Math 123 HW 3

1. In class we defined the cut locus of a polygon to be the set of centers of maximal disks contained in the (solid) polygon. See Figure 1.

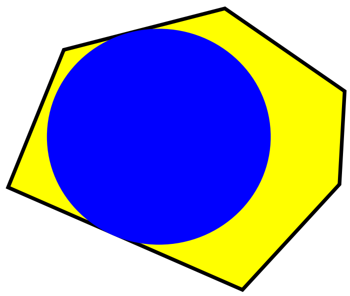

Figure 1: A maximal disk in a convex hexagon

Here *maximal* means that the disk is not contained in a larger disk also contained in the polygon. Prove that the cut locus for a convex hexagon is a tree. Here *convex* means that every line segment intersects the hexagon in a connected segment. What are the possible trees you can get?

2: Prove, for any n, that there exists n convex polygons, all having the same number of sides, such that the corresponding cut loci are pairwise non-isomorphic and trivalent. (All vertices have degree 1 or 3.) You can assume the theorem that the cut locus is always a tree (though after Problem 1 you might see how the proof goes in general.) The point of this problem is for you to look for distinguishing features of the trees.

3. Recall that the 2-adic integers are identified with the set of directed paths in a rooted binary tree. As explained in class, the distance between two such paths is 2^{-n}, where n is the number of edges common to both paths. This gives the 2-adic integers the structure of a metric space. Prove that the maps $f(x) = x + 1$ and $f(x) = 3x$ are both continuous maps on the metric space of 2-adic integers. In other words, for each point p and each $\epsilon > 0$ you want to show that there is some $\delta > 0$ such that the distance from $f(p)$ to $f(q)$ is less than ϵ provided that the distance from p to q is less than δ.

4. Formulate and prove a 3 dimensional version of Sperner's Lemma. Obviously, the formulation should involve a tetrahedron that is subdivided into smaller tetrahedra.

5. This is a problem from the book. A *caterpillar* is a tree which has a path such that every vertex is at most one unit away from the path. In other words, you get a caterpillar by starting with a path and attaching some edges directly to the vertices along the path. See Figure 2. Prove that every caterpillar has a graceful labeling. In fact, you can color the vertices black and white in a bipartite way so that the labels of all the black vertices are larger than the labels of all the white vertices.

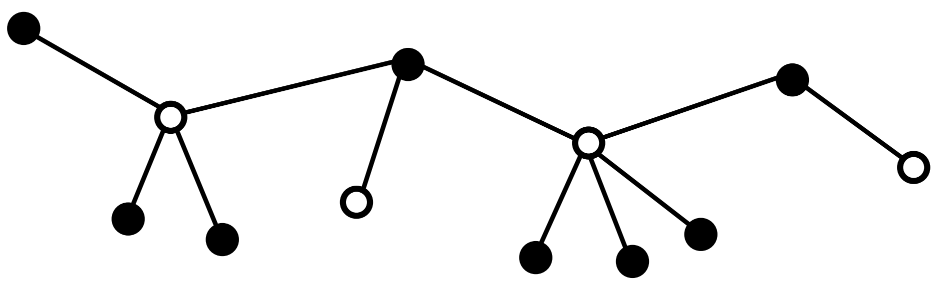

Figure 2: A caterpillar