To all my math teachers.
When I looked at the pages that mention atoms and special relativity, I wondered if I had gone too far.

But then I went a little crazy and put in all kinds of fancy ideas that I found interesting.

When I started writing this book, I planned to make it simple enough so that anyone old enough to read could enjoy it.

I ended up making both a simple version and an advanced version, but I couldn’t decide which one to show you.

Then I had an idea about how to show you both versions at the same time.

This book has all the pages, simple and advanced, but I put little black markers in the corners of all the advanced ones.

If you are feeling adventurous and want the advanced version, then just read it all.

If you want to read the simple version, then skip the pages with the markers. The book will still make sense.
Who are the animals on the infinite farm?
Gracie is one of the animals. She is an infinite cow.
Gracie loves shoes. She has one on every foot ...
and her feet go on forever!
Flambeau, an infinite sheep, mulls over shearing season while Simon, an infinite donkey, chats with Gerry,...
an infinite gopher.
Ezekiel is a goat with an infinite horn, but he thinks of himself as mostly a zebra.
Hammerwood is an infinite crocodile. Let's zoom out to see him better.
Look at those teeth!

They go on and on. And so do the bubbles.
Hammerwood loves bubble gum, even though it is hard for him to chew it all at once.
Penn also lives on the farm. He is an infinite chicken. On the infinite farm, chickens have teeth too.
Penn has staring contests with his sister Victoria, and also with...
Bill, an owl with an infinite eye.
Delores, an infinite squid, lives in the farm's pond.
Her tentacles branch out endlessly, and she decorates them with jewelry. She is crazy about rings and bracelets. She likes to swim on Saturday nights with her friend...
Nelson, a shark whose head extends forever in both directions.
The farm has lots of other animals too. Some of them are so strange I don't know what to call them. Maybe YOU can think of names for them.
Do the animals have any problems on the farm?
Gracie has a problem with shoes. Her friends know that she loves shoes, so they bring her new ones as presents. Oh yes, lots of problems!
She wants to wear the NEW shoes ...

But she doesn't want to stop wearing any of her OLD shoes.
She already has a shoe on every foot, so what is she going to do?
Hammerwood has problems with his teeth. He has so much fun blowing bubbles ...
...that he often forgets to brush his teeth.

To be honest, bad things sometimes happen to Hammerwood's teeth.

It isn't pretty up close.

It can get so bad that ...
He loses most of his teeth.

He needs those teeth for chewing and he can't grow new ones.

How can he fix his teeth?
Delores has a problem with her jewelry. Every week her cousin Bin swims over and asks to borrow some of it.
Bin wants to decorate all his tentacles for his Saturday night dances...
but Delores wants to decorate all of HER tentacles for her Saturday night swims with Nelson. How can they split the jewelry so that each has enough for all of their tentacles?
Here is how the animals solve their problems...
Gracie prepares to put on her new shoes by standing behind them on an infinitely long sticky strip.
She gives each left leg a little shake ...

then steps out of each left shoe ...
and into the one in front of it.
Then she does the same thing with all her right feet.
Gracie doesn't have a back end!

Now she wears the new shoes AND all of her old shoes. There aren't any shoes left over at the back end because...
Gracie moves off the sticky tape and then... she is ready for another present.
Hammerwood snaps into action once he loses about 9 out of 10 teeth.
The first thing he does is brush his remaining teeth.
Then his friends from the infinite ant colony put powerful rubber bands around his teeth.
For technical reasons, they anchor the first two bands to a stick.

At night, the rubber bands contract ...
and slide his teeth back into place.
The ants carry off the rubber bands and then his teeth are as good as new.
Delores agrees to lend Bin some jewelry.

Their tentacles branch out in the same way, so on Saturday afternoon...

they line them up like two infinite hands pressing together.
Little fish come and unclasp the jewelry.

The fish move each piece one step towards Delores' head.
Each piece of jewelry moves inward...

and is replaced by TWO pieces that come from behind it!
Something fancier happens to the piece of jewelry around Delores' neck. A fish unwinds it and moves it to her forehead. It becomes her tiara for the evening.
The fish move one piece of each pair over to Bin... and then swim away.
Except for the tiara, Delores has the same pattern of jewelry as in the beginning. And now Bin has it too.

