Planar Tilings and Wallpaper Groups:

These are some notes I made for an undergraduate class year. One of the topics was planar tilings. I thought that some of you might like the notes. The notes cover some of the things that I said on the first day, and also somethings for the second lecture. I also slightly modified them for this class. Feel free to ignore the exercises.

Isometries: A *metric space* is a set X equipped with a distance function $d : X \times X \to \boldsymbol{R}$ that satisfies the following rules.

- $d(x, y) \geq 0$ with equality if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) + d(y, z) \geq d(x, z)$ for $x, y, z \in X$.

A metric space is a generalization of Euclidean space. In Euclidean space, the metric is given by

$$d(x, y) = \sqrt{(x - y) \cdot (x - y)}.$$

The symbol $(\cdot)$ is the dot product. The reason I'm bringing general metric spaces is that it is a good context in which to define an orbifold. I'll talk more about that below.

In general, an *isometry* between metric spaces X and Y is a bijection $T : X \to Y$ which respects distances:

$$d_Y(T(x), T(y)) = d_X(x, y).$$

Here d_X is the distance on X and d_Y is the distance on Y. In case $X = Y$ it is usually said that f is an *isometry of* X. The *fixed point set* of an isometry $T : X \to X$ is the set of $p \in X$ such that $T(p) = p$.

Exercise 1: Prove that the fixed point set of an isometry of $\boldsymbol{R}^2$ is either the empty set, a point, a line, or all of $\boldsymbol{R}^2$.

There are 4 special kinds of isometries of $\boldsymbol{R}^2$:

1. **translations:** These have the form $T(v) = v + v_0$ for some vector v_0. Their fixed point set is empty.

2. **reflections:** These are maps whose fixed point set is a line. They act by reflecting trough the fixed line.

3. **rotations:** These are maps whose fixed point set is a single point.

4. **glide reflections** These are maps which are the composition of a reflection in a line L and translation by a nonzero amount along L. These are orientation-reversing maps having no fixed point set.

Exercise 2: Prove that the composition of a rotation and a translation is again a rotation.

Tilings: A *general tiling* is a map $f : \boldsymbol{R}^2 \to C$, where C is some set. This looks like a crazy general definition, but what is going on is that you should think of C as a set of possible colors. If $f(x) = c \in C$ if means that the value c is the color of the point x.

A *symmetry* if the tiling f is an isometry $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that

$$f \circ T = f.$$

In other words, T moves the plane in such a way that it does not disturb the colors. The points x and $T(x)$ always get the same color. If T is a symmetry of the tiling then so is T^{-1}. Also, if S and T are symmetries of the tiling, then so is $S \circ T$. In short, the set of symmetries of a tiling forms a group with respect to composition.

Let f be a tiling and let G be the group of symmetries of f. The *orbit* of a point $x \in \boldsymbol{R}^2$ is the set

$$\bigcup_{g \in G} g(x).$$

In other words, you move x around by the entire group. Intuitively, the tiling looks the same at each point in the orbit.

The tiling is called a *wallpaper tiling* if there is a polygon U in the plane such that

- Every orbit intersects U.
- An orbit can only intersect the interior of U once.

The polygon U is called a *unit* or a *fundamental domain*. It is meant to be a solid (i.e. filled-in) polygon and not just the boundary. (Some people in the audience will object to this definition, but let it stand.)

Note that a wallpaper tiling might have more than one unit. The group of symmetries of a wallpaper tiling is called a *wallpaper group*.

Exercise 3: Give an example of a wallpaper tiling which has units U_1 and U_2 which are not isometric to each other. That is, they have genuinely different shapes. Also, give an example of a wallpaper tiling in which all units are isometric to each other.

In a wallpaper tiling, the tiling group gives a way to kind of fold up the unit into a new space. The rule is that if two points on the boundary of the unit U are in the same orbit, these points should be glued together. For instance, one of the wallpaper tilings is the one in which there is just translation symmetry and the unit U is a square. Corresponding points on opposite sides of U are glued together because they belong to the same orbit. The resulting space (usually called the quotient space) is a torus. Informally, an *orbifold* is any space you get by folding up the unit corresponding to a walpaper tiling.

Orbifolds: It is tricky to define orbifolds straight away, so I'm first going to define a somewhat looser kind of object. An orbifold will turn out to be this kind of object, provided that it has additional properties.

First of all we're going to talk about several kinds of metric spaces.

