The Polygonal Jordan Curve Theorem

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0.1 Polygons and Cycles

Let $\boldsymbol{R}^2$ be the plane. A *directed segment* is a segment in $\boldsymbol{R}^2$ together with a direction – e.g., a choice of a tail vertex and a head vertex. The direction points from the head to the tail. The directed segment B *follows* the directed segment A if the head of A equals the tail of B. Also, B *nicely follows* A if B follows A and $A \cap B$ is the common vertex. Now we make 4 definitions:

- A *polygonal path* is a finite sequence $A_1, ..., A_n$ of directed arcs such that A_{i+1} follows A_i for all $i = 1, ..., n-1$.
- A *polygonal loop* is a finite sequence $A_1, ..., A_n$ such that A_j follows A_i provided that $j - i$ is congruent to 1 mod n.
- The polygonal path $A_1, ..., A_n$ is *embedded* if the specified followings are nice and if $A_i \cap A_j = \emptyset$ except when $|i - j| \leq 1$.
- The polygonal loop $A_1, ..., A_n$ is embedded if all the specified followings are nice and $A_i \cap A_j = \emptyset$ unless $i = j$ or $i - j$ is congruent to ± 1 mod n.

Figure 1 shows examples of each kind of object.

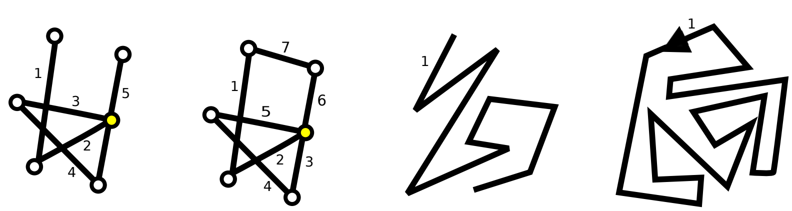

Figure 1: paths and loops

When the polygonal paths and loops are not embedded, there might be several ways to trace them out and get the same underlying set in the plane. The left two pictures suggest this. In all cases we have included just enough information to determine exactly which directed segments are involved.

Here is the polygonal Jordan Curve Theorem

Theorem 0.1 *Let J be any polygon. Then $\boldsymbol{R}^2 - J$ consists of exactly two sets A and B such that*

- *Any two points of A can be joined by a polygonal path in A.*
- *Any two points of B can be joined by a polygonal path in B.*
- *No point of A can be joined to a point of B by a polygonal path that avoids J.*

One often abbreviates this by saying that $\boldsymbol{R}^2 - J$ has exactly 2 *path components*.

Here is the polygonal Jordan Arc Theorem

Theorem 0.2 *Let A be any embedded polygonal. Then any two points of $\boldsymbol{R}^2 - A$ can be joined by a polygonal path that avoids A.*

One often abbreviates this by saying that $\boldsymbol{R}^2 - A$ has exactly 1 *path component*.

We'll first prove the Jordan Curve Theorem and then use the same method to prove the Jordan Arc Theorem. We just sketch the proof in the Arc Case.

0.2 A Parity Argument

We say that two directed segments A and B *intersect transversely* if either $A \cap B = \emptyset$ or $A \cap B$ is a single point which is not an endpoint of either A or B. We write $I(A, B) = 0$ or 1 according as there is no intersection point or one. Note that if B' is obtained from B by making a sufficiently small rotation and/or translation, A and B' still intersect transversely, and $I(A, B') = I(A, B)$. Call this the *wiggle property* for segments.

Let L and M be polygonal paths or loops made respectively with ℓ and m directed segments. We write $L = L_1, ..., L_\ell$ and $M = M_1, ..., M_m$. We say that L and M *intersect transversely* if L_i and M_j intersect transversely for all relevant indices. In this case, we define

$$I(L,M)=\sum_{i=1}^{\ell}\sum_{j=1}^{m} I(L_i,M_j). \tag{1}$$

We call $I(L,M)$ the *intersection number* of L and M. When L and M are not embedded, this could be different from the actual number of points of $L\cap M$. When L and M are embedded $I(L,M)$ is exactly the number of points in $L\cap M$.

In all the results to follow, L and M are assumed to have a transverse intersection.