Bin thanks Delores for being so amenable and then the cousins swim off to their Saturday night fun.
Any questions?
How does Delores get her jewelry back from Bin?

Easy. On Sunday morning they line up again and reverse the whole process.
Where does Hammerwood get his gum?

ALICE'S GUM SHOP

Credit cards accepted.
Simple. He points his mouth up an infinite branch of a gum tree and takes what he wants.
Can Gracie swing all the way around, like the hand on a clock?
No. She would get stuck on the trees.

But otherwise she is quite flexible.
How do the animals get around each other?
That is a very interesting question. For one thing, the animals often have very different sizes.

Gracie walks underneath Penn and they hardly notice each other.
Also, the animals are surprisingly acrobatic. Gracie's cousin Boopis can roll right over her.
Some of the animals have special tricks. Nelson's head can swing open like a door in some places...
and Delores is built like a flexible puzzle. She can pull herself apart to let another animal through.
How do the infinite animals fit in the pond?
Like this! The space inside the pond is warped in a way that makes it infinite from the inside but finite from the outside. To us outsiders, Delores' spots appear to shrink as they approach the edge, but they are really all the same size. The same goes for Nelson's teeth and eyes.
Flambeau and his herd live in a crater that is like the pond, except that they walk on the bottom.

From the outside, Flambeau's friends Bolyai, Gauss, and Lobachevsky seem to be different sizes, but they are really all the same size.
Bill's eye is also one of these enclosed infinite spaces. He loves watching Flambeau and the other sheep. He can see their whole crater perfectly.
Craters could have their own craters...
And so on.
Can the infinite farm fit on Earth?
The infinite farm doesn't close up like Earth, so how could it fit on Earth? It seems impossible.
But think back to how the pond and those craters only take up a finite amount of space from the outside.
Oh, I see. It isn't such a stretch to imagine that the farm itself is one of those craters, and that the crater is on Earth.
Yes, precisely, but there is a catch. On Earth the physical laws would prevent one of those craters from existing. So, the infinite farm can't REALLY fit on Earth.

*A crater like this seems to require matter to exist on all scales, which is incompatible with the atomic nature of our world. Even setting this aside, the crater might generate a gravitational field that would crush our entire universe.
How does day change to night on the farm?

Yeah, how does the sun get around the farm?
You could imagine that there is an infinite grid of suns and moons hovering over the farm... and that these celestial lights change their color and brightness to suit the time of day...
or you could imagine that the suns and moons dance around the farm in a complicated, multi-dimensional pattern...
I like to think that light appears on the farm without explanation or cause, like a new idea. So, if there are no further questions...

or you could imagine that there are no suns and moons, and instead that light is brought each day by an infinite flock of glowing birds.
How did Gracie get on the farm?

How can Ezekiel's neck support his infinite horn?

Would the farm have infinite atoms and/or infinite biomolecules?

Since most of his teeth would have to move faster than the speed of light, isn't Hammerwood's teeth-fixing process incompatible with the constraints of special relativity?
OK, OK, slow down! Let me try to answer all these questions at the same time. In our universe the sciences are relentless enforcers. They seem to look into every situation, no matter how large or small, and make sure that it unfolds according to the law...
On the infinite farm, the sciences are more like negligent babysitters who check in from time to time but mostly just play around.

Meanwhile, out on the farm,...
GEOMETRY makes the rules.

Well, I guess there's time for one last question.
Can we VISIT the infinite farm?
It depends on what you mean by "visit". You won't find the infinite farm on Earth...
but if you read between the lines of geometry books...

...you might find it there.

GALLERY OF THE INFINITE
THE GEOMETRY AND TOPOLOGY OF 3-MANIFOLDS
GEOMETRIC GROUP THEORY
1978
About the author: I am the Chancellor’s Professor of Mathematics at Brown University. In my spare time I like to draw, write computer programs, listen to music, walk on the beach, cycle, or go to the gym.

About the illustrations: I drew the pictures for this book using Inkscape, a great drawing program.

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