1. The plane: $\boldsymbol{R}^2$.
2. The upper half plane in $\boldsymbol{R}^2$. These are all the points (x, y) with $y \geq 0$.
3. A positive sector in $\boldsymbol{R}^2$: These are all the points of the form $rV + sW$ where V and W are linearly independent vectors having positive dot product and r, s are non-negative real numbers.
4. A positive cone: This is the space obtained by gluing together two positive sectors along their common boundaries. This space is sort of like an ice-cream cone.

A *ball* in a metric space is the set of points within r of a given point. That is, all the balls in a metric space X have the form $\{q|\ d(p,q) < r\}$ for some $p \in X$ and some $r > 0$. The balls in $\boldsymbol{R}^2$ are just open disks.

Exercise 4: Draw pictures of the kinds of spaces mentioned above: the half plane, positive sectors, and positive cones. Also, draw enough pictures of balls in these spaces so that you see what they look like.

Now for one of the main definitions: A *mirrored flat cone surface* is a metric space X which has the following property: There is some $\epsilon > 0$ such that every ball of radius ϵ in X is isometric to a ball in one of the spaces mentioned above. An *orbifold* is a mirrored flat cone surface that comes from taking a unit for a wallpaper tiling and making identifications on the boundaries. Here are the 17 examples.

Example A1: The mirrored square. The metric space is just a square. Small balls in the interior are isometric to balls in the plane. Small balls centered at points on the edge are isometric to balls in the upper half plane. Small balls centered at the corner points are isometric to balls centered at the apex of a right angled sector. The reason why the edges are called *mirrored* is that in the corresponding wallpaper tilings the unit is a square and the tiling has reflection symmetry across each edge. So, in the tiling, the edges function as mirrors.

Example A2: The mirrored equilateral triangle.

Example A3: The mirrored $45-45-90$ triangle.

Example A4: The mirrored $30-60-90$ triangle.

Example B1: The doubled square: Take two copies of the square and glue them along their boundary. Every small is isometric either to a ball in the plane or to a ball in a cone.

Example B2: The doubled equilateral triangle.

Example B3: The doubled $45-45-90$ triangle.

Example B4: The doubled $30-60-90$ triangle.

Example C1: The mirrored cylinder.

Example C2: The mirrored Mobius band.

Example D1: The torus. This is a square with the opposite sides identified straight across. In this space, every small ball is isometric to a ball in the plane.

Example D2: The Klein bottle. This is a square with the vertical sides identified straight across and the horizontal sides identified with a twist, as shown in Figure 1. In this space, every small ball is isometric to a ball in the plane.

Example D3: The projective plane. This is a square with the vertical sides identified with a twist and the horizontal sides identified with a twist.

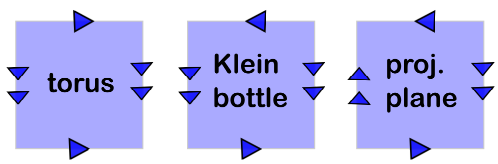

Figure 1: Torus, Klein bottle, Projective plane

Example E1: The 2-fold rotational quotient of the mirrored square. Take half the mirrored square and glue up the bottom edge as indicated in Figure 2.

Example E2: The 4-fold rotational quotient of the mirrored square. See the middle part of Figure 2.

Example E3: The 3-fold rotational quotient of the mirrored equilateral triangle. See the right part of Figure 2.

Example E4: The reflection quotient of the doubled square. This is sort of like a taco with the top edge mirrored. This one is hard to draw. Basically, start with the doubled square, then cut it in half vertically through the middle. The place where you cut is considered mirrored.

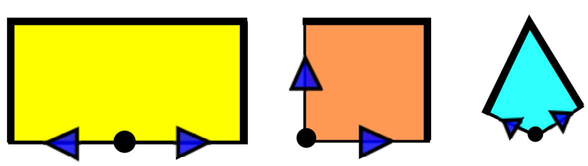

Figure 2: The E-series of orbifolds

Exercise 5: Take 3 examples above (from different series) and find a tiling which gives rise to the example you picked.

Exercise 6: Any triangle counts as a mirrored flat cone surface. Show that the only orbifolds like this are the ones in the A-series. In other words, you can't have a wallpaper tiling whose unit is a triangle, such that the tiling has reflection symmetries across all the edges of the triangle.

Exercise 7: Go around campus and take 3 pictures of things which (if infinitely extended) would be wallpaper tilings. Your goal is to pick 3 things which have different underlying orbifolds.