Lemma 0.3 *If L is a polygonal loop made of 2 segments then $I(L,M)$ is even.*

Proof: We have $L=L_1\to L_2$ where $L_1=L_2$ except that the direction of L_2 is opposite that of L_1. We immediately have

$$I(L,M)=I(L_1,M)+I(L_2,M)=2I(L_1,M).$$

This completes the proof. ♠

Lemma 0.4 *If L is a polygonal loop made of 3 segments then $I(L,M)$ is even.*

Proof: Note that L is either embedded or else the union of two of the edges of L equals the third one. In the latter case, the same argument as in the previous lemma proves the result. In case L is embedded, L is just a triangle. The Jordan Curve Theorem is trivial when L is a triangle. One of the components of $\boldsymbol{R}^2-L$ is the solid triangle bounded by L, the "inside", and the complementary component is the "outside". Consider what happens when we trace around M. Each time M intersects L, we either move from the inside of L to the outside, or from the outside to the inside. If we start on the inside, we end on the inside when we go all the way around. Likewise if we start on the outside, we end on the outside when we go all the way around. Hence the number of switches is even. But this is exactly $I(L,M)$. ♠

Now we come to the main result which subsumes the previous ones. Once again we assume that L and M intersect transversely. Note that L and M' also intersect transversely, when M' is the result of translating/rotating M by a sufficiently small amount. Furthermore, for all sufficiently small such perturbations, we have $I(L,M) = I(L,M')$. This new *wiggle property* is a consequence of the wiggle property for segments.

Lemma 0.5 *$I(L,M)$ is even for any two polygonal loops.*

Proof: Let $L = L_1, ..., L_\ell$. It suffices to consider the case when $\ell \geq 4$.

Let Λ be the straight line segment directed from the head of L_3 to the tail of L_1. By the wiggle property, we can assume that Λ has transverse intersection with M.

Consider the two new polygonal loops

$$L' = L_1 \to L_2 \to \Lambda, \qquad L'' = (-\Lambda) \to L_3 \to ... \to L_\ell. \tag{2}$$

Here $-\Lambda$ is the same segment as Λ but with the direction reversed. Both L' and L'' have transverse intersection with M.

We have

$$I(L,M) = I(L',M) + I(L'',M) - 2I(\Lambda,M),$$

because when we compute $I(L',M)$ and $I(L'',M)$ we are counting $I(\Lambda, M_j)$ twice for each $j = 1, ..., m$. By induction on the number of segments in L, both $L(L',M)$ and $I(L'',M)$ are even. Hence $I(L,M)$ is also even. ♠

0.3 At Least Two Components

Let J be a polygon. We prove in this section that $\boldsymbol{R}^2 - J$ has at least 2 components. Any two points $a, b \in \boldsymbol{R}^2 - J$ can be connected by a polygonal path which has transverse intersection with J. Just join a and b by a line segment, and put little kinks in the segment if it does not already have transverse intersection with J. Since J has only finitely many segments, this is easy to do.

Given any such path L joining a to b we define

$$I(a,b,J) = I(L,J) \bmod 2. \tag{3}$$

Lemma 0.6 *$I(a,b,J)$ is well defined, independent of all choices.*

Proof: If L' is some other path joining a to b and transverse to J, then $L \cup (-L')$ is a polygonal loop transverse to J and by the result in the previous section we have $I(L \cup L', J)$ is even. But then $I(L,J)$ and $I(L',J)$ have the same parity. Hence $I(a,b,J)$ is well defined, independent of all choices. ♠

Note that if $I(a,b,J) = 1$ it is impossible to join a to b by a path which avoids J, because this path would give us $I(a,b,J) = 0$. If we choose points a and b that lie very close to each other and on opposite sides of some segment of J, then we have $I(a,b,J) = 1$ because the line segment joining a to b intersects J only once. This shows that not all points in $\boldsymbol{R}^{-}J$ can be joined by polygonal paths which avoid J.

0.4 At Most Two Components

Let J be a polygon, as above. In this section we produce 2 points $a, b \in \boldsymbol{R}^2 - J$ such that every point of $\boldsymbol{R}^2 - J$ can be joined to either a or b. We let A and B denote the set of points in $\boldsymbol{R}^2 - J$ which can be joined to a and b respectively. Then a and b are "hubs" for A and B respectively: Any two points of A can be joined by a polygonal path which avoids J and goes through a. Likewise for B. Also, a and b cannot be joined together because this would contradict the result in the previous section. So, once we show the existence of a and b we are done with the proof of the Jordan Curve Theorem.

