Tangent Spaces and Orientations

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February 18, 2015

These notes assume that you have read the previous handout, about the definition of an abstract manifold.

1 Smooth Curves

Let M be a smooth manifold and $p \in M$ be a point. A *curve* on M through p is a smooth map

$$\phi : (-\epsilon, \epsilon) \to M \tag{1}$$

with $\phi(0) = p$. To say that ϕ is smooth in the neighborhood of some point s is to say that $h^{-1} \circ \phi$ is smooth at s, where (h, U) is a coordinate chart and $\phi(s) \in U$. This definition does not depend on the coordinate chart, because the overlap functions are all diffeomorphisms.

Let ϕ_1 and ϕ_2 be two smooth curves through p. We write $\phi_1 \sim \phi_2$ if

$$d(h^{-1} \circ \phi_1)|_0 = d(h^{-1} \circ \phi_2)|_0.$$

Again, this is independent of the choice of coordinate chart used. The equivalence class of ϕ is denoted $[\phi]$, so we are saying that $[\phi_1] = [\phi_2]$.

We say that a *tangent vector* at $p \in M$ is an equivalence class of regular curves through p. We let $T_p(M)$ be the set of tangent vectors at p.

2 Vector Space Structure

We would like to show that $T_p(M)$ is a vector space, and not just a set. Suppose that M is k-dimensional, so that our coordinate charts are maps from $\boldsymbol{R}^k$ to M. Given a vector $V \in \boldsymbol{R}^k$, let L_V denote the parametrized straight line through the origin whose velocity is V. That is $L_V(t) = tV$.

Let (U, h) be a coordinate chart with $h(0) = p$. We define a map

$$dh : \boldsymbol{R}^k \to T_p(M)$$

by the rule

$$dh(V) = [h \circ L_V].$$

Lemma 2.1 *dh is injective.*

Proof: Suppose that $dh(V) = dh(W)$. Then $[h \circ L_V] = [h \circ L_W]$. But we can use the chart (h, U) to measure the equivalence. So,

$$d(h^{-1} \circ h \circ L_V)|_0 = d(h^{-1} \circ h \circ L_W)|_0.$$

But then

$$V = d(L_V)|_0 = d(L_W)|_0 = W.$$

This completes the proof. ♠

Lemma 2.2 *dh is surjective.*

Proof: Let $[\phi] \in T_p(M)$ be some tangent vector. Let V be the velocity of the curve $h^{-1} \circ \phi$. By construction $dh(V) \sim \phi$. ♠

Now we know that dh is a bijection from $\boldsymbol{R}^k$ to $T_p(M)$. We define the vector space on $T_p(M)$ in the unique way which makes h a vector space isomorphism. That is,

$$dh(V) + dh(W) = dh(V + W), \qquad r\ dh(V) = dh(rV).$$

Lemma 2.3 *The vector space structure on $T_p(M)$ is independent of the choice of coordinate chart.*

Proof: Suppose that h_1 and h_2 are two coordinate charts having the property that $h_1(0) = h_2(0) = p$. Let

$$\phi = h_2^{-1} \circ h_1$$

be the overlap function. Since ϕ is a diffeomorphism, $d\phi|_0$ is a vector space isomorphism. We just have to check that

$$d(h_2 \circ \phi) = dh_2 \circ d\phi.$$

Choose some vector $V \in \boldsymbol{R}^k$ and consider the two curves

1. $h_2 \circ \phi(L_V)$
2. $h_2 \circ L_W$, where $W = d\phi_0(V)$.

We want to show that these curves are equivalent. We can measure this equivalence using the chart (U_2, h_2). We want to see that $\phi \circ L_V$ and L_W have the same velocity at 0. The velocity of L_W at 0 is just W. The velocity of $\phi \circ L_V$ at 0 is, by definition, $d\phi(V)$. This is W. So, these two curves are equivalent. ♠

Now we know that $T_p(M)$ is a k-dimensional real vector space at each point $p \in M$.

3 The Tangent Map

Suppose M and N are smooth manifolds and $f : M \to N$ is a smooth map. Given $p \in M$ let $q = f(p)$. We have the differential map:

$$df|_p : T_p(M) \to T_q(N),$$

defined as follows: Given any $[\phi] \in T_p(M)$ define

$$df([\phi]) = [f \circ \phi].$$

Lemma 3.1 *This definition is independent of all choices.*

Proof: Suppose that ϕ_1 and ϕ_2 are two curves with $\phi_1 \sim \phi_2$. We want to see that $f \circ \phi_1 \sim f \circ \phi_2$. Let (U, g) be a coordinate chart for M with $p = g(0)$ and let (V, h) be a coordinate chart for N with $q = h(0)$. We are trying to show that

$$d(h^{-1} \circ f \circ \phi_1)|_0 = d(h^{-1} \circ f \circ \phi_2)|_0.$$

Note that
$$h^{-1} \circ f \circ \phi_j = (h^{-1} \circ f \circ g) \circ (g^{-1} \circ \phi_j).$$
The maps on the right hand side are maps between Euclidean spaces, and the chain rule applies. Since $\phi_1 \sim \phi_2$, we know that
$$d(g^{-1} \circ \phi_1)|_0 = d(g^{-1} \circ \phi_2)|_0,$$
because $\phi_1 \sim \phi_2$. The desired equality now follows from the chain rule. ♠

Let's check that our new definition of df gives us the same definition in cases we have already worked out.

