The Polygonal Jordan Curve Theorem

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1 Main Result

A *polygonal loop* is a finite union of line segments $S_1, ..., S_n$ in the plane such that

- S_i and S_{i+1} share a common vertex for all i.
- S_i and S_j are disjoint if $i \neq j \pm 1$.

The indices are taken cyclically, so that $n+1$ is the same as 1. In other words, a polygonal loop is an embedded cycle, in which all the edges are straight lines. A *polygonal path* is defined in the same way, except that S_1 and S_n are also disjoint.

An open subset $U \subset \boldsymbol{R}^2$ is *path connected* if every two points $p, q \in U$ can be joined by a polygonal path. Here is the main result.

Theorem 1.1 (Polygonal Jordan Curve) *If P is any polygonal loop then $\boldsymbol{R}^2 - P$ consists of exactly two path connected sets U_1 and U_2. That is, U_1 and U_2 are path connected, and no point in U_1 can be joined to a point in U_2 by a path that does not cross P.*

The Jordan Curve Theorem is certainly true for triangles. The proof in the general cases uses this special case.

2 Intersections of Polygonal Loops

A polygonal path or loop P *cleanly crosses* a polygonal path or loop Q if

1. No vertex of P lies in Q.
2. No vertex of Q lies in P.
3. $P \cap Q$ consists of finitely many points.

Here is the main result in this section.

Lemma 2.1 *If P and P are two polygonal loops which interesect cleanly, then the number of intersection points is even.*

Proof: Let's first prove the result when P is a triangle. Since P satisifies the Jordan Curve Theorem, we can say that P has an inside and an outside. So, as we travel around Q, each intersection point represents a switch from outside to inside, or *vice versa*. Since we end up at the same place we started, there are an even number of switches.

The general case goes by induction in the number of sides of P. By considering all the lines emanating from a vertex of P we can find an edge e which joins two vertices of P and does not otherwise intersect P. This is shown in Figure 1.

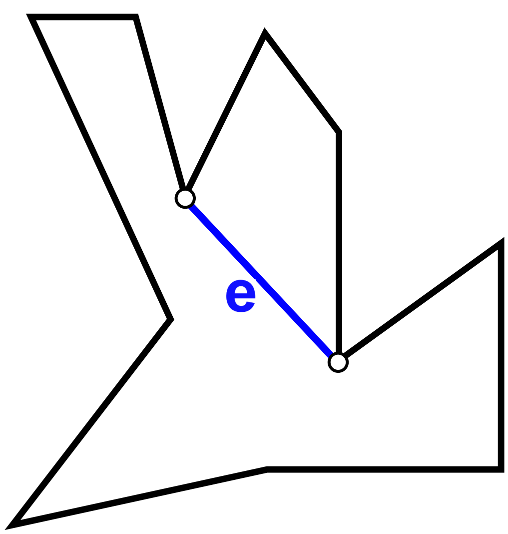

Figure 1: Dividing P into P_1 and P_2.

e divides P into two smaller polygonal loops, P_1 and P_2. Each of these loops uses some consecutive sides of P and then has e as the last side. The intersection $P_1 \cap P_2$ is exactly e. By rotating Q slightly, so as to leave the number of intersection points unchanged, we can arrange that all of P, P_1, P_2 have a clean intersection with Q. Let N, N_1, N_2, N_e denote the number of times that Q intersects P_1, P_2, P, e respectively. By induction, N_1 and N_2 are even. But also

$$N = (N_1 - N_e) + (N_2 - N_e) = N_1 + N_2 - 2e.$$

Therefore N is also even. ♠

3 The Main Argument

Now I'll give the argument I gave in class. For each point $p \in \boldsymbol{R}^2 - P$ consider any ray emanating from p that intersects P cleanly and let E_p denote the parity of the number of intersection points with this ray.

