Tensor Transformations

Rich Schwartz

February 26, 2015

I found the material in §12.4 really hard to read. I re-wrote the material in a way which seems easier to get through. One thing about my notes is that I switch the roles of V and W, because we've all had years of experience thinking about linear transformations from V into W and not the reverse. (This is just a psychological preference, of course.)

1 Pulling Back a Tensor

Let V and W be vector spaces and let $M : V \to W$ be a linear transformation. The map M gives a linear transformation

$$M^* : T^r(W) \to T^r(V). \tag{1}$$

Note that V and W have switched. Let $T : W^r \to \boldsymbol{R}$ be a tensor of type r. We have the tensor $M^*T : V^r \to \boldsymbol{R}$ defined by the equation

$$M^*(T)(V_1, ..., V_r) = T(M(V_1), ..., M(V_r)). \tag{2}$$

In other words, we map $V_1, ..., V_r$ into W and then apply the tensor to them. Everything involved is linear, so M^* is a linear map. The goal of these notes is to explain the action of M^*.

Let $\{v_1, ..., v_m\}$ is a basis for V and $\{w_1, ..., w_n\}$ is a basis for W. We have the formula

$$M(v_i) = \sum_{k=1}^{n} M_{ik} w_k. \tag{3}$$

The goal is to express the map M^* in terms of these coefficients.

There are three cases, the first of which is just a warm-up: the linear functional case, the general case, and the alternating case.

2 Linear Functional Case

We are interested in $M^* : W^* \to V^*$. We have the dual bases $\{v_1^*, ..., v_m^*\}$ and $\{w_1^*, ..., w_n^*\}$. Here $v_i^*(v_j) = 1$ if $i = j$ and 0 otherwise. Same goes for w_i^*. The matrix for M^* is just the transpose of the matrix for M.

To figure out the matrix for M^*, we just have to see that $M^*(w_j^*)$ does to v_i. We compute

$$M^*(w_j^*)(v_i) =$$

$$w_j^*(M(v_i)) =$$

$$w_j^*(\sum_{k=1^n}(M_{ik}w_k)) =$$

$$\sum_{k=1}^{n} w_j^*(M_{ik}w_k) =$$

$$M_{ij}.$$

In short,

$$M^*(w_j^*)(v_i) = M_{ij}. \tag{4}$$

But that means that

$$M^*(w_j^*) = \sum_{k=1}^{m} M_{ij}v_i^* \tag{5}$$

This is why the matrix for M^* is just the transpose of M_{ij}.

3 General Case

Let's introduce the multi-index notation. Let $I = (i_1, ..., i_r)$ be an r-tuple of numbers. We write

$$v_I^* = v_{i_1}^* \otimes ... \otimes v_{i_r}^*. \tag{6}$$

We write the same thing for w_I^*. Also, we write

$$v_I = (v_{i_1}, ..., v_{i_r}).$$

This is just an r-tuple of vectors. We have $v_I^*(v_J) = 1$ if $I = J$ and 0 otherwise.

We want to figure out what $M^*(w_J^*)$ does to v_I. This gives the component M_{IJ}^* of the giant matrix representing M^*.

We compute

$$M^*(w_J)(v_I) =$$
$$w_J(M(v_I)) =$$
$$w_J(M(v_{i_1}), ..., M(v_{i_r})) =$$
$$w^*_{j_1} \otimes ...w^*_{j_r}(M(v_{i_1}), ..., M(v_{i_r})) =$$
$$w^*_{j_1}(M(v_{i_1})) \times ... \times w^*_{j_r}(M(v_{i_r})) =$$
$$M_{i_1 j_1} ... M_{i_r j_r}.$$

So, the bottom line is that

$$M_{IJ} = M_{i_1,j_1} ... M_{i_r j_r}. \tag{7}$$

4 Alternating Case

The basis elements for $\wedge^r(V^*)$ are given by

$$[v^*_I] = A(v^*_I) = v_{i_1} \wedge ... \wedge v_{i_r}.$$

Sinilarly for $\wedge^r(W^*)$. The tensor $M^*([w_J]^*)$ is some linear combination of the various $[v_I]^*$. We want to find the coefficients. We have

$$[w^*_J] = \sum_\sigma \epsilon(\sigma) w^*_{\sigma J}. \tag{8}$$

Here σ is a permutation, and $\epsilon(\sigma)$ is the sign of σ, and σJ denotes the multi-index you get when you permute the entries of J according to the action of σ.

Now let's take I to be an increasing multi-index: $i_1 < ... < i_r$. From the previous case, and linearity, we have

$$M^*([w^*_J])(v_I) = \sum_\sigma \epsilon(\sigma) M_{I,\sigma J} = \sum_\sigma \epsilon(\sigma) M_{i_1 \sigma(j_1)}, ..., M_{i_r,\sigma(j_r)}). \tag{9}$$

This last expression is just the determinant of the $r \times r$ matrix you get by taking I rows of M and the J columns.

5 Crucial Special Case

Suppose that $V = W$ and $r = n = \dim(V)$. Then the transformation law tells us that M^* is just multiplication by $\det(M)$. In particular, M^* is the identity map if $\det(M) = 1$.