Integrating Functions on Riemannian Manifolds

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1 Introduction

Section 9.4 in the book deals with the theory of integrating a function over a smooth k-dimensional manifold embedded in $\boldsymbol{R}^n$. We already know how to integrate k-forms on a k-manifold but the topic here is how to deal with functions. The purpose of these notes is to clarify what is going on by explaining things in terms of abstract manifolds.

The general way it works is that one can integrate functions on a *Riemannian manifold*, because the Riemannian metric defines a canonical volume form locally. The canonical form is defined everywhere, up to a sign. The sign can't work out globally if the manifold is non-orientable, but there is a trick using partitions of unity to make use of these local volume forms even in the non-orientable case.

When one has a submanifold in $\boldsymbol{R}^n$, there is a canonical Riemannian metric which just comes from the restriction of the dot product. So, you can use the abstract theory to integrate functions submanifolds of $\boldsymbol{R}^n$. The final theory turns out to be equivalent to what is done in the book.

2 Inner Products and Volume Forms

Let V be a finite dimensional real vector space. An *inner product* on V is a map $Q : V \times V \to \boldsymbol{R}$ such that

1. Q is a symmetric 2-tensor.

2. $Q(w,w) > 0$ for all $w \neq 0$.

Lemma 2.1 *There exists an orthonormal basis for Q.*

Proof: Given a basis $\{v_1, ..., v_n\}$ for V we can perform the usual Gram-Schmidt process for creating an orthonormal basis with respect to Q. The procedure works like this.

- Replace v_1 by
$$w_1 = v_1/\sqrt{Q(v_1, v_1)}$$
so arrange that $Q(w_1, w_1) = 1$.

- Assuming that $w_1, ..., w_k$ have been constructed, let
$$w'_{k+1} = v_{k+1} - \sum_{i=1}^{k} Q(v_{k+1}, w_i) w_i.$$
This guarantees that $Q(w'_{k+1}, w_i) = 0$ for all $i = 1, ..., k$.

- Replace w'_{k+1} by
$$w_{k+1} = w'_{k+1}/\sqrt{Q(w'_{k+1}, w'_{k+1})}.$$

This produces $w_1, ..., w_n$ such that $Q(w_i, w_j) = 1$ if $i = j$ and 0 otherwise. ♠

Remark: Notice that each w_j varies smoothly as a function of $v_1, ..., v_n$. That is, we can think of w_j as a function from V^n to V, and it is a smooth function.

Lemma 2.2 *Assume that $\boldsymbol{R}^n$ is equipped with the dot product. There is a linear transformation $T : \boldsymbol{R}^n \to V$ which is an isometry between $\boldsymbol{R}^n$ and V.*

Proof: Let $e_1, ..., e_n$ be the standard basis for $\boldsymbol{R}^n$ and let $w_1, ..., w_n$ be an orthonormal basis for V. The map $T(e_j)$ does the trick. ♠

Definition: The *adapted volume forms* on V are the two forms
$$\pm (T^{-1})^* (dx_1 \wedge ... \wedge dx_n).$$
If V also has an orientation, we can "prefer" one of these over the other.

3 Riemannian Manifolds

A *Riemannian metric* on a smooth manifold M is a smoothly varying choice of inner product Q_p on each tangent space $T_p(M)$. The smoothness has the following explanation. If $\alpha : \boldsymbol{R}^n \to M$ is any smooth coordinate chart, then the pullback inner product $\alpha^*(Q)$ is given by a symmetric matrix at each point of $\boldsymbol{R}^n$. We want the entries of this matrix to be smooth functions. This is the usual way we talk about smooth tensor fields on manifolds.

