Partitions of Unity

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1 The Result

Let M be a smooth manifold. This means that

- M is a metric space.
- M is a countable union of compact subsets.
- M is locally homeomorphic to $\boldsymbol{R}^n$. These local homeomorphisms are the coordinate charts.
- M has a maximal covering by coordinate charts, such that all overlap functions are smooth.

Let $\{\Theta_\alpha\}$ be an open cover of M. The goal of these notes is to prove that M has a partition of unity subordinate to $\{\Theta_\alpha\}$. This means that there is a countable collection $\{f_i\}$ of smooth functions on M such that:

- $f_i(p) \in [0,1]$ for all $p \in M$.
- The support of f_i is a compact subset of some Θ_α from the cover.
- For any compact subset $K \subset M$, we have $f_i = 0$ on K except for finitely many indices i.
- $\sum f_i(p) = 1$ for all $p \in M$.

The *support* of f_i is the closure of the set $p \in M$ such that $f_i(p) > 0$.

These notes will assume that you already know how to construct bump functions in $\boldsymbol{R}^n$. Note: I deliberately picked a weird letter for the cover, so that it doesn't interfere with the rest of the construction.

2 The Compact Case

As a warm-up, let's consider the case when M is compact. For every $p \in M$ there is some open set V_p such that

- $p \in V_p$.
- $V_p \subset \Theta_\alpha$ for some Θ_α from our cover.
- V_p is contained in a coordinate chart.

Using the fact that we are entirely inside a coordinate chart, we can construct a bump function $f : M \to [0,1]$ such that $f(p) > 0$ and the support of f is contained in a compact subset of V_p. Let $W_p \subset V_p$ denote the set of points where $f > 0$. Then W_p is an open set which contains p. Call W_p a *nice open set*.

The set $\{W_p | \; p \in M\}$ is an open covering of M. Since M is compact, we can find a finite number $W_1, ..., W_m$ of nice open sets such that $M = \bigcup W_i$. Let $g_1, ..., g_m$ be the functions associated to these open sets. By construction, $g_i > 0$ on W_i. This means that the sum $\sum g_i$ is positive on M. Define

$$f_i = \frac{g_i}{\sum g_i}. \tag{1}$$

Then $f_1, ..., f_m$ make the desired partition of unity.

The rest of the notes deal with the case when M is not compact.

3 Fattening Compact Sets

We need two technical lemmas.

Lemma 3.1 *Let $p \in M$ be any point. For all sufficiently small ϵ, the ball of radius ϵ has compact closure in M.*

Proof: There is some neighborhood U of p which is homeomorphic to $\boldsymbol{R}^n$. Let $\phi : U \to \boldsymbol{R}^n$ be a homeomorphism. Choose some closed ball $B \subset \boldsymbol{R}^n$ which contains $\phi(p)$. Consider $\phi^{-1}(B)$. This is a compact subset of M, and it contains the open set $U' = \phi^{-1}(\text{interior}(B))$. Any sufficiently small open ϵ ball Δ about p will be contained in U' and hence will have closure contained in the compact set $\phi^{-1}(B)$. A closed subset of a compact set is compact. Hence, the closure of Δ is compact. This is what we wanted to prove. ♠

Lemma 3.2 *If $X \subset M$ is compact, then there exists some compact subset Y such that X is contained in the interior of Y.*

Proof: For each $p \in X$, there is some ϵ ball Δ_p whose closure in M is compact. The union of such balls covers X. Since X is compact, we can take a finite subcover. That is, $X \subset \Delta_1 \cup ... \cup \Delta_m$. Let Y be the union of the closures of these balls. Since Y is a finite union of compact sets, Y is compact. The interior of Y contains the union of these open balls, and hence contains X. ♠

4 Cleaning up the Compact Sets

Lemma 4.1 *There exists a countable collection $\{K_i\}$ of compact sets such that K_i is contained in the interior of K_{i+1} for all i, and $M = \bigcup K_i$.*

Proof: We know already that $M = \bigcup K_i$, where K_i is compact and the union is countable. Replacing K_m by $K_1 \cup ... \cup K_m$, it suffices to consider the case when $K_1 \subset K_2 \subset K_3 ....$

Suppose we know already that K_i is contained in the interior of K_{i+1} for $i = 0, ..., m$. By the preceding lemma, we can replace K_{m+2} by a larger compact set L_{m+2} which contains K_{m+2} in its interior. Now we redefine $K_{m+3} = L_{m+2} \cup K_{m+3}$ and $K_{m+4} = L_{m+2} \cup K_{m+3} \cup K_{m+4}$, etc. The new collection of compact sets has $K_i \subset K_{i+1}$ for all $i = 0, ..., m+1$. By induction, we can get this property for all i. ♠

Lemma 4.2 *We can write $M = \bigcup L_i$, where L_i is compact for all i, and $L_i \cap L_j = \emptyset$ if $j < i - 1$.*

Proof: We know that $M = \bigcup K_i$, where each K_i is compact, and K_i is contained in the interior of K_{i+1} for all i. Define

$$L_i = K_i - \text{interior}(K_{i-1}). \tag{2}$$

Note that L_i is disjoint from K_j for $j < i - 1$. Hence L_i is disjoint from L_j for $j < i - 1$. By construction L_i is a compact set minus an open set. In other words, L_i is the intersection of a compact set and a closed set. Hence L_i is compact. Also, $M = \bigcup L_i$. ♠

5 The Main Construction

We keep the notation from the previous section. Consider L_i. Each $p \in L_i$ has an open metric ball U such that

- U is disjoint from L_j for all $j < i - 1$. This uses the fact that there is a minimum positive distance between U_i and U_j for all $j < i - 1$.
- U is contained in some Θ_α from our cover.
- U is contained in a coordinate chart.

As in the compact case, we can construct a bump function f such that $f(p) > 0$ and the support of f is contained in a compact subset of U. Let $W \subset U$ denote the set where $f > 0$. Call W a nice set. Since L_i is compact, we can cover L_i by finitely many nice sets, say $W_{i1}, ..., W_{im_i}$. (The number depends on i.)

Now we consider the covering

$$W_{11}, ..., W_{1i_1}, W_{21}, ..., W_{2i_2}, ...$$

We rename these sets $X_1, X_2, X_3, ...$ and let g_1, g_2, g_3 be the associated functions. These functions have the following properties.

- For every $p \in M$, there is some g_i such that $g_i > 0$. This comes from the fact that $p \in L_j$ for some j, and then p is contained in some nice set on our list.
- Any compact set only intersects finitely many X_i. The point is that any compact set is contained in the union of finitely many L_i.
- The support of each g_i is contained in some Θ_α from the original cover. This comes from the fact that the support of g_i is the closure of a nice set.

Now we define $f_i = g_i / \sum g_j$, as in the compact case. The sum is locally finite at each point. This gives us the partition of unity.