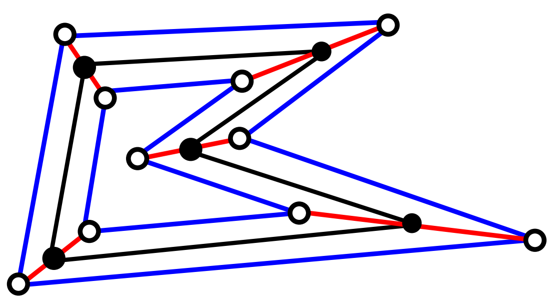

Figure 2: Building the fence

Our construction depends on some small $\epsilon > 0$. Let $v_1, ..., v_n$ be the consecutive vertices of J. Let $A_1, ..., A_n$ be the angle bisectors at $v_1, ..., v_n$. We choose points $v_{k,+}$ and $v_{k,-}$ along A_k so that consecutive pairs of points may be joined in pairs by (blue) segments which are parallel to the corresponding side of J and exactly ϵ away. If we choose ϵ small enough, these segments are all disjoint from J. By symmetry these segments piece together at their boundaries and form a union of 1 or 2 polygons. We call the union of all these connecting edges the *fence*. Figure 2 shows the construction. The angle bisectors are colored red and the fence is colored blue.

One of two things happens. Either the fence is connected or it is not. In Figure 2 it is not connected. If the fence is connected then we can join vertex $v_{1,-}$ to vertex $v_{i,+}$ by a (blue) path which avoids J. This gives $I(v_{1,-}, v_{1,+}, J) = 0$. But on the other hand, one can join these points by a path (very close to the angle bisector) which intersects J once, giving $I(v_{1,-}, v_{1,+}, J) = 1$. This is a contradiction.

So, the fence is disconnected and consists of 2 distinct polygons $J(-, \epsilon)$ and $J(+, \epsilon)$ which are disjoint from J. We define

$$a = v_{1,+}, \qquad b = v_{1,-}.$$

Now we are going to vary the choice of ϵ. If we replace ϵ by $t\epsilon$ with $t \in (0, 1)$, the same construction works. We choose the signs for $J(+, t\epsilon)$ and $J(-, t\epsilon)$ so that the fences vary continuously.

Lemma 0.7 *For any $t < 1$, any point of $J(+, t\epsilon)$ can be joined to a by a path which avoids J. Likewise, any point of $J(-, t\epsilon)$ can be joined to b by a path which avoids J.*

Proof: We prove the first statement. The second is similar. Pick the point c and follow along $J(+, \epsilon)$ until you hit A_1. Then go out along A_1 until hitting a. ♠

Lemma 0.8 *Let $c \in \boldsymbol{R}^2 - J$. Then there is some $t \in (0, 1)$ so that c may be joined to one of $J(+, t\epsilon)$ or $J(-t\epsilon)$ by a polygonal path that avoids J.*

Proof: Choose some line from c to a point on J which contains no vertices of J. Taking the first point where this line hits J we produce a point m in the interior of some segment J_j of J such that $\overline{cm}$ hits J only at the endpoint m. The intersection of $\overline{cm}$ with the union $X_t = J(+, t\epsilon) \cup J(-, t\epsilon)$ is non-empty for sufficiently small t because near m the set X_t is just 2 parallel line segments which are converging to J_j as $t \to 0$. So, if we choose t small enough then $\overline{cm}$ hits X_t before hitting J. ♠

Combining the previous two results, we see that any point of $\boldsymbol{R}^2 - J$ can be joined to one of a or b. This completes the proof of the Jordan Curve Theorem for polygons.

0.5 Polygonal Jordan Arc Theorem

Let A be such an arc. We choose ϵ small and make the same kind of fence construction as in the previous section. Figure 3 shows that we mean.

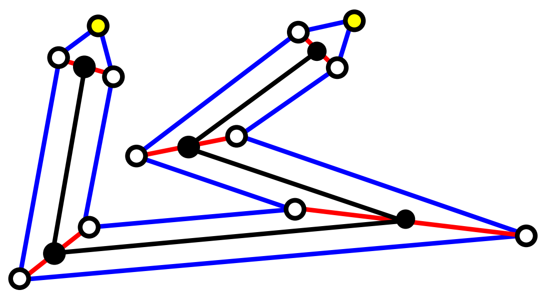

Figure 3: Building the fence for an arc

The resulting set $A(\epsilon)$ is a polygon provided that ϵ is small enough. We let a be some vertex of this polygon. The same two lemmas from the previous section work here. For any $t < 1$ any point of $A(t\epsilon)$ can be joined to a by a path that avoids A. Likewise any point of $\boldsymbol{R}^2 - A$ can be joined to $A(t\epsilon)$, for some $t > 0$, by a path that avoids A. Hence every two points of $\boldsymbol{R}^2 - A$ can be connected by a path that avoids A.