Suppose that M has a Riemannian metric Q. For each $p \in M$ there are two adapted volume forms associated to Q_p, and they differ only by sign. Call these two volume forms $\pm\omega_p$. Let V be a coordinate patch in M. Note that V has one of two local orientations, regardless of whether or not M is orientable. We say that the assignment $p \to \omega_p$ is *continuous* if ω_p defines the same orientation at each $p \in V$. In other words, ω_p is either always positive or always negative when evaluated on a positively oriented basis, as p varies throughout V. Notice that there are exactly 2 continuous adapted volume forms on each coordinate chart.

If M is orientable, we can make a consistent choice of a continuous adapted volume form on M. Otherwise, we have to be content with a system of continuous adapted volume forms, one per coordinate chart.

4 Integration of Functions

Let's continue with the same notation. Suppose that $V \subset M$ is a coordinate chart. Suppose that $f : M \to \boldsymbol{R}$ is a non-negative Borel measurable function whose support is contained in V.

We choose an orientation on V, as well as the corresponding adapted volume form ω. We then define

$$\int_M f = \int_M f\omega.$$

Notice that this is a non-negative number, and strictly positive if $f > 0$ on some open set. Were we to pick the opposite orientation, we would be integrating $-f\omega$ with respect to an oppositely oriented coordinate chart, and we would get the same answer. So, the integral is completely well defined.

Now suppose that $f : M \to \boldsymbol{R}$ is any non-negative Borel function whose support is compact. (This is automatic if M is a compact manifold.) We choose a partition of unity $\{\phi_i\}$ subordinate to some open cover by coordinate charts, and we define

$$\int_M f = \sum \int_M \phi_i f.$$

The compactness guarantees that this is just a finite sum. The same argument as for the integration of forms shows that this definition is independent of the choice of partition of unity.

Remark: If you don't like working with Borel measurable functions, you can restrict your attention to continuous functions. This is all we really need for applications in the book. For continuous functions, the integrals involved can be done by the usual Riemann integral.

Suppose now that $f : \boldsymbol{M} \to \boldsymbol{R}$ is a compactly supported function We can write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$, and $f_- = f - f_+$. Then we define

$$\int_M f = \int_M f_+ - \int_M f_-.$$

5 Euclidean Submanifolds

Suppose now that M is an n-dimensional submanifold of $\boldsymbol{R}^N$. There is a canonical Riemannian metric on M, namely

$$Q_p(V, W) = V \cdot W, \qquad \forall V, W \in T_p(M).$$

We then integrate functions on M with respect to the system of volume forms adapted to M on coordinate charts.

It is worth pointing out why these volume forms are smooth. Let $V \subset M$ be a coordinate patch on M and let $\alpha : \boldsymbol{R}^n \to V$ be a coordinate map. We can get a basis at each point $p \in V$ using $\alpha_*(e_1), ..., \alpha_*(e_n)$. This basis varies smoothly. We can then perform Gram-Schmidt to get a smoothly varying orthonormal basis. The matrix entries of the adapted quadratic form are rational-function entries of the coefficients of the orthonormal bases, to they vary smoothly as well.

6 Reconciling This Approach with the Book

Suppose that $f : M \to \boldsymbol{R}$ is a positive function whose support is contained in the coordinate patch V. Let α be a coordinate chart whose image if V.

Then the expression

$$\sqrt{\det(A^t A)}, \qquad A = D\alpha$$

computes the infinitesimal volume multiplier under the action of α. That is, in each tangent space, the differential map A multiplies volume by $\det(A^t A)$, as explained in the book.

But that means that

$$\alpha^*(\omega) = \sqrt{\det(A^t A)} dx_1 \wedge ... \wedge dx_n.$$

Hence

$$\alpha^*(f\omega) = f\sqrt{\det(A^t A)} dx_1 \wedge ... \wedge dx_n.$$

So,

$$\int_M f = \int_M f\omega = \int_{\boldsymbol{R}^n} \alpha^*(f\omega) = \int_{\boldsymbol{R}^n} f\sqrt{\det(A^t A)}\ dx_1 ... dx_n.$$

This last expression is what is in the book.