Math 123 HW 1

1. The 3-peg Towers of Hanoi problem works like this. You have 3 pegs, and N rings distributed over the pegs. No two rings have the same size. Figure 1 shows an example when $N = 6$.

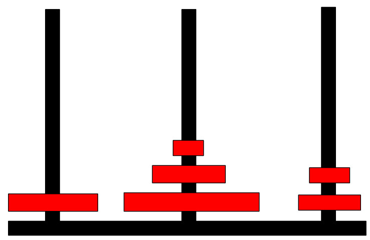

Figure 1: A sequence of graphs

The only rule is that a larger ring cannot sit on top of a smaller ring. A legal move consists of transfering a ring from one peg to another peg, keeping it legal. The *state graph* for the game is the graph whose vertices are the legal states of the game. Two vertices are connected by an edge if and only if they are connected by a legal move. Figure 2 suggests a sequence of graphs, $G_1, G_2, G_3, ...$

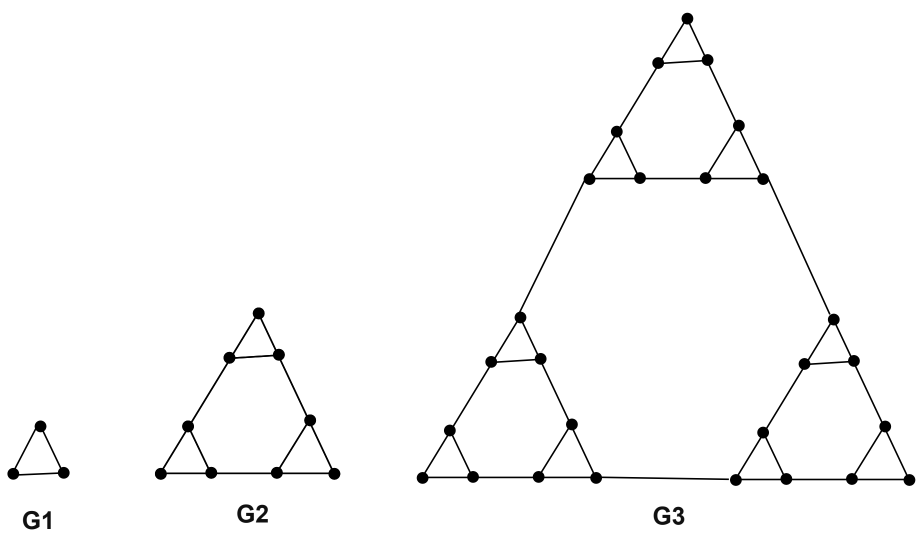

Figure 2: A sequence of graphs

The graph G_{n+1} is obtained from the graph G_n by adding chords to all the little triangles and placing new vertices at the intersection points. Prove that the state graph for the N-ring game is isomorphic to the graph G_N.

2. Say that a *trivalent tree* is a tree, all of whose vertices have degree either 1 or 3. Prove that there is some constant $C > 1$ so that there are at least $C^N - 1$ trivalant trees with $2N$ vertices, up to isomorphism.

3. Let $S_k(N)$ denote the set of finite trees having N vertices and exactly k vertices of degree 1. Prove, for each fixed k, that there is some polynomial P_k so that the number of non-isomorphic trees in $S_k(N)$ is less than $P_k(N)$.

4. Let G be a planar graph. Say that G is *nicely drawn* if all the edges of G are line segments, if no two edges cross, and if there is some K such that

- Every edge of G has length at least $1/K$ and at most K.
- The small angle between any two edges of the graph (which are incident to the same vertex) is at least $1/K$.

For instance, the graph you get by taking the edges of the usual infinite square grid is nicely drawn. In this example, all edges have length 1 and all the angles are either $\pi/2$ or π.

Prove that the infinite trivalent tree, shown in Figure 3, cannot be nicely drawn in the plane. All the vertices in this infinite tree have valence 3.

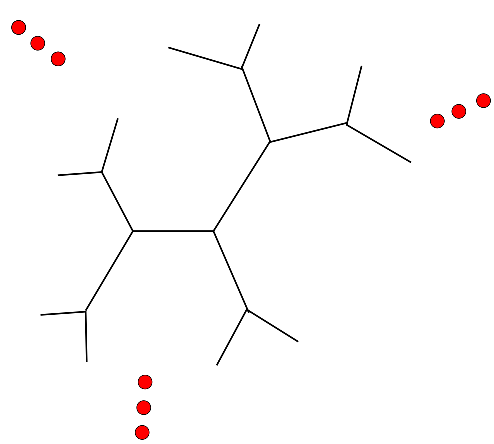

Figure 3: The beginning of the infinite trivalent tree

Hint: Exponential crowding.

5. Think of the number line as the x-axis in the plane. Consider the graph whose vertices are the rational numbers and also the point ∞. First join the integers to ∞ using vertical rays, as shown in Figure 4.

