphisms. One related notion is that of *homotopy equivalence*. For the sake of simplicity, I won't discuss homotopy equivalence in these notes.

Surfaces: One can ask the general question: Which spaces are homeomorphic to which others? This question is too vague, but special cases of it have interesting answers. A classic special case is that of *surfaces*. Most people have an intuitive notion of a surface – e.g. a sphere or a torus – but one can give a precise definition along the following lines. A surface is a finite union of (solid) triangles, glued together edge-to-edge such that each triangle is glued to a distinct triangle along each of its edges. Figure 1 shows an example.

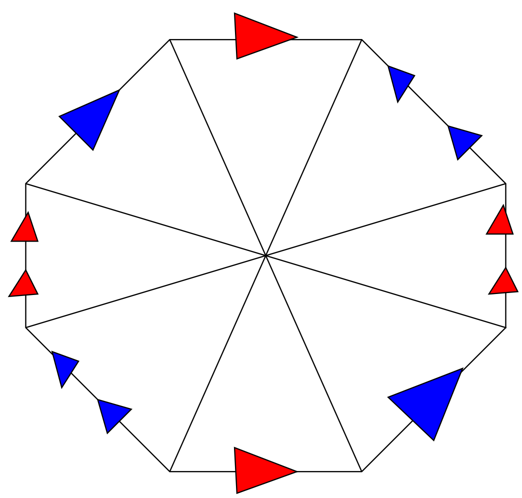
Figure 1

The arrow notation is supposed to indicate additional gluings that cannot be accurately represented on the page. For instance, the top edge is glued to the bottom edge in a left-to-right fashion and the left edge is glued to the right edge in a bottom-to-top fashion. In M141 this definition is made precise. The surface defined by Figure 1 is not homeomorphic to either the sphere or the torus. In M141 you will see how to determine exactly when two of these kinds of surface are homeomorphic to each other.

One way to tell apart these surfaces is the *Euler characteristic*,

$$\chi = F - E + V.$$

Here F is the number of triangles, and E is the number of edges, and V is the number of vertices. In our example, we have $\chi = 8 - 12 + 2 = -2$. If one puts the additional restriction that the surfaces are oriented, then two surfaces are homeomorphic if and only of they have the same Euler characteristic.

Knots: We say that a *knot* is an embedded closed loop in S^3, the 3-dimensional sphere. In other words, we have a continuous and injective map $\gamma : S^1 \to S^3$, where S^d denotes the d-dimensional sphere. Two knots K_0 and K_1 are *equivalent* if there is a homeomorphism from S^3 to S^3 that carries K_0 to K_1. This turns out to be equivalent to the condition that there exists a continuous 1-parameter family of knots $\{K_t|\ t \in [0,1]\}$. Informally, if you make K_0 and K_1 out of string, they are equivalent if and only if you can jiggle K_0 around until it looks like K_1. The simplest knot is just a round circle in S^3. This is known as the *unknot*. Amazingly, there are many inequivalent knots. Figure 2 shows that trefoil knot, which is not equivalent to the unknot.

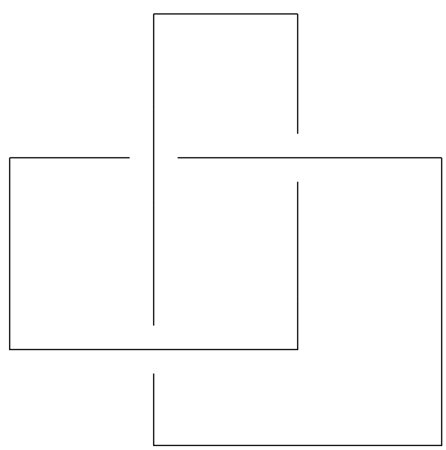

Figure 2

Figure 2 shows a picture of the trefoil knot projected into the plane. The little gaps in the picture indicate where one strand of the loop passes over another one. It turns out that one can work with these *knot projections* and get a lot of information about when knots are equivalent. One typical thing is to attach a number, polynomial, or group to a knot based on one of its projections. Then one shows by combinatorial means that the quantity is the same for all possible projections. This means that two knots are equivalent only if they have the same attached quantity. Such an attached quantity is called a *knot invariant*.

Fundamental Group Attaching auxilliary objects such as numbers and groups to a space is one of the themes of topology. Here I'll explain one of the central constructions along these lines. In this construction, you attach a group to a pair (X, x), where x is a distingished point in X. Assuming that X is a path connected space – meaning that any two points can be joined by a continuous curve – the choice of point x is not so important. Any other choice leads to an isomorphic group.

Here is the construction. Let S^1 be the circle, as above, and let 0 be some point of S^1. Let's say that a *loop* is a continuous map

$$f : S^1 \to X$$

such that $f(0) = x$. We call 2 loops f_0 and f_1 *equivalent* if there exists a continuous family $\{f_t|\ t \in [0,1]\}$ of loops. We write $f_0 \sim f_1$ in this case.

