

Math 9 Course Summary/Study Guide — Fall, 2005

[1] Limits

- Definition of Limit: We say that L is the limit of $f(x)$ as x approaches a if $f(x)$ gets closer and closer to L as x gets closer and closer to a . We write $\lim_{x \rightarrow a} f(x) = L$.
- Definition of Continuity: The function $f(x)$ is *continuous* at $x = a$ if $f(a)$ exists and $\lim_{x \rightarrow a} f(x) = f(a)$.
- Limits at Infinity: We say that L is the limit of $f(x)$ as x approaches ∞ if $f(x)$ gets closer and closer to L as x increases without bound. We write $\lim_{x \rightarrow \infty} f(x) = L$. Similarly for $x \rightarrow -\infty$.
- Rules for Computing Limits: Assuming that all the limits exist:

$$\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) \quad \text{Sum}$$

$$\lim_{x \rightarrow a} f(x)g(x) = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right) \quad \text{Product}$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{provided that } \lim_{x \rightarrow a} g(x) \neq 0. \quad \text{Quotient}$$

- A Special Limit: $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.
- L'Hôpital's Rule: If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.
Similarly, if $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

[2] Derivatives

- Definition of Derivative: The *derivative* of the function $y = f(x)$ is defined to be

$$\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative $f'(x)$ measures the rate at which the function $f(x)$ is changing.

- Basic Differentiation Rules: If u and v are functions of x , and if c is a constant, then

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \quad \text{Addition Rule}$$

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \quad \text{Product Rule}$$

$$\frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad \text{Quotient Rule}$$

$$\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx} \quad \text{Power Rule}$$

- The Chain Rule: If $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$ may be viewed as a function of x . The *Chain Rule* says that

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

If we let $h(x) = f(g(x))$, then another way to write the same rule is $h'(x) = f'(g(x))g'(x)$.

- Higher Order Derivatives: The *second derivative* of $y = f(x)$ is the derivative of the derivative of $f(x)$. Common notations for the second derivative are $\frac{d^2y}{dx^2}$ and $f''(x)$. Similarly, the n^{th} derivative of $y = f(x)$ is obtained by differentiating n times. Notations for the n^{th} derivative are $\frac{d^n y}{dx^n}$ and $f^{(n)}(x)$.

- Implicit Differentiation: If x and y are related by a equation, you can find a formula for dy/dx by differentiating both sides of the equation using differentiation rules and the chain rule, and then solving for dy/dx . The answer will generally be an expression that involves both x and y .

- Logarithm Differentiation: If $y = f(x)$, sometimes it is easier to first write $\ln(y) = \ln(f(x))$, use rules of logarithms to simplify $\ln(f(x))$, and then differentiate implicitly. In cases where y has the form $f(x)^{g(x)}$, this may be the only method that works.

- Derivatives of Some Special Functions:

$\frac{d \sin(x)}{dx} = \cos(x)$	$\frac{d \cos(x)}{dx} = -\sin(x)$
$\frac{d \tan(x)}{dx} = \sec^2(x)$	$\frac{d \sec(x)}{dx} = \sec(x) \tan(x)$

Trigonometric Functions

$\frac{d \sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$	$\frac{d \cos^{-1}(x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d \tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$	$\frac{d \sec^{-1}(x)}{dx} = \frac{1}{x\sqrt{x^2-1}}$

Inverse Trigonometric Functions

$\frac{d \ln(x)}{dx} = \frac{1}{x}$	$\frac{de^x}{dx} = e^x$
$\frac{d \log_a(x)}{dx} = \frac{1}{x \ln a}$	$\frac{da^x}{dx} = a^x \ln(a)$

Logs and Exponentials

[3] Further Differential Calculus Topics

- The Mean Value Theorem: The *Mean Value Theorem* says that if $f(x)$ is differentiable, then there is always some c between a and b for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The geometric interpretation of this formula is that there is a point on the curve $y = f(x)$ between $x = a$ and $x = b$ where the tangent line is parallel to the line connecting $(a, f(a))$ to $(b, f(b))$.