Lemma 3.1 *E_p is well defined.*

Proof: Let R_1 and R_2 be two rays emanating from P. By choosing points on R_1 and R_2 that are very far away from P and joining them by a line segment, we can find a triangle Q that only interesects P on $R_1 \cup R_2$. But $P \cap Q$ has an even number of intersection points. Hence, the parity of the number of intersection points of P with $R_1 \cup R_2$ is even. ♠

Lemma 3.2 *If $E_p \neq E_q$, then p and q cannot be joined by a polygonal loop in $\boldsymbol{R}^2 - P$.*

Proof: Suppose this is false. Then we can make a polygonal loop which intersects P an odd number of times. To make the loop, we connect p to q in $\boldsymbol{R}^2 - P$, then adjoin rays emanating from p and q way outside of P, then connect points on these rays. This is a contradiction. ♠

Lemma 3.3 *If $E_p = E_q$ then p and q can be joined by a polygonal loop in $\boldsymbol{R}^2 - P$.*

Proof: Consider a polygonal path Q which joins p to q and intersects P cleanly in the fewest possible number of points. The number must be even, by the same argument as above. Let a be the first intersection point of $Q \cap P$ we reach as we go from p to q along Q. We make a new path as follows. Just before reaching a we veer off and follow P around until we come to another intersection point of $P \cap Q$. This detour must hit another intersection point b in $P \cap Q$ because otherwise we would have a loop that intersects P an odd number of times. Taking the detour, we get a new polygonal path which joins p to q and intersects P fewer times. This is a contradiction. ♠

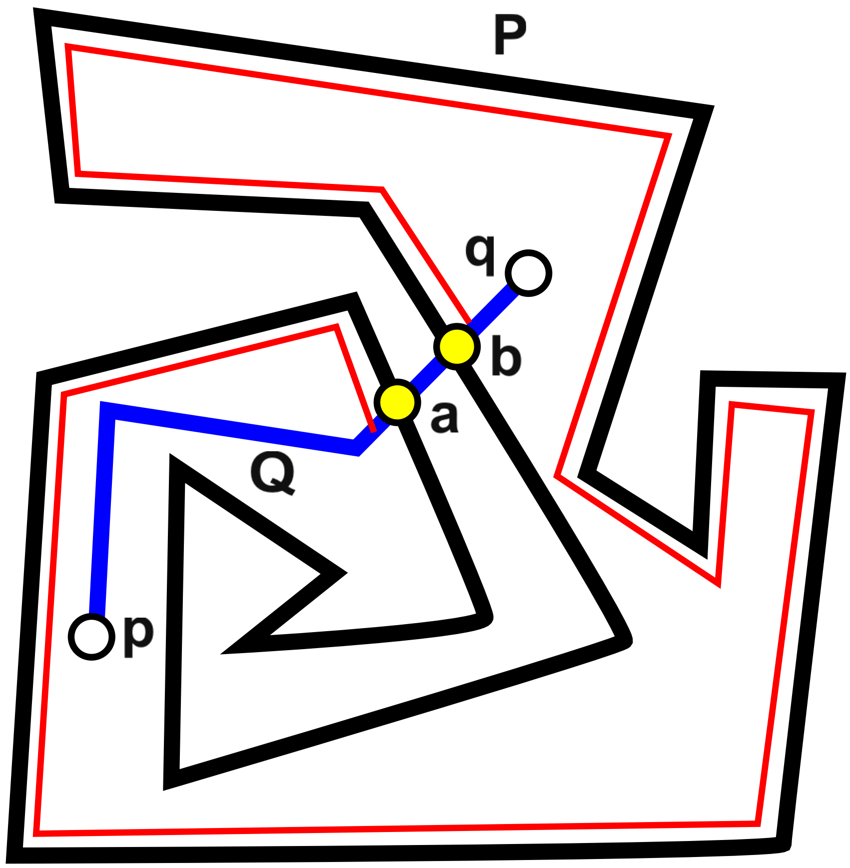

Figure 2: Decreasing the number of intersection points

The theorem follows from the lemmas above.