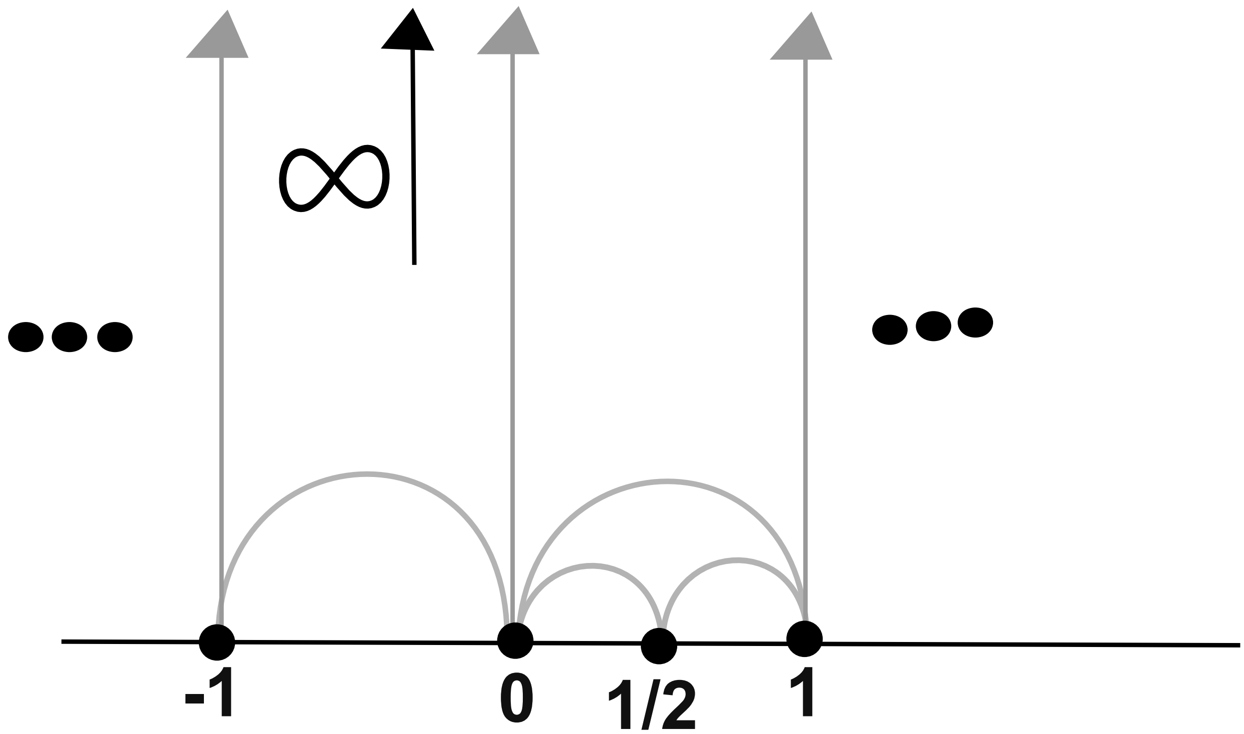

Figure 4: Part of the graph

Next, join two rational numbers (including integers) p_1/q_1 and p_2/q_2 by an edge if and only if the matrix

$$\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix}$$

has determinant ± 1. (Integers are represented as $p/1$.) The edges you should use are semicircles which meet the x-axis in right angles. Prove that no two edges cross, so that you get a planar graph. (Well, it's not quite planar because it has the extra point ∞, but just go with it.)

Extra Credit: This is a continuation of Problem 5, really. Start with the graph constructed in Problem 5, and construct a new graph where you put one vertex in each region bounded by the edges and join two vertices if the regions are adjacent. Prove that the resulting graph is the infinite trivalent tree. The red tree in Figure 5 is the beginning of this tree.

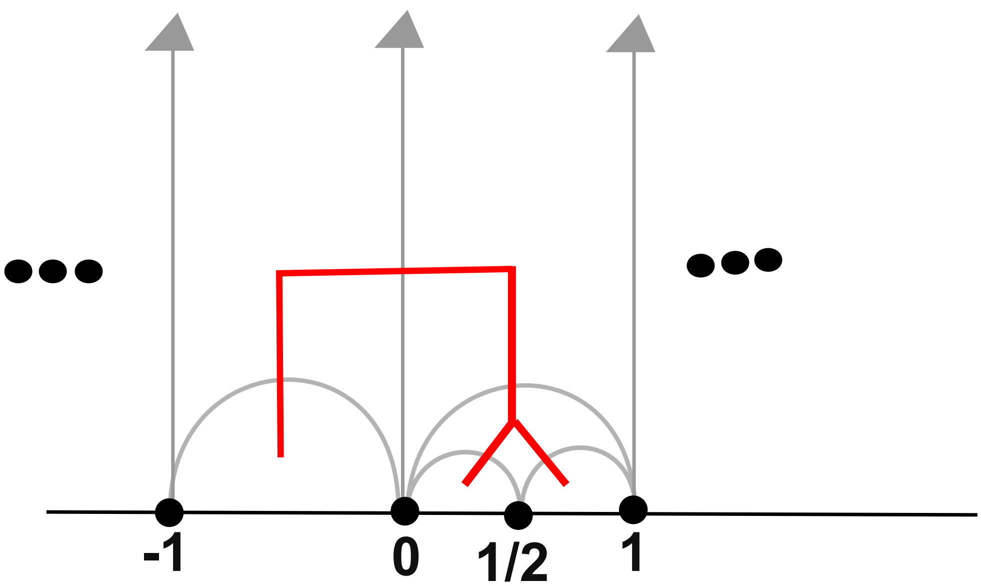

Figure 5: The red tree

Hints: Here are some facts from elementary number theory you might like to know, both for Problem 5 and for the extra credit.

- Two integers p and q are relatively prime if and only if there are integers a and b such that $ap + bq = 1$.
- If you have a 2×2 integer matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and a rational number t you can define $M(t) = (at + b)/(ct + b)$. You might want to prove that t_1 and t_2 are joined by an edge if and only if $M(t_1)$ and $M(t_2)$ are joined by an edge. This will let you move the graph around using matrices.