- Differentials: The *differential of* $y = f(x)$ is $dy = f'(x) dx$. The quantity dx is an independent variable which we usually take to be quite small. The differential dy is an approximation of how much f changes when x increases by dx . This gives the linear approximation formula

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

[4] Applications of the Derivative

- Slopes and Tangent Lines: The *slope* of the curve $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$. The *tangent line* to the curve $y = f(x)$ at the point $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$.

- Related Rates Problems: In a related rates problem, two or more quantities which are changing in time are related by equations. The first step is to give names to quantities and write down relations (a sketch is often helpful). You are generally told how fast some of the quantities are changing, and are asked to find how fast the remaining quantity is changing. This is done by differentiating the relations with respect to time (using the chain rule), substituting the known values into the differentiated equations, and then solving for the unknown value.

- Maximum/Minimum Problems: In a max/min problem, you are given a quantity, say Q , which is to be maximized for minimized. Do the following steps:

- (1) Use the given information (a sketch is often helpful) to express Q in terms of a single variable, say x .
- (2) Find the critical values by computing the derivative $Q'(x)$ and solving the equation $Q'(x) = 0$.
- (3) For each critical value a , determine whether a is a maximum, a minimum, or neither. This can be done by the *change-of-sign test*:
 - If $Q'(x)$ goes from plus to minus as x passes a , then a gives a local maximum.
 - If $Q'(x)$ goes from minus to plus as x passes a , then a gives a local minimum.
 - If $Q'(x)$ doesn't change sign as x passes a , then a is neither a max nor a min.

Another way to check is using the *second derivative test*:

- If $Q''(a) > 0$, then a gives a local minimum.
 - If $Q''(a) < 0$, then a gives a local maximum.
 - If $Q''(a) = 0$, then the second derivative test gives no information.
- (4) If the variable x is required to be between a and b , compute $Q(a)$ and $Q(b)$, the values at the endpoints.
 - (5) Compare the values of $Q(x)$ at the critical points and at the endpoints and pick out the absolute maximum and/or the absolute minimum.

- Curve Sketching: To sketch the graph of a function $y = f(x)$, do the following steps:

- (1) Compute $f'(x)$ and determine where:
 - $f'(x) > 0$ — the curve is rising.
 - $f'(x) < 0$ — the curve is falling.
 - $f'(x) = 0$ — the curve may have a local max or min.

- (2) Compute $f''(x)$ and determine where:
- $f''(x) > 0$ — the curve is concave up.
 - $f''(x) < 0$ — the curve is concave down.
 - $f''(x) = 0$ — the curve may have an inflection point.
- (3) Determine values of a for which $\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$. These are vertical asymptotes. They often occur at points where the denominator of $f(x)$ vanishes.
- (4) Compute $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. If these limits exist, they give horizontal asymptotes.
- (5) Draw and label all maxima, minima, inflection points, and asymptotes. Add a few other points to your graph (for example, the point $(0, f(0))$ and points where $y = 0$).
- (6) Using these marked points and your knowledge of where the graph is rising, falling, concave up, and concave down, sketch the graph.

- Newton's Method: Newton's Method is a way of finding solutions to $f(x) = 0$. Starting with an initial guess $x = x_1$, one produces a list of values x_1, x_2, x_3, \dots by the general formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This list of values (usually) converges quite rapidly to a solution.

[5] Integration

- Riemann Sums and Definite Integrals: First break the interval from a to b into n pieces, each of width $\Delta x = (b - a)/n$. Next choose an x -value in each little piece, say x_1^* in the first piece, x_2^* in the second piece, and so on. The *Riemann sum of f* is then

$$\sum_{i=1}^n f(x_i^*) \Delta x.$$

The *definite integral of f from a to b* is the limit of the Riemann sums as the number of pieces goes to infinity and the width of each little piece goes to zero:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

- Basic Integration Rules:

$$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

- Anti-Derivatives and Indefinite Integrals: A function $F(x)$ is called an *anti-derivative* for $f(x)$ if $F'(x) = f(x)$. If $F(x)$ is one anti-derivative of $f(x)$, then every anti-derivative

has the form $F(x) + C$ for some constant C . This most general anti-derivative is called the *indefinite integral of f* and is written $\int f(x) dx$.