Lemma 3.2 *If $M = R^k$ and $N = R^m$ and $f(0) = 0$, then the definition of df agrees with the usual one.*

Proof: For Euclidean spaces, we can always use the identity coordinate charts. There is a canonical isomorphism from $T_p(M)$ and $\boldsymbol{R}^k$ which maps $[\phi]$ to the velocity of ϕ at 0. Note that df_{old} maps V to the velocity of $f \circ \phi$. But this is just the velocity of $df_{\text{new}}([\phi])$. ♠

Lemma 3.3 *If $M = \boldsymbol{R}^k$ and $f : M \to N$ is a coordinate chart, then df agrees with the initial definition of df given in terms of straight lines.*

Proof: The previous definition tells us that $df(V) = [f \circ L_V]$. But this matches the new definition, since every tangent vector in $T_p(M)$ can be represented by some L_V. ♠

Now let's talk about the Chain Rule.

Lemma 3.4 *Smooth maps between manifolds obey the chain rule.*

Proof: Suppose $f_{12} : M_1 \to M_2$ and $f_{23} : M_2 \to M_3$ are smooth maps. Then
$$d(f_{23} \circ d_{12})$$
maps the tangent vector $[\phi]$ to $[f_{23} \circ f_{12} \circ \phi]$. But this is clearly the same as $df_{23} \circ df_{12}[\phi]$. ♠

Even though we have established the chain rule, we don't yet know that df is a linear map. So, here's this final result.

Lemma 3.5 *df is a linear map.*

Proof: Let g and h be coordinates for M and N, as above. Introduce the map

$$\psi = h^{-1} \circ f \circ g.$$

Note that

$$f = h \circ \psi \circ g^{-1}.$$

By the Chain Rule, we have

$$df|_p = (dh) \circ (d\psi) \circ (dg)^{-1}.$$

Here $d\psi$ means $d\psi|_0$. All three of the maps on the right are linear maps, so df is as well. ♠

4 Orientations on Manifolds

Orientations on a Vector Space: Let V be a finite dimensional vector space over $\boldsymbol{R}$. Let $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ be two bases for V. We have the transition matrix T_{ij} which expresses the identity map $I : V \to V$ relative to these two bases. Call this matrix T. We call the two bases equivalent if $\det(T) > 0$. By construction, this is an equivalence relation, and there are precisely two equivalence classes. An *orientation* of V is a choice of one of the equivalance classes.

Behavior under Linear Isomorphism: If V and W are vector spaces and $T : V \to W$ is a vector space isomorphism, then T respects the equivalence relations used to define orientations. So, T maps the set of two orientations on V to the set of two orientations on W.

Pointwise Orientations: Let M be a smooth manifold and $S \subset M$ be some set. A *pointwise orientation* on S is a choice of orientation on $T_p(M)$ for each $p \in S$.

Suppose that M and N are smooth manifolds and $f : M \to N$ is a smooth and injective map. Let $S \subset M$ be some set and let $T = f(M)$. Let $p \in S$ and $q = f(p) \in T$. The differential df_p is linear, and hence induces a map from the set of (two) orientations on $T_p(M)$ to the set of (two) orientations on

$T_q(N)$. So, df maps a pointwise orientation on M to a pointwise orientation on N.

Constant Orientations: When $M = \boldsymbol{R}^k$, there are two *constant orientations*. In either case, we just identify all the tangent spaces of M by translation, and take the same orientation at each point. If $U, V \subset \boldsymbol{R}^n$ is an open set and $h : U \to V$ is a diffeomorphism, then dh maps a constant orientation on U to a constant orientation on V. The point is that the determinant of dh never changes sign.

The result here is worth pondering. Even though dh could vary from point to point, on the level of orientations it is always a constant map.

Local Orientations: Let M be a manifold and let $U \subset M$ be an open set. A pointwise orientation on U is a *local orientation* if the orientation is the image of a constant orientation under a coordinate chart. It follows from the chain rule, and from the facts already mentioned about constant orientations, that this definition is independent of coordinate chart.

Global Orientations: A global orientation on M is a pointwise orientation which is a local orientation relative to every coordinate chart. If M has a global orientation, then M is said to be *orientable/*