$\pi_1(X, x)$ is the set of equivalence classes of loops. The group law for loops is "concatenation". Given loops f_1 and f_2, the loop $f_1 \circ f_2$ is obtained by first "going around" f_1 and then "going around" f_2. Setting $g = f_1 \circ f_2$, the map g restricted to the interval $[0, 1/2]$ agrees with f_1, and the map g restricted to the interval $[1/2, 1]$ agrees with f_2. The map g does f_1 in half the time and then does f_2 in half the time. One needs to check that this operation is well defined on equivalence classes. That is, we would need to check that $f_1 \circ f_2 \sim f_1' \circ f_2'$ if $f_1 \sim f_1'$ and $f_2 \sim f_2'$. The identity element of this group is the equivalence class of the constant loop. The inverse of any loop is the loop one gets by tracing the loop out in the opposite direction. Everything works out, and $\pi_1(X, x)$ is a well-defined group called the *fundamental group*. Here are some examples.

- $\pi_1(S^2) = \{e\}$, the trivial loop. This result reflects the fact that any loop on S^2 can be shrunk down to a point.
- $\pi_1(S^1) = \boldsymbol{Z}$.
- $\pi_1(T^n) = \boldsymbol{Z}^n$, for any choice of point x. Here T^n is the n-torus, namely the n-fold product of circles.

- $\pi_1(S^3 - S_1) = \boldsymbol{Z}$. Here S^1 is a round circle sitting in S^3.
- Let K be the trefoil knot. Then $\pi_1(S^3 - K)$ is infinite and nonabelian.

Suppose X and Y are spaces and f is a homeomorphism from X to Y. Then f induces an isomorphism from $\pi_1(X)$ to $\pi_1(Y)$. Hence, two spaces with different fundamental groups are not homeomorphic. In particular, the trefoil knot is not equivalent to the unknot.

Triangulations and the Fundamental Group When the space X is given as a union of glued-together triangles or tetrahedra, one can compute $\pi_1(X)$ in a combinatorial way. The rough idea is that one can replace any loop with one that stays in the edges of the triangulation. One can then work out the equivalence relations between loops by looking at the pattern of edges and triangles that bound them. This kind of thing is worked out in M141.

Van Kampen's Theorem One of the most powerful computational results about the fundamental group is *Van Kampen's Theorem*. This result describes $\pi_1(X \cup Y)$ in terms of $\pi(X)$ and $\pi_1(Y)$ and $\pi_1(X \cap Y)$, provided that $X \cap Y$ is connected. Here we think of $X \cup Y$ as a more complicated space that is built out of simpler components X and Y. For instance, you could glue two annuli together to build a torus. If you have a complicated space that is built out of simple pieces (like tetrahedra), you can use Van Kampen's theorem in an inductive way to effectively compute the fundamental group.

One nice application of Van Kampen's Theorem is a formula, known as the *Wirtinger presentation*, for the fundamental group of $S^3 - K$, where K is a knot. The Wirtinger presentation is computed from any knot projection. Different projections of K lead to different-looking descriptions of the same group, but the underlying group is always the same. Thus, one way to tell knots K_1 and K_2 apart is to compute $\pi_1(S^3 - K_1)$ and $\pi_1(S^3 - K_2)$ and see that the groups are different. This isn't always easy to do, but sometimes it is possible.

Brouwer Fixed Point Theorem: Topological ideas such as the fundamental group can be used to establish some classical results about maps between spaces. Here is an example:

Theorem 0.1 *Let D^2 be the unit disk, with boundary S^1. There is no continuous map $f : D^2 \to S^1$ that is the identity on the boundary.*

Proof: (sketch) Suppose $f: D^2 \to S^1$ exists, with f being the identity on S^1. Consider the chain of maps $S^1 \to D^2 \to S^1$. The first map is just inclusion. The second map is f. This chain of maps sets up a sequence of homomorphisms $\pi_1(S^1) \to \pi_1(D^2) \to \pi_1(S^1)$. The composed map is the identity. This gives us a chain of homomorphisms $\boldsymbol{Z} \to \{e\} \to \boldsymbol{Z}$, where the middle group is the trivial group and the composed map is the identity. This is impossible. ♠

The result we just sketched has a classical corollary, known as the Brouwer Fixed Point Theorem.

Corollary 0.2 *Any continuous mapping $f: D^2 \to D^2$ has a fixed point.*

Proof: Suppose that such an g exists, with no fixed point. Then one can define a retraction $f: D^2 \to S^1$ as follows. $f(p) = R \cap S^1$, where R is the ray that emanates from $g(p)$ and contains p. (Draw a picture!) One checks that f violates the previous result, giving a contradiction. ♠

These kinds of results are proved in much more detail in M141.

Some Classic Results: In M141 you prove a number of classic results that are similar in spirit to the Brouwer Fixed Point Theorem.

- The Brouwer Fixed Theorem for D^n, the n-dimensional ball. This requires different ideas than those mentioned above.
- The Hairy Ball Theorem: Any continuous vector field on the sphere vanishes at some point.
- The Borsuk-Ulam Theorem: Any continuous function from S^n to $\boldsymbol{R}^n$ maps a pair of antipodal points to the same point.
- The Index Theorem: Given a continuous vector field on a surface, this result relates the number of places where the vector field vanishes (suitably counted) to the Euler Characteristic of the surface.