- Basic Indefinite Integrals: The differentiation formulas for powers, trig functions, logs, exponentials, inverse trig functions, etc. all give corresponding indefinite integral formulas.
- The Fundamental Theorem of Calculus (one part): Let $F(x)$ be any antiderivative for $f(x)$. That is, $F'(x) = f(x)$. Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

- The Fundamental Theorem of Calculus (the other part): Define a function $F(x)$ by

$$F(x) = \int_a^x f(t) dt. \quad \text{Then} \quad F'(x) = f(x).$$

More generally, if $g(x)$ is a function of x , define a function

$$F(x) = \int_a^{g(x)} f(t) dt. \quad \text{Then} \quad F'(x) = f(g(x))g'(x).$$

- Evaluation of Integrals by Substitution: In an integral $\int f(x) dx$, we can substitute $u = g(x)$ provided we also use the relation $du = g'(x) dx$ to express everything in terms of u . For the definite integral $\int_a^b f(x) dx$, the limits of integration in the new u integral are $g(a)$ and $g(b)$, because as x goes from a to b , the value of $u = g(x)$ goes from $g(a)$ to $g(b)$.

[6] Applications of Integration

- Area Between Curves: The area below the curve $y = f(x)$ and above the x -axis and lying between $x = a$ and $x = b$ is

$$A = \int_a^b f(x) dx.$$

(This formula is valid provided $f(x) \geq 0$ for all x between a and b .)

The area below the curve $y = f(x)$ and above the curve $y = g(x)$ and lying between $x = a$ and $x = b$ is

$$A = \int_a^b (f(x) - g(x)) dx.$$

(This formula is valid provided $f(x) \geq g(x)$ for all x between a and b .)

- Volume by Cross Sections: Suppose that when a solid is cut at x , the area of the cross section is $A(x)$. Then the volume of the solid is

$$V = \int_a^b A(x) dx.$$

- Volume of Revolution using Disks or Washers: If the region below the curve $y = f(x)$ and above the x -axis and lying between $x = a$ and $x = b$ is rotated about the x -axis, the volume swept out is

$$V = \int_a^b \pi f(x)^2 dx$$

(This formula is valid provided $f(x) \geq 0$ for all x between a and b .)

If the region below the curve $y = f(x)$ and above the curve $y = g(x)$ and lying between $x = a$ and $x = b$ is rotated about the x -axis, the volume swept out is

$$V = \int_a^b (\pi f(x)^2 - \pi g(x)^2) dx$$

(This formula is valid provided $f(x) \geq g(x)$ for all x between a and b .)

- Work: *Work* is force applied over a distance. If the force is constant, then the work done is simply

$$\text{Work} = \text{Force} \times \text{Distance}.$$

If the force varies with position, say the force at the point x is given by the function $F(x)$, then the work done in moving from $x = a$ to $x = b$ is

$$W = \int_a^b F(x) dx.$$

A spring exerts a force proportional to the distance it is stretched (this is known as Hooke's Law). So if $x = 0$ is the unstretched position, then the force is

$$F = kx.$$

The spring constant k can be determined if you are told the force at some point $x = x_1$. The work done in stretching the spring from $x = a$ to $x = b$ is

$$W = \int_a^b kx dx = \frac{1}{2} kx^2 \Big|_a^b = \frac{1}{2} kb^2 - \frac{1}{2} ka^2.$$

- Average Value of a Function: The average value of the function $f(x)$ for $a \leq x \leq b$ is

$$f^{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$